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A TRANSLATION PLANE OF ORDER 81 AND ITS FULL COLLINEATION GROUP

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In this paper a new translation plane of order 81 is constructed. Its collineation group is solvable and acts on the line at infinity as a permutation group K which is the product of a group of order 5 belonging to the center of K with a group of order 48 . A 2-Sylow subgroup of K is the direct product of a dihedral group of order 8 with a group of order 2 . K admits six orbits. They have lengths 4, 6, 12, 12, 24, 24 .

1. Introduction

Let $GF(q)$ be a Galois field of odd order q (≥ 5) and let F be the regular spread of $PG(3, q)$. Suppose that F contains a set $\mathcal{R}$ of reguli satisfying the properties

(i) $\mathcal{R}$ consists of $(q+3)/2$ reguli, $R_1, R_2, \ldots, R_{(q+3)/2}$,

(ii) any two reguli of $\mathcal{R}$ have exactly two lines in common,

(iii) no three reguli of $\mathcal{R}$ have a line in common.

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Let U be the partial spread consisting of all the lines of the reguli R_i $\big(i = 1, 2, \ldots, (q+3)/2\big)$ and let I_R denote the point set covered by U. If R_i' denotes the opposite regulus of R_i, the set U' of all the lines of R_i' is not a spread of I_R because any point P of I_R belongs to two lines of U'. According to a result of Bruen [4], if a subset V of U' is a partial spread then V contains $(q+1)/2$ lines of each R_i' such that the union of these half-reguli is V. If there exists such a partial spread V then one can obtain a new spread $G = (F-U) \cup V$ by replacing U with V. G will be called β-derived from F $\big($with respect to $(U, V)\big)$. We can try to obtain a new partial spread $V = \frac{1}{2}U'$ of I_R by choosing only a suitable half of the lines of each opposite regulus. As for β-derived spreads the existence problem is yet unsolved except for $q = 5, 7, 11, 13$. Specific examples are given in Bruen's paper [4] when $q = 5, 7$, in [5] and [12] when $q = 11$ and in [13] when $q = 13$.

In [4] Bruen also proved that if $q \geq 7$ the collineation group of a β-derived spread is the inherited group, that is, it is the subgroup of the collineation group of $PG(3, q)$ which leaves F invariant. The collineation groups of the translation planes arising from the above mentioned examples are determined [5], [12], [13].

In this paper the first example of a β-derived spread is constructed for $q = 9$ and the collineation group of the corresponding translation plane is determined. It is a solvable group satisfying the properties listed in the summary.

2. Notation and terminology

Notation and terminology are the same as in Bruen's paper [4]. The reader is assumed to be familiar with the theory of projective planes given in Dembowski [6] and in Hughes and Piper [7] and with the theory of permutation groups given in Wielandt [14].

Let $GF(9)$ be the quadratic extension of $GF(3) = \{0, 1, 2\}$. In particular let us consider the polynomial $x^2 + x + 2$ $\big($which is irreducible in $GF(3)\big)$, $i \notin GF(3)$ such that $i^2 + i + 2 = 0$ and

$3 = i$, $4 = i + 1$, $5 = i + 2$, $6 = 2i$, $7 = 1 + 2i$, $8 = 2 + 2i$; so $GF(9) = \{0, 1, 2, 3, 4, 5, 6, 7, 8\}$.

The homogeneous coordinates (x, y, z, u) denote points in $PG(3, q)$. The line joining the point (x_1, y_1, z_1, u_1) to the point (x_2, y_2, z_2, u_2) is denoted by $\langle (x_1, y_1, z_1, u_1), (x_2, y_2, z_2, u_2) \rangle$; moreover if $g, h, \ldots, w$ are elements of a group, then the subgroup generated by $g, h, \ldots, w$ is denoted by $\langle g, h, \ldots, w \rangle$.

3. Preliminaries

Before the example, the general relation between the spread F and the Miquelian inversive plane $M(q)$ over $GF(q)$ is discussed briefly.

Assume that q is odd and denote by s a non-square element of $GF(q)$, so $GF(q^2) = \{a+bt \mid a, b \in GF(q), t^2 = s\}$. We define norm $N(a+bt)$ of $a + bt$ of $GF(q^2)$, $N(a+bt) = a^2 - sb^2$. Clearly $N(a+bt) \in GF(q)$; moreover set $N(\rho) = \{a+bt \mid N(a+bt) = \rho\}$. $M(q)$ is considered the incidence structure whose points are the elements of $GF(q^2) \cup \{\infty\}$ and whose circles are the subsets of $GF(q^2) \cup \{\infty\}$ of the following types:

(I) $a + bt + N(\rho) = \{x+yt \mid N((a-x)+(b-y)t) = \rho\}$;

(II) $C(a+bt, c+dt) = \{(a+bt)+\lambda(c+dt) \mid \lambda \in GF(q) \cup \{\infty\}\}$.

The spread F is the union of all the lines

$$\langle (1, 0, 0, 0), (0, 1, 0, 0) \rangle, \langle (a, sb, 0, 1), (b, a, 1, 0) \rangle$$

where a, b run over $GF(q)$. By a theorem of Bruck's ([3], also [4]), the map

$$\psi : \begin{array}{l} \infty \to \langle (1, 0, 0, 0), (0, 1, 0, 0) \rangle \\ a + bt \to \langle (a, sb, 0, 1), (b, a, 1, 0) \rangle \end{array}$$

is an incidence preserving isomorphism between $M(q)$, with its points and circles, and F , with its lines and reguli.

By Bruck's theorem, the problem of finding sets R of reguli satisfying the properties (i), (ii), (iii) is equivalent to that of constructing chains C of circles in $M(q)$ satisfying the properties

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(i)' $\mathcal{C}$ consists of $(q+3)/2$ circles, $C_1, C_2, \ldots, C_{(q+3)/2}$,

(ii)' any two circles of $\mathcal{C}$ have exactly two points in common,

(iii)' no three circles of $\mathcal{C}$ have a point in common.

Let W be the collineation group of $\mathrm{PG}(3, q)$ mapping F onto itself. Every $w \in W$ determines a map w' of $M(q)$ onto itself defined as follows: let P be any point of $M(q)$ and let r be the line of F which corresponds to P by Ψ; then $w'(P)$ is the point which corresponds to $w(r)$ by Ψ. Therefore a map Φ is defined putting $\Phi(w) = w'$. Of course Φ is a homomorphism of W into W', where W' denotes the automorphism group of $M(q)$. The kernel of Φ is the group M of all collineations of $\mathrm{PG}(3, q)$ fixing F linewise. M is a cyclic group of order $q+1$. Moreover $W \simeq \Gamma\mathrm{L}\left(2, q^2\right)$ and $W' \simeq \mathrm{P}\Gamma\mathrm{L}\left(2, q^2\right)$.

Let H be the subgroup of W which leaves invariant a set $\mathcal{R}$ of reguli satisfying the properties (i), (ii), (iii). If H' denotes the subgroup of W' which leaves invariant the chain $\mathcal{C}$ of circles of $M(q)$ corresponding to $\mathcal{R}$ by Ψ, then $\Phi(H) = H'$. Hence $H/M \simeq H'$.

4. Two partial spreads U and V of PG(3, 9) covering the same point set

Using the notation of Bruen's paper, our construction can be described as follows.

If $s = 3$, the norm-classes of GF(9) are

$N(0) = \{0\}$,
$N(1) = \{1, 2, 4+4t, 8+4t, 3+5t, 6+5t, 3+7t, 6+7t, 4+8t, 8+8t\}$,
$N(2) = \{5, 7, 4+t, 8+t, 4+2t, 8+2t, 3+3t, 6+3t, 3+6t, 6+6t\}$,
$N(3) = \{5+3t, 7+3t, 3+4t, 6+4t, 5t, 5+6t, 7+6t, 7t, 3+8t, 6+8t\}$,
$N(4) = \{3+t, 6+t, 3+2t, 6+2t, 3t, 1+5t, 2+5t, 6t, 1+7t, 2+7t\}$,
$N(5) = \{4, 8, 1+3t, 2+3t, 5+5t, 7+5t, 1+6t, 2+6t, 5+7t, 7+7t\}$,
$N(6) = \{t, 2t, 4+3t, 8+3t, 1+4t, 2+4t, 4+6t, 8+6t, 1+8t, 2+8t\}$,
$N(7) = \{3, 6, 1+t, 2+t, 1+2t, 2+2t, 5+4t, 7+4t, 5+8t, 7+8t\}$,
$N(8) = \{5+t, 7+t, 5+2t, 7+2t, 4t, 4+5t, 8+5t, 4+7t, 8+7t, 8t\}$.

An easy calculation shows that the following circles of $M(q)$ form a chain $\mathcal{C}$ satisfying the properties (i)', (ii)', (iii)':

$$C_1 : 2t + N(7) \ , \quad C_2 : t + N(7) \ , \quad C_3 : 7t + N(5) \ ,$$

$$C_4 : 5t + N(5) \ , \quad C_5 : (0,\ 3+t) \ , \quad C_6 : (0,\ 3+2t) \ .$$

The corresponding reguli $R_1, R_2, R_3, R_4, R_5, R_6$ contained in F form a set $\mathcal{R}$ satisfying the properties (i), (ii), (iii). The line $\langle(1, 0, 0, 0), (0, 1, 0, 0)\rangle$ is denoted by r_∞ and the line $\langle(a, 3b, 0, 1), (b, a, 1, 0)\rangle$ by $r(a, b)$, then

$$R_1 = \{r(1, 0), r(2, 0), r(1, 1), r(2, 1), r(3, 2), r(6, 2), r(5, 3), r(7, 3), r(5, 7), r(7, 7)\} \ ,$$

$$R_2 = \{r(1, 0), r(2, 0), r(3, 1), r(6, 1), r(1, 2), r(2, 2), r(5, 5), r(7, 5), r(5, 6), r(7, 6)\} \ ,$$

$$R_3 = \{r(5, 0), r(7, 0), r(1, 1), r(2, 1), r(1, 4), r(2, 4), r(5, 5), r(7, 5), r(4, 7), r(8, 7)\} \ ,$$

$$R_4 = \{r(5, 0), r(7, 0), r(1, 2), r(2, 2), r(4, 5), r(8, 5), r(5, 7), r(7, 7), r(1, 8), r(2, 8)\} \ ,$$

$$R_5 = \{r(0, 0), r(3, 1), r(6, 2), r(7, 3), r(1, 4), r(4, 5), r(5, 6), r(8, 7), r(2, 8), r_\infty\} \ ,$$

$$R_6 = \{r(0, 0), r(6, 1), r(3, 2), r(5, 3), r(2, 4), r(8, 5), r(7, 6), r(4, 7), r(1, 8), r_\infty\} \ .$$

Therefore

$$\begin{aligned} U = \{&r(1, 0), r(2, 0), r(1, 1), r(2, 1), r(3, 2), \\ &r(6, 2), r(5, 3), r(7, 3), r(5, 7), r(7, 7), r(3, 1), r(6, 1), r(1, 2), \\ &r(2, 2), r(5, 5), r(7, 5), r(5, 6), r(7, 6), r(5, 0), r(7, 0), r(1, 4), \\ &r(2, 4), r(4, 7), r(8, 7), r(4, 5), r(8, 5), \\ &r(1, 8), r(2, 8), r(0, 0), r_\infty\} \ . \end{aligned}$$

Denote by V the union of the following half-reguli

$$\tfrac{1}{2}R'_i = \{r'(i, 1), r'(i, 2), r'(i, 3), r'(i, 4), r'(i, 5)\} :$$

$$\begin{aligned} \tfrac{1}{2}R'_1 = \{&\langle(2, 3, 0, 1), (0, 1, 1, 0)\rangle, \langle(2, 0, 0, 1), (1, 1, 1, 0)\rangle, \\ &\langle(6, 6, 0, 1), (2, 3, 1, 0)\rangle, \langle(7, 8, 0, 1), (3, 5, 1, 0)\rangle, \\ &\langle(7, 7, 0, 1), (7, 5, 1, 0)\rangle\} \ , \end{aligned}$$

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$$\tfrac{1}{2}R'_2 = \{\langle(1, 0, 0, 1), (2, 2, 1, 0)\rangle, \langle(1, 6, 0, 1), (0, 2, 1, 0)\rangle, \langle(3, 3, 0, 1), (1, 6, 1, 0)\rangle, \langle(5, 5, 0, 1), (5, 7, 1, 0)\rangle, \langle(5, 4, 0, 1), (6, 7, 1, 0)\rangle\} ,$$

$$\tfrac{1}{2}R'_3 = \{\langle(1, 1, 0, 1), (1, 2, 1, 0)\rangle, \langle(1, 3, 0, 1), (4, 2, 1, 0)\rangle, \langle(7, 4, 0, 1), (0, 5, 1, 0)\rangle, \langle(7, 0, 0, 1), (5, 5, 1, 0)\rangle, \langle(8, 8, 0, 1), (7, 4, 1, 0)\rangle\} ,$$

$$\tfrac{1}{2}R'_4 = \{\langle(2, 6, 0, 1), (8, 1, 1, 0)\rangle, \langle(2, 2, 0, 1), (2, 1, 1, 0)\rangle, \langle(4, 4, 0, 1), (5, 8, 1, 0)\rangle, \langle(5, 0, 0, 1), (7, 7, 1, 0)\rangle, \langle(5, 8, 0, 1), (0, 7, 1, 0)\rangle\} ,$$

$$\tfrac{1}{2}R'_5 = \{\langle(4, 1, 0, 0), (0, 0, 1, 0)\rangle, \langle(7, 1, 0, 0), (0, 0, 1, 1)\rangle, \langle(1, 0, 0, 0), (0, 0, 1, 2)\rangle, \langle(8, 1, 0, 0), (0, 0, 1, 3)\rangle, \langle(5, 1, 0, 0), (0, 0, 1, 6)\rangle\} ,$$

$$\tfrac{1}{2}R'_6 = \{\langle(0, 1, 0, 0), (0, 0, 1, 4)\rangle, \langle(6, 1, 0, 0), (0, 0, 1, 5)\rangle, \langle(1, 1, 0, 0), (0, 0, 1, 7)\rangle, \langle(3, 1, 0, 0), (0, 0, 1, 8)\rangle, \langle(2, 1, 0, 0), (0, 0, 0, 1)\rangle\} .$$

Note that $\tfrac{1}{2}R'_i$ is contained in the opposite regulus R'_i of R_i. An explicit calculation shows that U and V cover the same subset I_R of PG(3, 9) and have no lines in common. Therefore $G = (F-U) \cup V$ is a spread of PG(3, 9). Clearly G is a β-derived spread from F.

5. The automorphism group of the chain C

In this section the automorphism group H' of $M(9)$ which leaves the point set I covered by C invariant is determined. As a model of $M(9)$ we will take the geometry $G(Q)$ of the plane sections of an elliptic quadric Q of PG(3, 9). The passage from $M(9)$ to $G(Q)$ can be realized as follows. Let Q be the elliptic quadric of PG(3, 9) whose equation is

$$Q : zu = x^2 + 6y^2 .$$

As it is well known, the map

$$\sigma : \begin{cases} \infty \to (0, 0, 1, 0) \\ a + bt \to (a, b, a^2+6b^2, 1) \end{cases}$$

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is an incidence preserving isomorphism between $M(9)$ and Q with their points and circles. Moreover (*cf.* [6], p. 274), if w' is an automorphism of $M(9)$, then there is a unique collineation $\bar{w}$ of PG(3, 9) leaving Q invariant such that w' acts on $M(9)$ as $\bar{w}$ on Q. Thus

$$\sigma(w'(P)) = \bar{w}(\sigma(P)) \quad \text{for every} \quad P \in M(9)$$

and

$$\sigma^{-1}(\bar{w}(P)) = w'(\sigma^{-1}(P)) \quad \text{for every} \quad P \in Q .$$

Therefore the problem of determining the automorphism group H' of $M(9)$ which leaves I invariant is equivalent to that of determining the collineation group $\bar{H}$ of PG(3, 9) which maps Q onto itself and leaves $\sigma(I)$ invariant, where $\sigma(I) = \{\sigma(P) \mid P \in I\}$. First some lemmas are needed.

LEMMA 5.1. *Let D be a circle of Q contained in $\sigma(I)$; then D coincides with a circle $\sigma(C_i)$.*

Proof. By way of contradiction, let D be distinct from any circle $\sigma(C_i)$; then for every $i = 1, 2, \ldots, 6$ we have $|D \cap \sigma(C_i)| \leq 2$ and from this

$$\sum_{i=1}^{6} |D \cap \sigma(C_i)| \leq 2 \cdot 6 = 12 .$$

On the other hand each point of D lies exactly on two circles $\sigma(C_i)$, so

$$\sum_{i=1}^{6} |D \cap \sigma(C_i)| = 2 \cdot 10 = 20 .$$

Therefore we would have $20 \leq 12$.

LEMMA 5.2. *Let Z be a collineation group of* PG(3, 9) *which maps Q onto itself leaving $\sigma(I)$ invariant; then Z preserves the chain $\sigma(C)$.*

Proof. Let $\alpha \in Z$ and $\sigma(C_i) \in \sigma(C)$; as $\alpha(\sigma(C_i))$ is a circle contained in $\sigma(I)$, the preceding Lemma 5.1 assures that $\alpha(\sigma(C_i))$ is a circle of $\sigma(C)$. This proves Lemma 5.2.

Let π_i $(i = 1, 2, \ldots, 6)$ be the plane of PG(3, 9) which meets Q in $\sigma(C_i)$. Denote by P_i the polo of π_i with respect to Q . Then, by Lemma 5.2, $\bar{H}$ is the collineation group of PG(3, 9) which maps Q onto itself and leaves the set $\{P_1, P_2, P_3, P_4, P_5, P_6\}$ invariant. An easy calculation shows that $P_1(0, 2, 2, 1)$, $P_2(0, 1, 2, 1)$, $P_3(0, 7, 1, 1)$, $P_4(0, 5, 1, 1)$, $P_5(1, 1, 0, 0)$, $P_6(2, 1, 0, 0)$.

LEMMA 5.3. *$\bar{H}$ fixes Y_∞ .*

Proof. The lines joining P_1 and P_2 , P_3 and P_4 , P_5 and P_6 pass through Y_∞ ; there exist only three planes joining Y_∞ and four points among P_i $(i = 1, 2, \ldots, 6)$. So $\bar{H}$ must fix Y_∞ .

Let $\bar{H}_i$ be the stabilizer of P_i . First we note that

LEMMA 5.4. $\bar{H}_1 = \bar{H}_2$, $\bar{H}_3 = \bar{H}_4$, $\bar{H}_5 = \bar{H}_6$.

LEMMA 5.5. *The identity collineation* 1 *is the only collineation of $\bar{H}$ fixing each P_i $(i = 1, 2, \ldots, 6)$.*

Proof. The only non-identity collineation which leaves each P_i invariant has equations

$$\rho x' = x^3 , \quad \rho y' = y^3 , \quad \rho z' = 2u^3 , \quad \rho u' = 2z^3 ;$$

but this collineation does not preserve Q .

LEMMA 5.6. *$\bar{H}$ acts faithfully on $\{P_1, P_2, \ldots, P_6\}$.*

LEMMA 5.7. *$\bar{H}_6$ is a dihedral group of order* 8 . *$\bar{H}_6$ is generated by the following collineations:*

$$\bar{h} : \rho x' = 5x , \quad \rho y' = 5y , \quad \rho z' = 2z , \quad \rho u' = u ,$$

$$\bar{s} : \rho x' = x , \quad \rho y' = y , \quad \rho z' = u , \quad \rho u' = z .$$

$\bar{h}$ and $\bar{s}$ act on $\{P_1, P_2, \ldots, P_6\}$ as follows:

$$\bar{h} : (P_1P_3P_2P_4)(P_5)(P_6) \quad \textit{and} \quad \bar{s} : (P_1P_2)(P_3)(P_4)(P_5)(P_6) .$$

Proof. By Lemmas 5.4 and 5.6 we have that $\bar{H}_6$ is a subgroup of the

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symmetric group S_4 on four objects. An easy calculation shows that $\bar{h}$ and $\bar{s}$ act on $\{P_1, P_2, \ldots, P_6\}$ as said before and $\bar{h}^4 = \bar{s}^2 = (\bar{h}\bar{s})^2 = 1$. So $\langle \bar{h}, \bar{s} \rangle$ is a dihedral group of order 8 contained in $\bar{H}_6$. As $\bar{H}_1 = \bar{H}_2$ we have $\bar{H}_1 \cap \bar{H}_6 = \bar{H}_2 \cap \bar{H}_6$ and so $\bar{H}_6$ cannot contain any element of order 3. Thus $|\bar{H}_6| \leq 8$ and therefore $\bar{H}_6 = \langle \bar{h}, \bar{s} \rangle$.

As $|\bar{H}_6| = 8$ and the orbit of P_6 has length at most 6, the following lemma holds.

LEMMA 5.8. $|\bar{H}| \leq 48$.

Let us consider the collineation

$$\bar{c} : \rho x' = 5z + 7u , \quad \rho y' = 5y , \quad \rho z' = x + z + u , \quad \rho u' = x + 2z + 2u .$$

An easy calculation shows

LEMMA 5.9. $\langle \bar{c} \rangle$ *has order* 6 *and is contained in* $\bar{H}$. $\bar{c}^3$ *centralizes* $\bar{H}_6$.

Next we prove

LEMMA 5.10. $\langle \bar{c} \rangle \cap \bar{H}_6 = \{1\}$.

Proof. By Lemma 5.9, $|\langle \bar{c} \rangle \cap \bar{H}_6| \leq 2$. If $|\langle \bar{c} \rangle \cap \bar{H}_6| = 2$, then $\bar{c}^3 \in \bar{H}_6$ and, by Lemma 5.7, $\bar{c}^3 = \bar{h}^2$, as $\bar{h}^2$ is the central involution of $\bar{H}_6$. But one can verify that $\bar{c}^3 \neq \bar{h}^2$.

THEOREM 5.11. $\bar{H} = \langle \bar{h}, \bar{s}, \bar{c} \rangle$, $|\bar{H}| = 48$ *and a* 2*-Sylow subgroup of* $\bar{H}$ *is the direct product of a dihedral group of order* 8 *with a group of order* 2.

Proof. By Lemmas 5.7, 5.10, $\langle \bar{h}, \bar{s}, \bar{c} \rangle$ is a subgroup of order at least 48 of $\bar{H}$. By Lemma 5.8, $|\bar{H}| \leq 48$, thus $\bar{H} = \langle \bar{h}, \bar{s}, \bar{c} \rangle$ and $|\bar{H}| = 48$. The latter assertion of the theorem follows from Lemmas 5.7 and 5.9.

6. The collineation group H preserving U

Let us denote by 1 the identity collineation of PG(3, 9) and put

(6.1) $m_\alpha : \rho x' = \alpha x + y$, $\rho y' = 3x + \alpha y$, $\rho z' = \alpha z + 3y$, $\rho u' = z + \alpha u$,

where $\alpha \in GF(9)$. Then

LEMMA 6.1. *The cyclic group of order* 10 ,

$$M = \{m_\alpha \mid \alpha \in GF(9)\} \cup \{1\} ,$$

fixes U *linewise; moreover* $M = \langle m_3 \rangle$. M *is the full collineation group of* PG(3, 9) *which fixes* U *linewise.*

Lemmas 6.2 and 6.3 follow from Lemma 6.1.

LEMMA 6.2. M *is the subgroup of* H *fixing each regulus* R_i . *Therefore* H/M *acts on* $\{R_1, R_2, \ldots, R_6\}$ *faithfully.*

Proof. Let Φ be the correspondence defined in Section 3. Then $\Phi(H) = \overline{H}$ and $\Phi(H)$ acts on $\{R_1, R_2, \ldots, R_6\}$ as $\overline{H}$ on $\{P_1, P_2, \ldots, P_6\}$. By Lemma 6.6, $\overline{H}$ is faithful on $\{P_1, P_2, \ldots, P_6\}$. Thus Ker Φ is the subgroup of H fixing each R_i . As we have shown in Section 3, Ker $\Phi = M$. This proves Lemma 6.2.

Since $|\overline{H}| = 48$ and $|M| = 10$, from Lemma 6.2 it follows that

LEMMA 6.3. $|H| = 480$.

Let us consider the collineations

(6.2) $h : \rho x' = 5x,\quad \rho y' = 5y,\quad \rho z' = z,\quad \rho u' = u$,

(6.3) $s : \rho x' = z,\quad \rho y' = 6u,\quad \rho z' = 6x,\quad \rho u' = y$,

(6.4) $c : \rho x' = 5y + 7z,\ \rho y' = 8x + 4u,\ \rho z' = 3x + 3u,\ \rho u' = 2y + 2z$.

It is easy to check that h, s, c have order 4, 2, 6 respectively and

$$\begin{cases} h(r_\infty) = r_\infty , \\ h\big(r(a, b)\big) = r(5a, 5b) , \end{cases} \tag{6.5}$$

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$$(6.6)\quad \begin{cases} s(r_\infty) = r(0,\ 0)\ , \\ s\big(r(0,\ 0)\big) = r_\infty\ , \\ s\big(r(a,\ b)\big) = r\big(a\big(a^2+6b^2\big)^{-1},\ b\big(a^2+6b^2\big)^{-1}\big) \quad \text{for} \quad (a,\ b) \neq (0,\ 0)\ , \end{cases}$$

$$(6.7)\quad \begin{cases} c(r_\infty) = r(7,\ 0)\ , \\ c\big(r(2,\ 0)\big) = r_\infty\ , \\ c\big(r(a,\ b)\big) = r\big(\big(7a^2+4b^2+5\big)\big((a+1)^2+6b^2\big)^{-1},\ 7b\big((a+1)^2+6b^2\big)^{-1}\big) \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{for} \quad (a,\ b) \neq (2,\ 0)\ . \end{cases}$$

Therefore

LEMMA 6.4. *$\langle h,\ s,\ c,\ M\rangle$ is a subgroup of H.*

From (6.5), (6.6) and (6.7) we infer that $h,\ s,\ c$ act on $\{R_1,\ R_2,\ \ldots,\ R_6\}$ as follows:

$$(6.9)\qquad h\ :\ \big(R_1R_3R_2R_4\big)\big(R_5\big)\big(R_6\big)\ , \quad s\ :\ \big(R_1R_2\big)\big(R_3\big)\big(R_4\big)\big(R_5\big)\big(R_6\big)\ ,$$
$$c\ :\ \big(R_1R_6R_3R_2R_5R_4\big)\ .$$

Thus $h,\ s,\ c$ act on $\{R_1,\ R_2,\ \ldots,\ R_6\}$ as $\bar{h},\ \bar{s},\ \bar{c}$ on $\{P_1,\ P_2,\ \ldots,\ P_6\}$. As the subgroup $\langle\bar{h},\ \bar{s},\ \bar{c}\rangle$ of S_6 gives a faithful representation of $\bar{H}$ on $\{P_1,\ P_2,\ \ldots,\ P_6\}$, we get $\langle h,\ s,\ c,\ M\rangle/M = \langle\bar{h},\ \bar{s},\ \bar{c}\rangle$. Therefore from Lemma 6.3 it follows that

LEMMA 6.5. *$H = \langle h,\ s,\ c,\ M\rangle$.*

Next we prove

LEMMA 6.6. *$\langle h,\ s,\ c\rangle \cap M = \{1\}$.*

Proof. The orbit of X_∞ in $\langle h,\ s,\ c\rangle$ consists of the points X_∞, Y_∞, Z_∞, 0, $(1,\ 0,\ 0,\ 1)$, $(0,\ 1,\ 1,\ 0)$, $(2,\ 0,\ 0,\ 1)$, $(0,\ 2,\ 1,\ 0)$, $(5,\ 0,\ 0,\ 1)$, $(0,\ 5,\ 1,\ 0)$, $(7,\ 0,\ 0,\ 1)$, $(0,\ 7,\ 1,\ 0)$. The collineation m_α maps X_∞ into $(\alpha,\ 3,\ 0,\ 0)$, so $m_\alpha \notin \langle h,\ s,\ c\rangle$ if $\alpha \neq 0$. Now let us consider m_0 : it maps $(1,\ 1,\ 0,\ 0)$ into $(4,\ 1,\ 0,\ 0)$; the orbit of $(1,\ 1,\ 0,\ 0)$ in $\langle h,\ s,\ c\rangle$ consists of the points $(1,\ 1,\ 0,\ 0)$, $(0,\ 0,\ 1,\ 1)$, $(1,\ 1,\ 1,\ 1)$, $(8,\ 1,\ 0,\ 0)$, $(0,\ 0,\ 6,\ 1)$, $(2,\ 2,\ 1,\ 1)$, $(5,\ 5,\ 1,\ 1)$, $(7,\ 7,\ 1,\ 1)$, $(1,\ 6,\ 6,\ 1)$, $(2,\ 3,\ 6,\ 1)$, $(7,\ 4,\ 6,\ 1)$, $(5,\ 8,\ 6,\ 1)$. Therefore $m_0 \notin \langle h,\ s,\ c\rangle$.

From the preceding lemmas we obtain

THEOREM 6.7. *The collineation group* H *of* PG(3, 9) *which maps* F *into itself and leaves* U *invariant is* $\langle h, s, c\rangle \cdot M$. *Moreover* $\langle h, s, c\rangle \simeq \overline{H}$ *(cf. Theorem* 5.11) *and* $\langle h, s, c\rangle$ *acts on* $\{R_1, R_2, \ldots, R_6\}$ *as* $\overline{H}$ *on* $\{P_1, P_2, \ldots, P_6\}$.

7. The inherited group K

Bruen [4] proved that if $q \geq 7$ the collineation group K preserving the partial spread $G = (F-U) \cup V$ is the inherited group, that is, the subgroup of H fixing G . In our case K is a proper subgroup of H . In fact one can check that m_3 does not preserve V . On the other hand $h, s, c, m_3^2 \in K$, as

$$h : \big(r'(1, 1)r'(3, 3)r'(2, 2)r'(4, 5)\big)\big(r'(1, 2)r'(3, 4)r'(2, 1)r'(4, 4)\big)\big(r'(1, 3)r'(3, 5)r'(2, 3)r'(4, 3)\big)\big(r'(1, 4)r'(3, 2)r'(2, 5)r'(4, 1)\big)\big(r'(1, 5)r'(3, 1)r'(2, 4)r'(4, 2)\big)\big(r'(5, 1)\big)\big(r'(5, 2)\big)\big(r'(5, 3)\big)\big(r'(5, 4)\big)\big(r'(5, 5)\big)\big(r'(6, 1)\big)\big(r'(6, 2)\big)\big(r'(6, 3)\big)\big(r'(6, 4)\big)\big(r'(6, 5)\big) ,$$

$$s : \big(r'(1, 1)r'(2, 1)\big)\big(r'(1, 2)r'(2, 2)\big)\big(r'(1, 3)r'(2, 3)\big)\big(r'(1, 4)r'(2, 4)\big)\big(r'(1, 5)r'(2, 5)\big)\big(r'(3, 1)r'(3, 2)\big)\big(r'(3, 3)r'(3, 4)\big)\big(r'(3, 5)\big)\big(r'(4, 1)r'(4, 2)\big)\big(r'(4, 3)\big)\big(r'(4, 4)r'(4, 5)\big)\big(r'(5, 1)r'(5, 3)\big)\big(r'(5, 2)r'(5, 4)\big)\big(r'(5, 5)\big)\big(r'(6, 1)r'(6, 5)\big)\big(r'(6, 2)\big)\big(r'(6, 3)r'(6, 4)\big) ,$$

$$c : \big(r'(1, 1)r'(6, 5)r'(3, 3)r'(2, 1)r'(5, 1)r'(4, 4)\big)\big(r'(1, 2)r'(6, 1)r'(3, 4)r'(2, 2)r'(5, 3)r'(4, 5)\big)\big(r'(1, 3)r'(6, 2)r'(3, 5)r'(2, 3)r'(5, 5)r'(4, 3)\big)\big(r'(1, 4)r'(6, 4)r'(3, 2)r'(2, 4)r'(5, 2)r'(4, 2)\big)\big(r'(1, 5)r'(6, 3)r'(3, 1)r'(2, 5)r'(5, 4)r'(4, 1)\big) ,$$

$$m_3^2 : \big(r'(1, 1)r'(1, 3)r'(1, 2)r'(1, 4)r'(1, 5)\big)\big(r'(2, 1)r'(2, 5)r'(2, 4)r'(2, 2)r'(2, 3)\big)\big(r'(3, 1)r'(3, 3)r'(3, 5)r'(3, 4)r'(3, 2)\big)\big(r'(4, 1)r'(4, 2)r'(4, 5)r'(4, 3)r'(4, 4)\big)\big(r'(5, 1)r'(5, 5)r'(5, 3)r'(5, 2)r'(5, 4)\big)\big(r'(6, 1)r'(6, 2)r'(6, 5)r'(6, 3)r'(6, 4)\big) .$$

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Therefore we can state the following:

LEMMA 7.1. $\langle h, s, c\rangle \leq K$, $m_3 \notin K$, $m_3^2 \in K$.

Thus $\langle h, s, c, m_3^2\rangle$ is a subgroup of K . Since $K \cap M = \langle m_3^2\rangle$, $\left|\langle m_3^2\rangle\right| = |M|/2$, it follows that $[H : K] = 2$. By $K \neq H$, $K = \langle h, s, c, m_3^2\rangle$. Moreover $\langle m_3^2\rangle$ is a normal subgroup of K . So we have the following:

THEOREM 7.2. *The collineation group* K *preserving the spread* $G = (F-U) \cup V$ *is the semidirect product of a group of order* 5 *lying in the center of* K *with a group isomorphic to* $\bar{H}$. *Therefore* K *is a solvable group of order* 24 . *Moreover the orbits of* K *are*

$\big(r(3, 0)r(4, 0)r(6, 0)r(8, 0)\big)$,

$\big(r(0, 1)r(0, 2)r(0, 3)r(0, 4)r(0, 5)r(0, 6)r(0, 7)r(0, 8)r(3, 3)$
$r(3, 5)r(3, 6)r(3, 7)r(4, 1)r(4, 2)r(4, 4)r(4, 8)r(6, 3)r(6.5)$
$r(6, 6)r(6, 7)r(8, 1)r(8, 2)r(8, 4)r(8, 8)\big)$,

$\big(r(1, 3)r(1, 5)r(1, 6)r(1, 7)r(2, 3)r(2, 5)r(2, 6)r(2, 7)r(3, 4)$
$r(3, 8)r(4, 3)r(4, 6)r(5, 1)r(5, 2)r(5, 4)r(5, 8)r(6, 4)r(6, 8)$
$r(7, 1)r(7, 2)r(7, 4)r(7, 8)r(8, 3)r(8, 6)\big)$,

$\big(r'(1, 1)r'(1, 2)r'(2, 1)r'(2, 2)r'(3, 3)r'(3, 4)r'(4, 4)r'(4, 5)$
$r'(5, 1)r'(5, 3)r'(6, 1)r'(6, 5)\big)$,

$\big(r'(1, 3)r'(2, 3)r'(3, 5)r'(4, 3)r'(5, 5)r'(6, 2)\big)$,

$\big(r'(1, 4)r'(1, 5)r'(2, 4)r'(2, 5)r'(3, 1)r'(3, 2)r'(4, 1)r'(4, 2)$
$r'(5, 2)r'(5, 4)r'(6, 3)r'(6, 4)\big)$.

8. The translation complement

As it is well known a translation plane τ of order q^2 arises from any spread of PG(3, q) (*cf.* [6], p. 220 and [11] and [2]). Here we give an outline for the case $q = 9$.

The points of τ are those of PG(4, 9) - PG(3, 9) , the lines of τ are the planes of PG(4, 9) which meet PG(3, 9) in a line of the spread and do not belong to PG(3, 9) . If we consider PG(4, 9) - PG(3, 9) as a vector space $V_4(9)$ of dimension 4 over GF(9), the points of τ can be identified with the vectors of $V_4(9)$, the lines through the zero vector $\mathbf{0}$ are some 2-dimensional subspaces (called also components), the other lines of τ are translates of the components. The group of collineations of τ fixing $\mathbf{0}$ is called the "translation complement" and consists of some linear or semilinear transformations of $V_4(9)$. In particular such a transformation α of $V_4(9)$ belongs to the translation complement of τ if and only if the collineation of PG(3, 9) defined by the matrix of α leaves the spread invariant. Moreover if we denote the translation complement of τ by Λ_0, the cyclic group of order 8 of all dilations with center 0 by Δ_0 and the collineation group of PG(3, 9) which leaves the spread invariant by K, then $K \simeq \Lambda_0/\Delta_0$.

As τ is the translation plane arising from the spread $G = (F-U) \cup V$, by Theorem 7.2 we get

THEOREM 8.1. *The translation complement* Λ_0 *of* τ *is* $\langle \lambda, \sigma, \gamma, \mu \rangle$, *where*

$$\begin{aligned}
\lambda &: x' = 5x, & y' &= 5y, & z' &= z, & u' &= u,\\
\sigma &: x' = z, & y' &= 6u, & z' &= 6x, & u' &= y,\\
\gamma &: x' = 5y + 7z, & y' &= 8x + 4u, & z' &= 3x + 3u, & u' &= 2y + 2z,\\
\mu &: x' = 8x + y, & y' &= 3x + 8y, & z' &= 8z + 3u, & u' &= z + 8u.
\end{aligned}$$

As $\langle \mu \rangle$ is a normal subgroup of Λ_0 and $\langle \lambda, \sigma, \gamma \rangle \cap \langle \mu \rangle = \{1\}$, Λ_0 is semidirect product of $\langle \lambda, \sigma, \gamma \rangle$ and $\langle \mu \rangle$. Therefore Λ_0 is a solvable group of order $240 \cdot 8 = 1920$.

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