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I_0 SETS FOR COMPACT, CONNECTED GROUPS: INTERPOLATION WITH MEASURES THAT ARE NONNEGATIVE OR OF SMALL SUPPORT

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Abstract

In the dual object of an infinite compact, connected group, every infinite Sidon set contains an infinite subset on which full interpolation can be performed using very small classes of measures (discrete measures on arbitrarily small sets or nonnegative discrete measures). In particular, the Figà-Talamanca–Rider subset of an infinite product of compact, connected, simple Lie groups has these kinds of interpolation. This substantially improves previous interpolation results.

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1. Introduction

1.1. Definitions and background Throughout this paper, G denotes a compact group with dual object $\widehat{G}$ consisting of a full set of inequivalent irreducible representations. If $U \subset G$, then $M(U)$ denotes the finite, regular, Borel measures concentrated on U; $M_d(U)$ denotes the discrete measures on U and $M_d^+(U)$ the nonnegative, discrete measures on U. We write $\widehat{\mu}$ for the Fourier–Stieltjes transform of the measure μ.

Given $\sigma \in \widehat{G}$ we write H_σ for any (fixed) space $\mathbb{C}^d$ on which $\sigma(G)$ is an irreducible group of operators, with (finite) dimension $d = d_\sigma$.

We investigate subsets of $\widehat{G}$ where every possible function can be interpolated by the Fourier–Stieltjes transform $\widehat{\mu}$ of a nonnegative, discrete measure μ. We say 'possible' because 'nonnegative' restricts what can be interpolated. Indeed, if μ

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is nonnegative (or even, real), then $\widehat{\mu}(\overline{\sigma}) = \overline{\widehat{\mu}(\sigma)}$. Furthermore, if $\sigma \sim \overline{\sigma}$ with say, $\overline{\sigma} = P^{-1}\sigma P$ (we write $\sigma \sim_P \overline{\sigma}$), then we also have

$$\widehat{\mu}(\sigma) = P \int \overline{\sigma}\, d\mu P^{-1} = P\overline{\widehat{\mu}(\sigma)}P^{-1},$$

which leads to the following definitions, in which $\sigma \in E \subset \widehat{G}$ and $U \subset G$ is open, and the norm of a matrix $A_\sigma \in \mathcal{B}(H_\sigma)$ is its usual norm $\|A_\sigma\|$ as an operator on H_σ and $l^\infty(E) = \{(A_\sigma)_{\sigma\in E} : \sup_\sigma \|A_\sigma\| < \infty\}$. These definitions are generalizations from the abelian case [2, 4, 5, 9].

DEFINITION 1. We say that $\varphi \in l^\infty(E)$ is *Hermitian* if $\varphi(\overline{\sigma}) = \overline{\varphi(\sigma)}$ whenever σ and $\overline{\sigma} \in E$, and $\varphi(\sigma) = P\overline{\varphi(\sigma)}P^{-1}$ whenever $\sigma \sim_P \overline{\sigma}$. We write $l_h^\infty(E)$ for the Hermitian elements in $l^\infty(E)$.

We say that $E \subset \Gamma$ is *symmetric* if the identity $\mathbf{1} \notin E$ and whenever $\sigma \in E$, then $\overline{\sigma} \in E$.

We say that E is *antisymmetric* if $\mathbf{1} \notin E$ and if $\sigma \in E$, then $\overline{\sigma} \notin E$ unless $\sigma \sim \overline{\sigma}$.

We say that E is a Sidon(U) *set* if, whenever $(A_\sigma)_{\sigma\in E} \in l^\infty(E)$, there is a measure μ supported on U satisfying $\widehat{\mu}(\sigma) = A_\sigma$ for all $\sigma \in E$. If $U = G$, E is *Sidon*.

We say that E is $I_0(U)$ if E is Sidon(U) and the interpolating measures concentrated on U can be chosen discrete. If $U = G$, E is I_0.

We say that E is *Fatou–Zygmund* $I_0(U)$ ($FZI_0(U)$, for short) if each Hermitian $\varphi \in \ell^\infty(E)$ can be interpolated by a discrete, *nonnegative* measure concentrated on U. If $U = G$, E is FZI_0.

We say that E is *cofinitely* $FZI_0(U)$ if there is a finite subset F of E such that $E \setminus F$ is $FZI_0(U)$.

Clearly, Sidon(U) ($I_0(U)$, $FZI_0(U)$) implies Sidon (I_0, FZI_0 respectively). Also, if E is $FZI_0(U)$ so is $E \cup \overline{E}$. As $\widehat{\mu}(\mathbf{1}) \geq 0$ for μ nonnegative, an $FZI_0(U)$ never contains $\mathbf{1}$. Clearly $I_0(U)$ implies Sidon(U). Less trivially, if E is $FZI_0(U)$, then E is $I_0(U)$ (Proposition 2.5). Finite sets in duals of compact, connected groups are $I_0(U)$ for all open sets U and $FZI_0(U)$ (Corollary 2.9 and Proposition 2.10). Sidon sets in the duals of compact connected abelian groups are Sidon(U) for all open U (see [2, 9]).

1.2. Statement of results We adapt the methods of [1, 7] to prove that there are as many FZI_0 sets as one could hope for.

THEOREM 1.1. *Let G be a compact connected group and $E \subset \widehat{G}$. If E is an infinite Sidon set, then there exists an infinite $F \subset E$ such that F is FZI_0 and cofinitely $FZI_0(U)$ for all open U.*

The sets $I_0(U)$ and $FZI_0(U)$ are related as follows.

THEOREM 1.2. *Let G be a connected, compact group and $E \subset \widehat{G}$. Let $F \subset \widehat{G}$ be finite. If E is cofinitely $FZI_0(U)$ for all open U then $E \cup F \cup \overline{E \cup F}$ is $I_0(U)$ for all open U. If $\mathbf{1} \notin F$ then $E \cup F$ is also FZI_0.*

COROLLARY 1.3. *If $E \subset \widehat{G}$ is cofinitely $FZI_0(U)$ for all open U, then E is $I_0(U)$ for all open U.*

REMARKS.

(i) Readers familiar with the union theorem for Sidon sets[1] may wonder why we need to prove that a union of a finite (!) set with an I_0 set is I_0. That is because the class of I_0 sets is *not* closed under unions (see [3, Example 5.1] for the standard example), and the same holds for the classes of $I_0(U)$, FZI_0, and $FZI_0(U)$ sets [4, 5].

(ii) 'Connected' is an essential hypothesis, even in the abelian case, for our conclusions. See [4, 5, 9].

In the course of the proof of Theorem 1.1 we prove a result about the *FTR* set of G (defined below). This is important because *FTR* sets are the basic examples of Sidon and I_0 sets of unbounded degree.

THEOREM 1.4. *Suppose that $G = \prod_{j \in J} G_j$ is a product of simple, simply connected, connected, compact Lie groups. Then $FTR(G) \setminus \{\mathbf{1}\}$ is FZI_0 and cofinitely $FZI_0(U)$ for all open U.*

REMARKS.

(i) We do not know whether all I_0 sets are $I_0(U)$, even for abelian groups. See [4, 5] for related results in the abelian case.

(ii) We also do not know whether one of the groups of Theorem 1.4 has a Sidon set that is not I_0.

1.3. Organization of this paper We give further background in the next subsection. Preliminary results are given in Section 2. Theorem 1.2 is proved in Section 2.5, Theorem 1.4 in Section 3.1, and Theorem 1.1 in Section 5.2.

1.4. Some further background In $\mathbb{Z}$ there are: Sidon sets not I_0 (see [3]) and I_0 sets not FZI_0 (see [5]). Hadamard sets in $\mathbb{Z}$ are FZI_0 (see [5]), as well as being $I_0(U)$ (see [4]) and cofinitely $FZI_0(U)$ for all open U (see [5]). Every infinite subset E of the dual of a compact abelian group contains an FZI_0 set of the same cardinality, and, if the compact abelian group is connected, E will contain an infinite subset that is cofinitely $FZI_0(U)$ for all open U (see [5]). We do not know whether 'of the same cardinality' correctly may be added in the nonabelian case: our combinatorial arguments do not permit us to select a large enough subset to which the technical Lemma 4.1 can be applied. On the other hand, if E is uncountable, then E does contain an uncountable set that is cofinitely $FZI_0(U)$ for all open U.

'Cofinitely' is forced upon us by the fact that no set, not even a singleton, is $FZI_0(U)$ for all open sets U: consider any $\{\lambda\} \subset \widehat{G}$ and suppose U is a neighbourhood of e with the property that $\|\lambda(x) - \lambda(e)\| < 1/2$ for all $x \in U$. As $\lambda(e) = I$, the

[1] See, for example, [12] and the references therein.

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diagonal elements of $\lambda(x)$ are at least $1/2$ for all $x \in U$. Thus, if $\mu = \sum a_k \delta(x_k) \in M_d^+(U)$, then the diagonal elements of $\widehat{\mu}(\lambda)$ are strictly positive and so arbitrary interpolation with positive, discrete measures supported on U is impossible.

In the nonabelian setting there are infinite, compact, connected groups whose duals do not admit *any* infinite Sidon sets. Cartwright and McMullen [1] effectively characterized those groups admitting infinite Sidon sets and described their Sidon sets in terms of *FTR* sets (defined below). This was used in [7] to prove that every infinite Sidon set contained an infinite I_0 set.

The *Figà-Talamanca and Rider set* (*FTR* set) of a group G that is a (product of) classical, simple, simply connected, compact Lie group(s) is denoted $FTR(G)$ and found as follows.

DEFINITION 2. If G is one of the matrix groups $\mathrm{SU}(n)$, $\mathrm{O}(n)$, $\mathrm{SO}(n)$ or $\mathrm{Sp}(n)$ let $\sigma : G \to \mathrm{U}(n)$ be the self-representation. For $G = \mathrm{Spin}(n)$ let $q : G \to \mathrm{SO}(n)$ be the canonical covering map and let σ denote the composition with the self representation of $\mathrm{SO}(n)$. Then $FTR(G) = \{\sigma, \overline{\sigma}, \mathbf{1}\}$. (Note that σ is equivalent to $\overline{\sigma}$ except if $G = \mathrm{SU}(n)$.)

If $G = \prod_{j \in J} G_j$ where the groups G_j are classical, simple, simply connected, connected, compact Lie groups (that is, the matrix groups above), then

$$FTR(G) \equiv \bigcup_j \{\sigma \circ P_j : \sigma \in FTR(G_j)\},$$

where $P_j : G \to G_j$ are the projection maps.

2. Preliminaries

2.1. Some properties of $FZI_0(U)$ sets Suppose that $\sigma \sim_P \overline{\sigma}$ and A is any matrix of size $d_\sigma \times d_\sigma$. Put $\mu = d_\sigma(TrA\sigma + \overline{TrA\sigma})$. Then μ is a real measure (a real-valued polynomial even) and $\widehat{\mu}(\sigma) = A + P\overline{A}P^{-1}$. Taking conjugates and noting that we must have $\widehat{\mu}(\sigma) = P\overline{\widehat{\mu}(\sigma)}P^{-1}$, it follows that $P^{-1}AP = \overline{P}A\overline{P}^{-1}$ for all A. Thus, if we set $\varphi(\sigma) = A + P\overline{A}P^{-1}$, then $\varphi(\sigma) = P\overline{\varphi(\sigma)}P^{-1}$. Conversely, if $B = P\overline{B}P^{-1}$, then $B = A + P\overline{A}P^{-1}$ for $A = B/2$.

These observations show that for an antisymmetric set E, $\varphi \in l^\infty(E)$ is Hermitian if and only if whenever $\sigma \sim_P \overline{\sigma}$, $\varphi(\sigma)$ has the form $A_\sigma + P\overline{A_\sigma}P^{-1}$ for $(A_\sigma)_{\sigma \in E} \in l^\infty(E)$. This can be used to give a characterization of FZI_0 sets, as follows, the proof of which is immediate.

PROPOSITION 2.1. *Let G be a connected compact group and $U \subset G$. An antisymmetric set E is FZI_0 if and only if whenever $(A_\sigma) \in l^\infty(E)$ and $\varphi = (B_\sigma)$ satisfies $B_\sigma = A_\sigma$ if $\sigma \nsim \overline{\sigma}$ and $B_\sigma = A_\sigma + P\overline{A_\sigma}P^{-1}$ if $\sigma \sim_P \overline{\sigma}$, there is $\mu \in M_d^+(U)$ such that $\widehat{\mu}(\sigma) = \varphi(\sigma)$ for all $\sigma \in E$.*

As with Sidon and I_0 sets it is enough to perform 'almost' interpolation. We make this precise in the next proposition, which requires some notation to be stated. Let

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$$D(N, U) = \left\{ \sum_{k=1}^{N} a_k \delta(x_k) : 0 \leq a_k \leq 1,\, x_k \in U \right\} \quad \text{and}$$

$$D^{+}(N, U) = \left\{ \sum_{k=1}^{N} a_k \delta(x_k) : 0 \leq a_k \leq 1,\, x_k \in U \right\}.$$

We also set

$$B(\ell^{\infty}(E)) = \{\varphi \in \ell^{\infty}(E) : \|\phi(\sigma\| \leq 1\} \quad \text{and}$$
$$B_h(\ell^{\infty}(E)) = \{\varphi \in \ell^{\infty}_h(E) : \|\phi(\sigma\| \leq 1\}.$$

PROPOSITION 2.2. *Let $E \subset \widehat{G}$ be an antisymmetric set and let U be a σ-compact subset of G. The following properties are equivalent:*

(1) *E is $I_0(U)$ (respectively $FZI_0(U)$);*
(2) *there is some $0 < \varepsilon < 1$ (equivalently, for all $0 < \varepsilon < 1$) such that whenever $\varphi \in l^{\infty}(E)$ (respectively $\varphi \in l^{\infty}_h(E)$), then there is $\mu \in M_d(U)$ (respectively $\mu \in M_d^{+}(U)$) satisfying*

$$\|\widehat{\mu}(\sigma) - \varphi(\sigma)\| < \varepsilon \quad \textit{for all } \sigma \in E;$$

(3) *for every $0 < \varepsilon < 1$ there is N such that for all φ in the unit ball of $l^{\infty}(E)$ (respectively φ in the unit ball of $l^{\infty}_h(E)$) there is $\mu \in D(N, U)$ (respectively $\mu \in D^{+}(N, U)$) with $\|\widehat{\mu}(\gamma) - \varphi(\gamma)\| < \varepsilon$ for all $\gamma \in E$;*
(4) *there is N such that for all φ in the unit ball of $l^{\infty}(E)$ (respectively φ in the unit ball of $l^{\infty}_h(E)$) there exists $\mu \in D(N, U)$ (respectively $\mu \in D^{+}(N, U)$) with $\widehat{\mu}(\gamma) = \varphi(\gamma)$ for all $\gamma \in E$ and $\|\mu\| \leq N$.*

PROOF. We prove that (2) implies (3) for the $FZI_0(U)$ case and leave the remainder to the reader. (That $M_d^{+}(G)$ is not a vector space slightly complicates the proof.) See [5, 7] for a similar characterization of $FZI_0(U)$ sets in abelian groups and I_0 sets in nonabelian groups, respectively.

One may view $B(\ell^{\infty}_h(E))$ as the product space $\prod_{\sigma \in E} B_\sigma$ where B_σ is the set of the norm at most one $d_\sigma \times d_\sigma$ matrices, of the form $A = P\overline{A}P^{-1}$ if $\sigma \sim_P \overline{\sigma}$. Let $U = \bigcup_1^{\infty} U_n$, where the U_n are compact and $U_1 \subset U_2 \subset \cdots$. For $1 \leq N$ let

$$W_N = \left\{ \varphi \in \prod_{\sigma \in E} B_\sigma : \exists \mu \in D^{+}(N, U_N) \text{ s.t. } \|\widehat{\mu}(\sigma) - \varphi(\sigma)\| \leq \varepsilon \text{ for all } \sigma \in E \right\}.$$

By assumption $\bigcup_{N=1}^{\infty} W_N = \prod_{\sigma \in E} B_\sigma$. The compactness of the U_n ensures that each W_N is closed. Since $\prod_{\sigma \in E} B_\sigma$ is compact (with the product topology), the Baire category theorem implies that some W_N has nonempty interior. Consequently, there is a finite set $F \subseteq E$ and a $\psi \in \prod_{\sigma \in F} B_\sigma$ such that $\psi \times \prod_{\sigma \in E \setminus F} B_\sigma \subseteq W_N$.

Consider the subset S of $l^{\infty}_h(E)$ consisting of the elements which vanish off F. Note that S is a finite-dimensional, real subspace. Take a basis, say $e_1, \ldots, e_\ell$,

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where $e_j \in B_h(l^\infty(E))$. Since all norms are comparable on a finite-dimensional space, there is some $C > 0$ such that $\|\sum b_j e_j\|_{l^\infty} \geq C \sum |b_j|$. Since each $\pm e_j$ is Hermitian, we can obtain $\mu_j, \nu_j \in M_d^+(U)$ such that for all $\sigma \in E$,

$$\|\widehat{\mu_j}(\sigma) - e_j(\sigma)\| < C\varepsilon/2N \quad \text{and} \quad \|\widehat{\nu_j}(\sigma) + e_j(\sigma)\| < C\varepsilon/2N.$$

By taking suitably large partial sums we can assume there exists some N' such that each $\mu_j, \nu_j \in D^+(N', U)$.

Let $\varphi \in B_h(l^\infty(E))$. Since φ coincides on $E \setminus F$ with an element of W_N, we can find $\mu \in D^+(N, U)$ such that $\|\widehat{\mu}(\sigma) - \varphi(\sigma)\| \leq \varepsilon$ for all $\sigma \in E \setminus F$. As μ is a positive measure, $(\varphi - \widehat{\mu})|_F$ (extended by 0 on $E \setminus F$) belongs to S and therefore equals $\sum b_j e_j$ for some b_j real. Write $b_j = b_j^+ - b_j^-$ where $b_j^\pm \geq 0$. Note

$$C \sum |b_j| \leq \|\varphi - \widehat{\mu}|_F\|_{l^\infty} \leq 1 + \|\mu\|_{M(U)} \leq 2N.$$

For $\sigma \in E$,

$$\begin{aligned}
&\left\| \varphi(\sigma) - \widehat{\mu}(\sigma) - \left(\sum b_j^+ \widehat{\mu_j} + b_j^- \widehat{\nu_j} \right)(\sigma) \right\| \\
&= \left\| (\varphi(\sigma) - \widehat{\mu}(\sigma))|_{E\setminus F} + \left(\sum b_j^+ (e_j - \widehat{\mu_j}) + b_j^- (-e_j - \widehat{\nu_j}) \right)(\sigma) \right\| \\
&\leq \sup_{\sigma \in E\setminus F} \|(\varphi - \widehat{\mu})(\sigma)\| + \sup_{\sigma \in E} \left\| \left(\sum b_j^+ (e_j - \widehat{\mu_j}) + b_j^- (-e_j - \widehat{\nu_j}) \right)(\sigma) \right\| \\
&\leq \varepsilon + \sum |b_j| C\varepsilon/(2N) \leq 2\varepsilon.
\end{aligned}$$

Finally, we note that $\mu + \sum b_j^+ \mu_j + \sum b_j^- \nu_j \in D^+(N + 2N', U)$ and, as N' is independent of the choice of φ, this proves that (2) implies (3). □

Definition 3. The set E is an $I_0(N, \varepsilon)$ (respectively, $FZI(N, \varepsilon)$) set if (3) holds.

Lemma 2.3. *The set $E \subseteq \widehat{G}$ is $I_0(U)$ (respectively $FZI_0(U)$) if and only if E is $I_0(Ux)$ (respectively $FZI_0(Ux)$) for each $x \in G$.*

Proof. Indeed, $\mu = \sum a_k \delta_{x_k}$ satisfies $\widehat{\mu}(\sigma) = \varphi(\sigma)\sigma(x)^{-1}$ if and only if

$$\sum \widehat{a_k \delta_{x_k x}}(\sigma) = \sum a_k \sigma(x_k)\sigma(x) = \varphi(\sigma). \qquad \square$$

Proposition 2.4. *Suppose that $q : G \longrightarrow H$ is a continuous, surjective homomorphism. Then $E \subset \widehat{H}$ is $FZI_0(q(U))$ if and only if $\widetilde{E} = \{\sigma \circ q : \sigma \in E\}$ is an $FZI_0(U)$ subset of $\widehat{G}$.*

Proof. This follows easily since the surjectivity of q ensures that $\sigma \circ q \sim_P \overline{\sigma \circ q}$ if and only if $\sigma \sim_P \overline{\sigma}$. □

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2.2. Antisymmetric $FZI_0(U)$ sets are $I_0(U)$

PROPOSITION 2.5. *Let G be a compact connected group and $E \subset \widehat{G}$ be an antisymmetric $FZI_0(U)$ set. Then $E \cup \overline{E}$ is $I_0(U)$.*

PROOF. Write $E = E_0 \cup E_1$ where $E_0 = \{\sigma \in E : \sigma \sim \overline{\sigma}\}$ and $E_1 = \{\sigma \in E : \sigma \nsim \overline{\sigma}\}$ and suppose that $\varphi \in l^\infty(E \cup \overline{E})$. Find $\mu_1, \nu_1, \mu_2, \nu_2, \in M_d^+(U)$, such that

$$\widehat{\mu_1}(\sigma) = \begin{cases} \varphi(\sigma) + P_\sigma \overline{\varphi(\sigma)} P_\sigma^{-1} & \text{if } \sigma \in E_0, \sigma \sim_{P_\sigma} \overline{\sigma}, \\ \varphi(\sigma) & \text{if } \sigma \in E_1, \end{cases}$$

$$\widehat{\nu_1}(\sigma) = \begin{cases} i\left(\varphi(\sigma) - P_\sigma \overline{\varphi(\sigma)} P_\sigma^{-1}\right) & \text{if } \sigma \in E_0, \sigma \sim_{P_\sigma} \overline{\sigma}, \\ i\varphi(\sigma) & \text{if } \sigma \in E_1, \end{cases}$$

$$\widehat{\mu_2}(\sigma) = \begin{cases} 0 & \text{if } \sigma \in E_0 \\ \overline{\varphi(\overline{\sigma})} & \text{if } \sigma \in E_1 \end{cases} \quad \text{and} \quad \widehat{\nu_2}(\sigma) = \begin{cases} 0 & \text{if } \sigma \in E_0, \\ \overline{i\varphi(\overline{\sigma})} & \text{if } \sigma \in E_1. \end{cases}$$

It is routine to verify that the discrete measure $(\mu_1 - i\nu_1 + \mu_2 - i\nu_2)/2$ does the desired interpolation. □

2.3. Finite sets are $I_0(U)$

LEMMA 2.6. *Let G be a compact, connected Lie group and U be an open subset of G. Let H be a finite-dimensional Hilbert space and suppose $\sigma : G \to B(H)$ (where $B(H)$ is the bounded operators on H) is an analytic map. Then $\sigma(U)$ spans $B(H)$ if and only if $\sigma(G)$ spans $B(H)$.*

PROOF. If $\sigma(U)$ does not span $B(H)$, then $\sigma(U)$ is contained in a hyperplane $L \subseteq B(H)$ of (complex) co-dimension at least one. Choose a nonzero vector $\zeta \in \mathbb{C}^{\dim B(H)}$ orthogonal to L with respect to a fixed inner product $(\cdot, \cdot)$ on $B(H)$.

Since the map $x \mapsto \sigma(x)$ is an analytic mapping of G into $B(H)$ [11, p. 102], the map $x \mapsto (\zeta, \sigma(x)) \equiv f(x)$ is also analytic on G. Because f vanishes on the open set U and G is a connected Lie group, f vanishes identically. However, then ζ is orthogonal to $\sigma(G)$ and hence $\sigma(U)$ cannot span $B(H)$.

The converse is trivial. □

COROLLARY 2.7. *Suppose that G is a compact, connected Lie group, E is a finite subset of $\widehat{G}$, and U is an open subset of G. Then E is $I_0(U)$.*

PROOF. All finite sets are I_0 [7, Proposition 2.2] thus the set $M_d(G)\widehat{\ }$ spans $B(H)$ where $H = \bigoplus_{\sigma \in E} H_\sigma$. Since the mapping $x \mapsto \bigoplus_{\sigma \in E} \sigma(x)$ is an analytic mapping of G into $B(H)$. By Lemma 2.6, $M_d(U)\widehat{\ }$ spans $B(H)$ for all open $U \subseteq G$. □

PROPOSITION 2.8. *Let $G = A \times \prod_{i \in I} G_i$ where A is a compact, connected, abelian group and the subgroups G_i are compact, connected Lie groups. Then each finite set $E \subseteq \widehat{G}$ is $I_0(U)$ for all open sets $U \subseteq G$.*

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PROOF. First, suppose that $E_1 \subseteq \widehat{A}$ and $E_2 \subseteq \widehat{\prod G_i}$ are finite sets. A dimension argument shows that $l^\infty(E_1 \times E_2) = l^\infty(E_1) \otimes l^\infty(E_2)$.

As each representation, σ, in E has finite degree, there is a finite index set I such that $E \subset (A \times \prod_{i\in I} G_i)\widehat{}$ (in the sense that σ restricted to $\prod_{i\notin I} G_i$ is trivial). If we put $E_1 = E|_A$ and $E_2 = E|_{\prod_{i\in I} G_i}$, then certainly $E \subseteq E_1 \times E_2$ and thus $l^\infty(E)$ is contained in $l^\infty(E_1) \otimes l^\infty(E_2)$. As E_1 is a finite set of representations on a compact, connected abelian group it is $I_0(U)$ for all open sets $U \subseteq A$ (see [4, Corollary 2.8]). A finite product of compact, connected Lie groups is again a compact, connected Lie group and thus E_2 is also $I_0(U)$ for all open sets $U \subseteq \prod_{i\in I} G_i$.

Let $U \subset G$ be open. We may assume that U is a neighbourhood of the identity by Lemma 2.3. Therefore, U contains a set of the form $U_1 \times U_2 \times \prod_{i\notin I} G_i$, where U_1 is open in A and U_2 is open in $\prod_{i\in I} G_i$. Since $M_d(U_j)\hat{}$ spans $l^\infty(E_j)$ for $j = 1, 2$, and each $\sigma \in E$ is trivial off $E_1 \times E_2$, $M_d(U)\hat{}$ spans $l^\infty(E)$. □

COROLLARY 2.9. *If G is a compact, connected group, then any finite set $E \subseteq \widehat{G}$ is $I_0(U)$ for all open sets U*

PROOF. By the structure theorem for compact, connected groups [10, Theorem 6.5.6], G is isomorphic to a quotient of $A \times \prod_{i\in I} G_i$ where G_i are compact, connected Lie groups, and A is a compact, connected, abelian group. If E is a finite subset of $\widehat{G}$, then E lifts to a finite set of irreducible representations of $A \times \prod G_i$. As the quotient map is open, it suffices to assume $G = A \times \prod G_i$ and thus the previous proposition applies. □

2.4. Finite sets are FZI_0

PROPOSITION 2.10. *If G is a compact group, then any finite set $E \subseteq \widehat{G}$ not containing $\mathbf{1}$ is FZI_0.*

PROOF. Without loss of generality we may assume that E is antisymmetric. Given any Hermitian φ in the unit ball of $l^\infty(E)$, put $\psi(\sigma) = \varphi(\sigma)$ if $\sigma \nsim \overline{\sigma}$ and $\psi(\sigma) = \varphi(\sigma)/2$ otherwise. Set

$$\rho = A + \sum_{\sigma\in E} d_\sigma (Tr\psi(\sigma)\sigma + Tr\overline{\psi(\sigma)}\overline{\sigma}),$$

where $A \geq 0$ is sufficiently large to ensure that $\rho \geq 0$. If $\sigma \sim_P \overline{\sigma}$, then as φ is Hermitian, $Tr\overline{\psi(\sigma)}\overline{\sigma} = TrP\psi(\sigma)P^{-1}P\sigma P^{-1} = Tr\psi(\sigma)\sigma$. Moreover, if $\sigma \in E$, then $\widehat{\rho}(\overline{\sigma}) = \overline{\varphi(\sigma)} = \varphi(\overline{\sigma})$, thus $\widehat{\rho}(\sigma) = \varphi(\sigma)$ for all $\sigma \in E$.

For each $x \in G$ choose a neighbourhood U_x of x such that

$$\|\sigma(x) - \sigma(y)\| < \varepsilon/\|\rho\|,$$

for all $y \in U_x$ and $\sigma \in E$. Choose a finite subcover $U_{x_1}, \ldots, U_{x_n}$ of G. For each j let $V_j = U_{x_j} \setminus \bigcup_{k=1}^{j-1} U_{x_k}$. The V_j form a finite (but not open) covering of G by disjoint sets. Put $\mu = \sum_{j=1}^n \rho(V_j)\delta(x_j)$, where by $\rho(V_j)$ we mean $\int_{V_j} \rho$ and note that $\|\widehat{\mu}(\sigma) - \varphi(\sigma)\| < \varepsilon$ for all $\sigma \in E$. Now an application of Proposition 2.2 (3) proves E is FZI_0. □

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2.5. Proof of Theorem 1.2 We may assume $E \cap F = \emptyset$ and that both E and F are symmetric. It does not matter whether F is empty. Let U be an open neighbourhood of the identity of G. Let U', W' be open neighbourhoods of the identity with $W'U' \subset U$. Let $F_1 \subset E$ be a finite set such that $E \setminus F_1$ is $FZI_0(U')$.

Let W be an open neighbourhood of the identity of G such that $W^3 \subset W'$ and $x \in W$ implies $\|\mathbf{1}_\sigma - \sigma(x)\|_{\mathcal{B}(H_\sigma)} \leq 1/2$ for $\sigma \in F \cup F_1$. Since E is cofinitely $FZI_0(W)$, we may choose a finite set $F_2 \subset E \setminus F_1$ such that $E \setminus (F_1 \cup F_2)$ is $FZI_0(W)$. Choose any $\tau \in E \setminus (F_1 \cup F_2)$. Then there exist $\mu_-, \mu_+ \in M_d^+(W)$ such that $\widehat{\mu}_\pm(\tau) = \widehat{\mu}_\pm(\overline{\tau}) = \pm \mathbf{1}_\tau$ and $\widehat{\mu}_\pm = 0$ elsewhere on $E \setminus (F_1 \cup F_2)$.

Let $\mu = \mu_+ + \mu_-$. Then $\mu \geq 0$ is supported on W, $\widehat{\mu} = 0$ on $E \setminus (F_1 \cup F_2)$, and $a = \|\mu\| \geq 2$ (this is where the nonnegativity is used). Furthermore, $\|\widehat{\mu}(\sigma) - a\mathbf{1}_\sigma\| \leq a/2$, by integration, for each $\sigma \in F \cup F_1$, so $\widehat{\mu}(\sigma)$ is invertible for each $\sigma \in F \cup F_1$.

Since $F \cup F_1 \cup F_2$ is finite, Corollary 2.9 implies there exists a discrete measure[1] ω supported in W such that $\widehat{\omega} = \widehat{\mu}^{-1}$ on $F \cup F_1$ and 0 on F_2. Then $\mu * \omega$ has transform the identity on $F \cup F_1$ and zero on $E \setminus F_1$.

Now let $\varphi \in \ell^\infty(E \cup F)$. Let $\omega_1 \in M_d(W)$ have $\widehat{\omega}_1 = \varphi$ on $F \cup F_1$ and $\omega_2 \in M_d(U')$ have $\widehat{\omega}_2 = \varphi$ on $E \setminus F_1$. Then $\omega_3 = \mu * \omega * \omega_1 + (\delta_e - \mu * \omega) * \omega_2$ has $\widehat{\omega}_3 = \varphi$ on $E \cup F$, and ω_3 is a discrete measure concentrated on $W^3 \cup WU' \subset U$. It now follows that $E \cup F$ is $I_0(U)$. Since E, F are symmetric, $E \cup F \cup \overline{E \cup F}$ is $I_0(U)$. That proves the $I_0(U)$ assertion.

The proof of the FZI_0 assertion follows similarly, with a call to Proposition 2.10 in place of Corollary 2.9. We observe that $\widehat{\mu}^{-1}$ can be interpolated on F by a nonnegative measure supported on W since the nonnegativity of μ implies $\widehat{\mu}^{-1}$ is Hermitian. □

2.6. Orthogonal representations and the padding property

DEFINITION 4. The nontrivial representations $\{\sigma_j\} \subseteq \widehat{G}$ are *mutually orthogonal* if $G = \prod_{j \in J} G_j$, the index set J is the disjoint union of sets J_k and $\sigma_k \in \widehat{\prod_{j \in J_k} G_j}$.

We say that $E \subseteq \widehat{G}$ has the *padding property* if for every $\varepsilon > 0$ there is $m = m(\varepsilon)$ and $x_0, \ldots, x_{m-1} \in G$ satisfying $(1/m)\|\sum_{j=0}^{m-1} \sigma(x_j)\| < \varepsilon$ for all $\sigma \in E$.

Padding was a key idea used in [7] to 'piece together' I_0 sets in a product group setting. It is shown in [7] that $FTR(G) \setminus \{\mathbf{1}\}$ and finite sets not including 1 are sets which have the padding property. Of course, if $\mathbf{1} \in E$, then E does not have the padding property.

LEMMA 2.11. *Let $G = \prod_{i \in J} G_i$ and $E_i \subset \widehat{G_i}$. Suppose there is some $\varepsilon < 1$ and N such that all E_i are $FZI(N, \varepsilon)$ sets and that $E = \bigcup E_i$ has the padding property. Then E is FZI_0.*

PROOF. Let $\{A_\sigma\}_{\sigma \in E}$ be a Hermitian function in the unit ball of $l^\infty(E)$ and for each i let $\mu_i = \sum_{k=1}^N a_{ki}\delta(g_{ki})$ be a positive measure on G_i with $0 \leq a_{ki} \leq 1$ and $\|\widehat{\mu}_i(\sigma) - A_\sigma\| < \varepsilon$ for all $\sigma \in E_i$. We can 'combine' these measures to produce a

[1] At this point we do not need nonnegativity.

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single positive discrete measure on G which will simultaneously interpolate all A_σ by imitating the proof of [7, Theorem 3.3] with one change: rather than choosing $s_1, \ldots, s_r$ an ε-net in the complex unit ball, we choose $s_i = i\varepsilon$ for $i = 0, \ldots, [1/\varepsilon]$. Given any $0 \le a_{ki} \le 1$ there is some s_j such that $|s_j - a_{ki}| \le \varepsilon$. Hence the measure ν constructed in [7, Theorem 3.3] does the appropriate interpolation and is positive as needed. □

3. Proof of the *FTR* Theorem 1.4

We use two lemmas, from which the theorem will follow easily. The first, Lemma 2.11 above, allows us to piece together the *FTR* sets of the factors; the second, which follows, shows that those *FTR* sets satisfy the conditions necessary for that piecing.

LEMMA 3.1. *If G is any one of the classical, compact, simple, simply connected Lie groups, then $FTR(G) \setminus \{\mathbf{1}\}$ is $FZI_0(N, \varepsilon)$ for some N and $0 < \varepsilon < 1$ independent of G (and so FZI_0).*

REMARKS.

(i) The choice of N and ε will be clear in the proof.
(ii) Since the *FTR* set (less $\mathbf{1}$) has at most two elements in the classical case, this would follow from Proposition 2.10, if we did not need N, ε independent of G.

PROOF. Denote the *FTR* set of G (excluding $\mathbf{1}$) by $\{\sigma\}$ or $\{\sigma, \overline{\sigma}\}$, as appropriate. We consider the classical matrix groups separately.

(i) For SU(2), SU(3) we appeal to the finite sets result.
(ii) For SU(n), $n \ge 4$ and O(n), $n \ge 1$, we essentially use the argument in [7, Proposition 3.2].
(iii) For SO(n), $n \ge 3$, it suffices to show each matrix in O(n) is a positive linear combination of a bounded number of matrices in SO(n), with the number independent of n. We need only consider orthogonal matrices with determinant -1 and these can be written as $P^{-1}NP$ with P special orthogonal and N block diagonal of the form

$$\begin{pmatrix} -I_j & 0 & 0 \\ 0 & I_k & 0 \\ 0 & 0 & T \end{pmatrix},$$

with j odd and T block diagonal with 2×2 blocks of the form

$$T_\varphi = \begin{pmatrix} \cos\varphi & \sin\varphi \\ -\sin\varphi & \cos\varphi \end{pmatrix}.$$

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Thus, N is one of the block diagonal matrices

$$\begin{pmatrix} -I_3 & 0 \\ 0 & R \end{pmatrix}, \quad \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & R \end{pmatrix}, \quad \text{or} \quad \begin{pmatrix} -1 & 0 & 0 \\ 0 & T_\varphi & 0 \\ 0 & 0 & R \end{pmatrix},$$

where R is special orthogonal. We can write

$$\begin{aligned} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & R \end{pmatrix} &= \frac{1}{4}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & R \end{pmatrix} + \frac{1}{4}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & R \end{pmatrix} \\ &\quad + \frac{1}{4}\begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & R \end{pmatrix} + \frac{1}{4}\begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & R \end{pmatrix}, \end{aligned}$$

and thus as a positive sum of four matrices in $\mathrm{SO}(n)$. The positive combinations of matrices for

$$\begin{pmatrix} -I_3 & 0 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} -1 & 0 & 0 \\ 0 & T_\varphi & 0 \\ 0 & 0 & R \end{pmatrix},$$

can be obtained in a similar manner. Here 0 denotes a zero matrix (possibly different in each instance) of dimensions required by the nonzero matrices of its row and column.

(iv) For $\mathrm{Sp}(n)$, $\sigma \sim_P \overline{\sigma}$ with intertwining operator

$$P = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix}.$$

Thus, we need only interpolate matrices of the form

$$\begin{pmatrix} A & B \\ -\overline{B} & \overline{A} \end{pmatrix}.$$

Any matrix of the form

$$\begin{pmatrix} U & 0 \\ 0 & \overline{U} \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 0 & V \\ -\overline{V} & 0 \end{pmatrix}$$

with U, V unitary belongs to $\mathrm{Sp}(n)$. Since any matrix can be written as the positive linear combination of four unitaries it is straightforward to perform the required interpolation.

(v) The $\mathrm{Spin}(n)$ case follows from $\mathrm{SO}(n)$ since the property of being $FZI_0(U)$ is preserved under quotients (Proposition 2.4). $\square$

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3.1. Proof of Theorem 1.4 Because $FTR(G) \setminus \{\mathbf{1}\}$ is known [7] to have the padding property, we can apply Lemma 3.1 and Lemma 2.11 to conclude that $FTR(G) \setminus \{\mathbf{1}\}$ is FZI_0.

We now show that $FTR(G) \setminus \{\mathbf{1}\}$ is cofinitely $FZI_0(U)$ for all open $U \subset G$. If J is finite, then $\#FTR(G) \le 1 + 2\#J$, so $FTR(G) \setminus \{\mathbf{1}\}$ is cofinitely $FZI_0(U)$ for all sets U, open or not.

We therefore assume that J is infinite. If $U \subset G$ is open, then U contains a set of the form

$$U_1 \times \cdots \times U_N \prod_{i \notin \{1,\ldots,N\}} G_i,$$

where $U_j \subset G_j$ is open, $1 \le j \le N$. Let $G' = \prod_{i \notin \{1,\ldots,N\}} G_i$ and $G'' = \prod_{i \in \{1,\ldots,N\}} G_i$. Suppose that $E = FTR(G) \setminus \{\mathbf{1}\}$ and take $F = FTR(G'') \setminus \{\mathbf{1}\}$. Of course, we can identity F with a subset of E, and $E \setminus F$ with $FTR(G')$, and we do so. With those identifications, $E \setminus F$ is FZI_0 by the preceding paragraph, and F is finite.

Let $\varphi \in B(\ell^\infty(E))$. Since $E \setminus F$ is FZI_0, we can obtain $\mu \in M_d^+(G')$ such that $\widehat{\mu} = \varphi$ on $E \setminus F$, say $\mu = \sum a_k \delta(x_k)$. (Here we think of $E \setminus F$ as $FTR(G')$.) Replace x_k by $x_k' = (y_{ki})_{i \in I}$ where $y_{ki} = x_{ki}$ if $i \neq 1, \ldots, N$ and $y_{ki} = e$ for $i = 1, \ldots, N$. Then $x_k' \in U$, and $\mu_k' = \sum a_k \delta(x_k')$ is a discrete positive measure supported on U whose Fourier transform coincides with that of μ on $E \setminus F$. That ends the proof of Theorem 1.4. $\square$

4. A technical lemma

If A and B are infinite sets, then $A \otimes B$ is never Sidon [1, p. 311], even in the abelian setting [9, Theorem 1.4]. It will be useful for us to know that certain infinite subsets of the product of two FZI_0 sets are FZI_0.

LEMMA 4.1. *Let $E_1 = \{\chi_j\}_{j \in J}$ be antisymmetric and assume all χ_j have the same degree. Assume that $E_1 \cup \overline{E_1}$ is $I_0(U_1)$. Let $E_2 = \bigcup_{j \in J} E_{2,j}$ where the sets $E_{2,j}$ are disjoint. Suppose that E_2 is antisymmetric and $FZI_0(U_2)$. Furthermore, assume that E_1 is orthogonal to E_2. Then $E = \bigcup_{j \in J} \chi_j \otimes E_{2,j}$ is $FZI_0(U_1 \times U_2)$.*

PROOF. Partition E as $F_1 \cup F_2 \cup F_3$ where

$$\begin{aligned} F_1 &= \{\chi \otimes \sigma \in E : \sigma \nsim \overline{\sigma}\}, \\ F_2 &= \{\chi \otimes \sigma \in E : \sigma \sim_{Q_\sigma} \overline{\sigma} \text{ and } \chi \sim_{P_\chi} \overline{\chi}\} \quad \text{and} \\ F_3 &= \{\chi \otimes \sigma \in E : \sigma \sim_{Q_\sigma} \overline{\sigma} \text{ and } \chi \nsim \overline{\chi}\}. \end{aligned}$$

Let $\varphi = (A(\chi \otimes \sigma)) \in B(l^\infty(E))$ and put

$$X(\chi \otimes \sigma) = \begin{cases} A(\chi \otimes \sigma) + R(\chi \otimes \sigma)\overline{A(\chi \otimes \sigma)}R(\chi \otimes \sigma)^{-1} \\ \qquad \text{if } \chi \otimes \sigma \sim_{R(\chi \otimes \sigma)} \overline{\chi \otimes \sigma} \\ A(\chi \otimes \sigma) \quad \text{otherwise.} \end{cases}$$

Our task is to interpolate $(X(\chi \otimes \sigma))$ on E.

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We produce three positive, discrete measures μ_ℓ whose transforms agree with $X(\chi\otimes\sigma)$ when $\chi\otimes\sigma\in F_\ell$ and 0 otherwise on E.

Let d be the common degree of the χ_j, and let $I_{k,\ell}$ be the $d\times d$ matrix with a 1 in the (k,ℓ) place and 0 elsewhere. For each $\chi\otimes\sigma\in E$, write $A(\chi\otimes\sigma) = \sum_{k,\ell=1}^{d} I_{k,\ell}\otimes a_{k,\ell}(\chi\otimes\sigma)$.

Case I, interpolation on F_1. Since E_1 is $I_0(U_1)$ there exists, for each $1\le k$, $\ell\le d$, $\mu_{k,\ell}=\sum_n \alpha_{n,k,\ell}\delta(x_{n,k,\ell})\in M_d(U_1)$ such that $\widehat{\mu_{k,\ell}}(\chi)=I_{k,\ell}$ for each $\chi\in E_1$. Observe that for each $\sigma\in E_2$ there exists a unique $\chi\in E_1$ such that $\chi\otimes\sigma\in E$. Since E_2 is $FZI_0(U_2)$, there exists $\nu_{n,k,\ell}\in M_d^+(U_2)$ such that for $\sigma\in E_2$,

$$\widehat{\nu_{n,k,\ell}}(\sigma)=\begin{cases}\alpha_{n,k,\ell}a_{k,\ell}(\chi\otimes\sigma) & \text{if there exists } \chi\in E_1 \text{ with } \chi\otimes\sigma\in F_1\\ 0 & \text{otherwise.}\end{cases}$$

(There is no problem doing this as $\sigma\not\sim\overline{\sigma}$ if $\chi\otimes\sigma\in F_1$.)

For $\chi\otimes\sigma\in F_1$ and $1\le k,\ \ell\le d$, we put $\omega_{k,\ell}=\sum_n\delta(x_{n,k,\ell})\otimes\nu_{n,k,\ell}$. Then

$$\begin{aligned}\widehat{\omega_{k,\ell}}(\chi\otimes\sigma) &= \sum_n \chi(x_{n,k,\ell})\otimes\widehat{\nu_{n,k,\ell}}(\sigma)=\sum_n\chi(x_{n,k,\ell})\otimes\alpha_{n,k,\ell}a_{k,\ell}(\chi\otimes\sigma)\\ &=\sum_n \alpha_{n,k,\ell}\widehat{\delta(x_{n,k,\ell})}(\chi)\otimes a_{k,\ell}(\chi\otimes\sigma)=I_{k,\ell}\otimes a_{k,\ell}(\chi\otimes\sigma),\end{aligned}$$

on F_1, and $\widehat{\omega_{k,\ell}}$ equals 0 otherwise on E. Thus $\mu_1=\sum_{k,\ell=1}^{d}\omega_{k,\ell}$ interpolates X on F_1 and is zero on $F_2\cup F_3$.

Case II, interpolation on F_2. We write $E_1=E_1'\cup E_1''$, where E_1' is the set of elements $\chi\sim\overline{\chi}$, and E_1'' is the rest of E_1.

We use the $I_0(U_1)$ property of E_1 to get $\mu_{k,\ell}=\sum_n\alpha_{n,k,\ell}\delta(x_{n,k,\ell})\in M_d(U_1)$ such that

$$\widehat{\mu}_{k,\ell}(\chi)=\begin{cases} I_{k,\ell} & \text{if } \chi\in E_1'\\ 0 & \text{if } \chi\in E_1''.\end{cases}$$

Now we obtain $\nu_{n,k,\ell}\in M_d^+(U_2)$ such that

$$\widehat{\nu_{n,k,\ell}}(\sigma)=\alpha_{n,k,\ell}a_{k,\ell}(\chi\otimes\sigma)+Q_\sigma\overline{(\alpha_{n,k,\ell}a_{k,\ell}(\chi\otimes\sigma))}Q_\sigma^{-1},$$

if there exists χ with $\chi\otimes\sigma\in F_2$.

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If we again put $\omega_{k,\ell} = \sum_n \delta(x_{n,k,\ell}) \otimes \nu_{n,k,\ell}$ we have for $\chi \otimes \sigma \in F_2$,

$$\begin{aligned}
\widehat{\omega_{k,\ell}}(\chi \otimes \sigma) &= \sum_n \chi(x_{n,k,\ell}) \otimes \widehat{\nu_{n,k,\ell}}(\sigma) \\
&= \sum_n \chi(x_{n,k,\ell}) \otimes [\alpha_{n,k,\ell} a_{k,\ell}(\chi \otimes \sigma) \\
&\quad + Q_\sigma(\overline{\alpha_{n,k,\ell} a_{k,\ell}(\chi \otimes \sigma)}) Q_\sigma^{-1}] \\
&= \sum_n \alpha_{n,k,\ell} \chi(x_{n,k,\ell}) \otimes a_{k,\ell}(\chi \otimes \sigma) \\
&\quad + \overline{\alpha_{n,k,\ell}} P_\chi \overline{\chi(x_{n,k,\ell})} P_\chi^{-1} \otimes Q_\sigma(\overline{a_{k,\ell}(\chi \otimes \sigma)}) Q_\sigma^{-1} \\
&= I_{k\ell} \otimes a_{k,\ell}(\chi \otimes \sigma) \\
&\quad + P_\chi \overline{\sum_n \alpha_{n,k,\ell} \chi(x_{n,k,\ell})} P_\chi^{-1} \otimes Q_\sigma(a_{k,\ell}(\chi \otimes \sigma)) Q_\sigma^{-1} \\
&= I_{k,\ell} \otimes a_{k,\ell} + (P_\chi \otimes Q_\sigma) \overline{I_{k,\ell} \otimes a_{k,\ell}(\chi \otimes \sigma)}\, (P_\chi \otimes Q_\sigma)^{-1}.
\end{aligned}$$

Consequently, $\widehat{\mu_2}(\chi \otimes \sigma) = \sum_{k,\ell=1}^d \widehat{\omega_{k,\ell}}(\chi \otimes \sigma) = A + R_{\chi\otimes\sigma} \overline{A} R_{\chi\otimes\sigma}^{-1}$ on F_2 and equals zero otherwise on E.

Case III, interpolation on F_3 (the final case). Here we need that $E_1 \cup \overline{E_1}$ is $I_0(U_1)$. That allows us to obtain[1] a real $\mu_{k,\ell} = \sum_n c_{n,k,\ell} \delta(x_{n,k,\ell}) \in M_d(U_1)$ such that $\widehat{\mu_{k,\ell}}(\chi) = I_{k,\ell}$ for all $\chi \in E_1$ with $\chi \nsim \overline{\chi}$ and 0 otherwise on E_1. Then obtain $\nu_{n,k,\ell} \in M_d^+(U_2)$ such that

$$\widehat{\nu_{n,k,\ell}}(\sigma) = \begin{cases} c_{n,k,\ell}(a_{k,\ell}(\chi \otimes \sigma) + Q_\sigma \overline{a_{k,\ell}(\chi \otimes \sigma)} Q_\sigma^{-1}) \\ \qquad \text{if there exists } \chi \text{ with } \chi \otimes \sigma \in F_3 \\ 0 \quad \text{otherwise on } E_2, \end{cases}$$

since $c_{n,k,\ell}$ is real. If $\omega_{k,\ell}^{(1)} = \sum_n \delta(x_{n,k,\ell}) \otimes \nu_{n,k,\ell} \in M_d^+(U_1 \times U_2)$, then

$$\widehat{\omega_{k,\ell}^{(1)}}(\chi \otimes \sigma) = I_{k,\ell} \otimes [a_{k,\ell}(\chi \otimes \sigma) + Q_\sigma \overline{a_{k,\ell}(\chi \otimes \sigma)} Q_\sigma^{-1}],$$

if $\chi \otimes \sigma \in F_3$ and 0 otherwise. Similarly, obtain

$$\widehat{\omega_{k,\ell}^{(1)}}(\chi \otimes \sigma) = i I_{k,\ell} \otimes [a_{k,\ell}(\chi \otimes \sigma)/i + Q_\sigma \overline{a_{k,\ell}(\chi \otimes \sigma)/i}\, Q_\sigma^{-1}],$$

on F_3 and add together to get $\mu_3 = \sum_{k,l=1}^d \omega_{k,\ell}^{(1)} + \omega_{k,\ell}^{(2)}$ noting that on F_3, $A(\chi \otimes \sigma)$ equals

$$\begin{aligned}
\frac{1}{2} \sum_{k,l=1}^d & I_{k,\ell} \otimes [a_{k,\ell}(\chi \otimes \sigma) + Q_\sigma \overline{a_{k,\ell}(\chi \otimes \sigma)} Q_\sigma^{-1}] \\
& + \frac{i}{2} I_{k,\ell} \otimes [a_{k,\ell}(\chi \otimes \sigma)/i + Q_\sigma \overline{a_{k\ell}(\chi \otimes \sigma)/i}\, Q_\sigma^{-1}].
\end{aligned}$$

[1] Just take any $\sum_n b_{n,k,\ell} \delta(x_{n,k,\ell}) \in M_d(U_1)$ that interpolates $I_{k,\ell}$ at both χ and $\overline{\chi}$ if they are not equivalent, and 0 else on E_1, and then take $\mu_{k,\ell} = \sum_n ((b_{n,k,\ell,} + \overline{b_{n,k,\ell}})/2) \delta(x_{n,k,\ell})$.

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Taking $\mu_1 + \mu_2 + \mu_3$ gives a positive, discrete measure concentrated on $U_1 \times U_2$ and doing the required interpolation. □

COROLLARY 4.2. *Suppose that $G = G_1 \times G_2$ is a compact, connected group, $\chi \in \widehat{G_1}$, $E_2 \subseteq \widehat{G_2}$ is antisymmetric and cofinitely $FZI_0(U)$. Then $E = \chi \otimes E_2$ is cofinitely $FZI_0(V \times U)$ for all open sets V.*

PROOF. Since $\{\chi\}$ is $I_0(U)$ for all open U, the conclusion follows from Lemma 4.1 where $E_1 = \{\chi\}$ and the $E_{2,j}$ are singletons with union E_2. □

A similar argument proves the following.

COROLLARY 4.3. *Suppose that $G = G_1 \times G_2$ is a compact, connected group, $U_j \subset G_j$ are open for $j = 1, 2$, I is an index set, $E_1 = \{\chi_i : i \in I\} \subset \widehat{G_1}$ is antisymmetric, $E_1 \cup \overline{E_1}$ is $I_0(U_1)$ and the elements of E_1 have the same degree. Suppose also that $E_2 = \{\sigma_i : i \in I\} \subset \widehat{G_2}$ is antisymmetric and cofinitely $FZI_0(U_2)$. Then $E = \{\chi_i \otimes \sigma_i : i \in I\}$ is cofinitely $FZI_0(U_1 \times U_2)$.*

5. Sets that are cofinitely $FZI_0(U)$

5.1. Products of Lie groups

THEOREM 5.1. *Let $G = \prod_{i \in I} G_i$ be a product of simple, simply connected, connected, compact Lie groups and suppose $E \subseteq \widehat{G}$ is an infinite set of representations of bounded degree. Then E contains an infinite subset F such that F is cofinitely $FZI_0(U)$ for all open sets U.*

PROOF. The boundedness of the degrees implies that up to isomorphism there are only finitely many choices for $\sigma|_{G_i}$ and that E has the padding property [7].

Therefore, we may assume that, for each i, $\{\sigma|_{G_i} : \sigma \in E\}$ has at most one element. A combinatorial argument shows that such a set contains an infinite subset of the form $\{\chi \otimes \sigma_i\}_i$ where the representations $\{\sigma_i\}$ are mutually orthogonal and orthogonal to χ (see [6, Theorem 2.7] or [7, Lemma 4.5] for a proof). By Corollary 4.2 $\{\chi \otimes \sigma_i\}_i$ is an infinite set which is cofinitely $FZI_0(U)$ for all open U. □

Using the preceding results in an analogous fashion to the proof of Theorem 4.9 in [7] we can obtain the following.

COROLLARY 5.2. *Suppose that $G = \prod G_i$ is a product of simple, simply connected, connected, compact Lie groups. Any infinite Sidon set in $\widehat{G}$ contains an infinite set which is cofinitely $FZI_0(U)$ for all open sets U.*

PROOF. By Cartwright and McMullen's characterization of Sidon sets (see [1, (5.1) and Proposition 5.5] or [7, Theorem 3.1] for details, including our notation), $E \subseteq E_1 \times E_2 \times E_3$ where $E_1 = \{\mathbf{1}\}$, $E_3 = FTR(\widetilde{G_3})$, E_2 is a set of representations on $\widetilde{G_2}$ of bounded degree and E_2 is orthogonal to E_3. Of course, if either $\#(E_2 \cap E)$ or $\#(E_3 \cap E)$ is infinite, then since the FTR set is cofinitely $FZI_0(U)$ set for all open

U, and since every infinite set of representations of bounded degree contains an infinite set that is cofinitely $FZI_0(U)$ for all open U, the proof is complete.

More generally, if there is some $\sigma \in E_2$ such that $\{\sigma \otimes \chi \in E : \chi \in E_2\}$ is infinite, then this set suffices, being a translate of an FTR set. Similarly, if there is some $\chi \in E_3$ such that $\{\sigma \otimes \chi \in E : \sigma \in E_2\}$ is infinite, then this set again contains an infinite set that is cofinitely $FZI_0(U)$ for all open U.

Otherwise, it is possible to choose infinite asymmetric sets $\{\sigma_i\} \subset E_2$, with all σ_i of the same degree, and $\{\chi_i\} \subset E_3$ such that $\{\sigma_i \otimes \chi_i\} \subset E$. By passing to a further infinite subset, if necessary, by Theorem 5.1 we can assume $\{\sigma_i\}$ is cofinitely $FZI_0(U)$ for all open U. By Theorem 1.2 $\{\sigma_i\} \cup \overline{\{\sigma_i\}}$ is $I_0(U)$ for all open U. As E_3 is cofinitely $FZI_0(U)$ for all open U, Corollary 4.3 implies $\{\sigma_i \otimes \chi_i\}$ is also cofinitely $FZI_0(U)$ for all open U. □

5.2. Cofinitely $FZI_0(U)$ sets, the general case Theorem 1.1 follows immediately from the following result.

THEOREM 5.3. *Suppose that G is a compact, connected group. Then every infinite Sidon set in $\widehat{G}$ contains an infinite set that is cofinitely $FZI_0(U)$ for all open sets U.*

PROOF. According to the structure theorem, G is isomorphic to a quotient of $\widetilde{G} = \prod_{i \in I} G_i \times A$, where A is compact, connected and abelian and G_i are classical, simple, connected, compact Lie groups. Let q be the quotient map and given $E \subseteq \widehat{G}$, put $\widetilde{E} = \{\sigma \circ q : \sigma \in E\}$. By Proposition 2.4 E is $FZI_0(U)$ if $\widetilde{E}$ is $FZI_0(q^{-1}(U))$. Thus, it suffices to prove that $\widetilde{E}$ contains an infinite set that is cofinitely $FZI_0(U)$ for each open U. In [5] it is shown that every infinite subset of the dual of a compact, connected, abelian group contains a cofinite $FZI_0(U)$ set. With these facts and the results of this section the proof can be completed in the same manner as the proof of Corollary 5.2. □

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