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AN ARITHMETICAL EXCURSION VIA STONEHAM NUMBERS

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To Professor Peter Borwein on his 60th birthday

Abstract

Let p be a prime and b a primitive root of p^2. In this paper, we give an explicit formula for the number of times a value in $\{0, 1, \dots, b-1\}$ occurs in the periodic part of the base-b expansion of $1/p^m$. As a consequence of this result, we prove two recent conjectures of Aragón Artacho *et al.* ['Walking on real numbers', *Math. Intelligencer* **35**(1) (2013), 42–60] concerning the base-b expansion of Stoneham numbers.

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1. Introduction

Let $b \geqslant 2$ be an integer. A real number $\alpha \in (0, 1)$ is called *b-normal* if in the base-b expansion of α the asymptotic frequency of the occurrence of any word $w \in \{0, 1, \dots, b-1\}^*$ of length n is $1/b^n$. A canonical example of such a number is Champernowne's number,

$$C_{10} := 0.123456789101112131415161718192021\cdots,$$

which, given here in base 10, is the size-ordered concatenation of $\mathbb{N}$ (each number written in base 10) proceeded by a decimal point. Champernowne's number was shown to be 10-normal by Champernowne [5] in 1933 and transcendental by Mahler [9] in 1937.

In 1973, Stoneham [12] defined the following class of numbers. Let $b, c \geqslant 2$ be relatively prime integers. The *Stoneham number* $\alpha_{b,c}$ is given by

$$\alpha_{b,c} := \sum_{n \geqslant 1} \frac{1}{c^n b^{c^n}}.$$

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Stoneham [12] showed that $\alpha_{2,3}$ is 2-normal. A new proof of this result was given by Bailey and Misiurewicz [4], and finally, in 2002, Bailey and Crandall [3] proved that $\alpha_{b,c}$ is b-normal for all coprime integers $b, c \geqslant 2$; see also Bailey and Borwein [2]. Transcendence of $\alpha_{b,c}$ follows easily by Mahler's method; the interested reader can see the details Appendix A.

Recently Aragón Artacho *et al.*[1] made two conjectures concerning properties of the base-4 expansion of the Stoneham number $\alpha_{2,3}$ and the base-3 expansion of $\alpha_{3,5}$, respectively. In this paper, we prove their conjectures, and as such they are stated here as theorems (we have fixed a few small typos in their published conjectures).

THEOREM 1.1. *Let the base-4 expansion of $\alpha_{2,3}$ be given by $\alpha_{2,3} := \sum_{k\geqslant 1} d_k 4^{-k}$, with $d_k \in \{0,1,2,3\}$. Then, for all $n \geqslant 0$:*

(i) $\sum_{k=\frac{3}{2}(3^n+1)}^{\frac{3}{2}(3^n+1)+3^n-1} (e^{\pi i/2})^{d_k} = -\begin{cases} i & \text{if } n \text{ is odd,} \\ 1 & \text{if } n \text{ is even;} \end{cases}$

(ii) $d_k = d_{3^n+k} = d_{2\cdot 3^n+k}$ *for* $k = \frac{3}{2}(3^n+1), \frac{3}{2}(3^n+1)+1, \ldots, \frac{3}{2}(3^n+1)+3^n-1$.

THEOREM 1.2. *Let the base-3 expansion of $\alpha_{3,5}$ be given by $\alpha_{3,5} := \sum_{k\geqslant 1} a_k 3^{-k}$, with $a_k \in \{0,1,2\}$. Then, for all $n \geqslant 0$:*

(i) $\sum_{k=1+5^{n+1}}^{1+5^{n+1}+4\cdot 5^n} (e^{\pi i/3})^{a_k} = (-1)^n e^{\pi i/3}$;

(ii) $a_k = a_{4\cdot 5^n+k} = a_{8\cdot 5^n+k} = a_{12\cdot 5^n+k} = a_{16\cdot 5^n+k}$ *for* $k = 5^{n+1} + j$, *with* $j = 1, \ldots, 4\cdot 5^n$.

We note here that the Stoneham numbers $\alpha_{b,c}$ are in some ways very similar to Champernowne's numbers. They are not concatenations of consecutive integers, but the concatenation of periods of certain rational numbers. Let $b, c \geqslant 2$ be coprime integers and let w_n be the word $w \in \{0,1,\ldots,b-1\}^*$ of minimal length such that

$$\left(\frac{1}{c^n}\right)_b = 0.\overline{w_n},$$

where $(x)_b$ denotes the base-b expansion of the real number x and $\overline{w}$ denotes the infinitely repeated word w. Then the Stoneham numbers are similar to the numbers

$$0.w_1w_2w_3w_4w_5\cdots w_n\cdots,$$

which are given by concatenating the words w_n. Indeed, the Stoneham number has this structure, but with the w_j repeated and cyclicly shifted.

REMARK. While we will be considering the base-4 expansion of $\alpha_{2,3}$ we are still dealing with a normal number; $\alpha_{2,3}$ is also 4-normal. This is given by a result of Schmidt [11] who proved in 1960 that the r-normal real number x is s-normal if $\log r/\log s \in \mathbb{Q}$.

2. Base-b expansions of rationals

To prove the above theorems in as much generality as possible we will need to consider how we write a reduced fraction a/k in the base b. Such an algorithm is well known, but we remind the reader here, as it will be useful to have the general

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Base-b Algorithm for $a/k < 1$.

Let $b, k \geqslant 2$ be integers and $a \geqslant 1$ be an integer coprime to k. Set $r_0 = a$ and write

$$\begin{aligned} r_0 b &= q_1 k + r_1 \\ r_1 b &= q_2 k + r_2 \\ &\vdots \\ r_{j-1} b &= q_j k + r_j \\ &\vdots \end{aligned}$$

where $q_j \in \{0, 1, \ldots, b-1\}$ and $r_j \in \{0, 1, \ldots, k-1\}$ for each j. Stop when $r_n = r_0$. Then

$$\left(\frac{a}{k}\right)_b = 0.\overline{q_1 q_2 \cdots q_n}.$$

FIGURE 1. The base-b algorithm for the reduced rational $a/k < 1$.

framework for the proofs of Theorems 1.1 and 1.2. To write a/k in the base b, we use a sort of modified division algorithm; see Figure 1.

We record here facts about the base-b algorithm which we will need.

LEMMA 2.1. *Suppose that $b, k \geqslant 2$ are coprime, and that r_j and q_j are defined by the base-b algorithm for a/k. Then* $\gcd(r_i, k) = 1$.

PROOF. Suppose that $p|k$, and proceed by induction on i. Firstly, $r_0 = a$ and by assumption $\gcd(r_0, k) = \gcd(a, k) = 1$.

Now suppose that $\gcd(r_i, k) = 1$, so that also $\gcd(r_i b, k) = 1$. Then

$$r_{i+1} = r_i b - q_{i+1} k \equiv r_i b \not\equiv 0 \mod p,$$

since $\gcd(b, k) = 1$. Thus $\gcd(r_{i+1}, k) = 1$. □

Also, we have that equivalent r_j give equal q_j.

LEMMA 2.2. *Suppose $b, k \geqslant 2$ are coprime, and that r_j and q_j are defined by the base-b algorithm for the reduced fraction a/k. Then $r_i \equiv r_j \pmod{b}$ if and only if $q_i = q_j$.*

PROOF. Suppose that $r_i \equiv r_j \pmod{b}$. By considering the difference between $r_{i-1} b = q_i k + r_i$ and $r_{j-1} b = q_j k + r_j$ modulo b, we see that $b|(q_i - q_j)k$, so that since $\gcd(b, k) = 1$, we have that $b|(q_i - q_j)$. Since $q_i, q_j \in \{0, 1, \ldots, b-1\}$, we thus have that $q_i = q_j$.

Conversely, suppose that $q_i = q_j$. Here, again, we can consider the difference between the defining equations for q_i and q_j modulo b; this gives the desired result. □

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Indeed, the value of q_j is determined by the residue class of r_j modulo b and the value of k^{-1} modulo b.

LEMMA 2.3. *Suppose that $b, k \geqslant 2$ are coprime, and that r_j and q_j are defined by the base-b algorithm for the reduced fraction a/k. Then $r_i \equiv j \pmod{b}$ if and only if $q_i \equiv -jk^{-1} (\bmod b)$, where $q_i \in \{0, 1, \ldots, b-1\}$.*

PROOF. If $r_i \equiv j \pmod{b}$, then the equation $r_{i-1}b = q_i k + r_i$ gives $q_i k \equiv -j \pmod{b}$, which in turn gives that $q_i \equiv -jk^{-1} \pmod{b}$. Since $q_i \in [0, b-1]$ we are done with this direction of proof.

Conversely, suppose that $q_i = (-jk^{-1} \bmod b)$. Then surely $q_i \equiv -jk^{-1} \pmod{b}$ and so $q_i k \equiv -j \pmod{b}$. Thus, again using $r_{i-1}b = q_i k + r_i$, we have that $r_i \equiv j \pmod{b}$. □

The following lemma is a direct corollary of Lemma 2.3.

LEMMA 2.4. *Suppose that $b, k \geqslant 2$ are coprime, and that r_j and q_j are defined by the base-b algorithm for the reduced fraction a/k. Then $r_i \equiv 0 \pmod{b}$ if and only if $q_i = 0$.*

PROOF. Apply Lemma 2.3 with $j = 0$. □

We will use the following classical theorem (see [10, Theorem 12.4]) and lemma.

THEOREM 2.5. *Let b be a positive integer. Then the base-b expansion of a rational number either terminates or is periodic. Further, if $r, s \in \mathbb{Z}$ with $0 < r/s < 1$ where $\gcd(r, s) = 1$ and $s = TU$, where every prime factor of T divides b and $\gcd(U, b) = 1$, then the period length of the base-b expansion of r/s is the order of b modulo U, and the preperiod length is N, where N is the smallest positive integer such that $T | b^N$.*

Theorem 2.5 tells us that the base-b expansion of a/k is purely periodic (recall that $\gcd(b, k) = 1$), and that the minimal period is $\mathrm{ord}_k b$, which divides $\varphi(k)$, so that this also is a period. This result can be exploited using the following number-theoretic result, a proof of which can be found in most elementary number theory texts; for example, see [10, Theorem 9.10].

LEMMA 2.6. *A primitive root of p^2 is a primitive root of p^k for any integer $k \geqslant 2$.*

Applying Lemma 2.6 gives the following result.

LEMMA 2.7. *Let $0 < a/p^m < 1$ be a rational number in lowest terms and let $b \geqslant 2$ be an integer that is a primitive root of p^2. Suppose that $(1/p^m)_b = .\overline{q_1 q_2 \cdots q_n}$ is given by the base-b algorithm. Then*

$$\left(\frac{a}{p^m}\right)_b = .\overline{q_{\sigma(1)} q_{\sigma(2)} \cdots q_{\sigma(n)}}$$

where σ is a cyclic shift on n letters.

PROOF. This is a direct consequence of the base-b algorithm. □

As a consequence of the above lemmas we are able to provide the following characterisation of certain base-b expansions.

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Proposition 2.8. *Let $m \geqslant 1$ be an integer, p be an odd prime, $b \geqslant 2$ be an integer coprime to p, and q_j and r_j be given by the base-b algorithm for the reduced fraction a/p^m. If b is a primitive root of p and p^2, then* $\operatorname{period}(a/p^m) = \varphi(p^m)$ *and*

$$\#\{j \leqslant \varphi(p^m) : q_j = 0\} = \left\lfloor \frac{p^m}{b} \right\rfloor - \left\lfloor \frac{p^{m-1}}{b} \right\rfloor.$$

Proof. The fact that $\operatorname{period}(a/p^m)_b = \varphi(p^m)$ follows directly from b being a primitive root of p and p^2, Lemma 2.6 and Theorem 2.5. This further implies that the $\varphi(p^m)$ values of r_i given by the base-b algorithm for a/p^m are distinct. Applying Lemma 2.1 gives that

$$\{r_1, r_2, \ldots, r_{\varphi(p^m)}\} = \{i \leqslant p^m : \gcd(i, p) = 1\}. \tag{2.1}$$

Also recall that

$$\left(\frac{a}{p^m}\right)_b = .\overline{q_1 q_2 \cdots q_{\varphi(p^m)}},$$

and that by Lemma 2.4, $q_i = 0$ if and only if $r_i \equiv 0 \pmod{b}$. Note that there are exactly

$$\left\lfloor \frac{p^m}{b} \right\rfloor - \left\lfloor \frac{p^m}{bp} \right\rfloor = \left\lfloor \frac{p^m}{b} \right\rfloor - \left\lfloor \frac{p^{m-1}}{b} \right\rfloor$$

elements of $\{i \leqslant p^m : \gcd(i, p) = 1\}$ which are divisible by b. Thus using the set equality (2.1), we have that there are exactly $\lfloor p^m/b \rfloor - \lfloor p^{m-1}/b \rfloor$ elements of $\{r_1, r_2, \ldots, r_{\varphi(p^m)}\}$ divisible by b. Appealing to Lemma 2.4, we then have that there are $\lfloor p^m/b \rfloor - \lfloor p^{m-1}/b \rfloor$ of $q_1, q_2, \ldots, q_{\varphi(p^m)}$ such that $q_j = 0$. □

Note that while we record the $q_i = 0$ case because of its simplicity, the method can be applied to count any value of q_i that is desired by using the appropriate case of Lemma 2.3. In fact, we will do this in a few special cases to prove Theorems 1.1 and 1.2.

3. The base-b expansion of the Stoneham number $\alpha_{b,p}$

We will need properties for both the base-b and base-b^2 expansions of the Stoneham number $\alpha_{b,p}$.

Proposition 3.1. *Let $b, p \geqslant 2$ be coprime integers with p a prime. Denote the base-b expansion of $\alpha_{b,p}$ as*

$$\alpha_{b,p} = \sum_{j \geqslant 1} \frac{1}{p^j b^{p^j}} = \sum_{k \geqslant 1} \frac{a_k}{b^k},$$

where $a_k \in \{0, 1, \ldots, b-1\}$, and write

$$\left(\frac{\sum_{j=0}^{m-1} p^j}{p^m}\right)_b = .\overline{q_1 q_2 \cdots q_n},$$

where q_i is determined by the base-b algorithm, for each i, so $n = \operatorname{ord}_{p^m} b$. Then $q_i = a_{p^m + jn + i}$ for each $i \in \{1, 2, \ldots, n\}$ and each $j \in \{0, 1, 2, \ldots, p \cdot \varphi(p^m)/\operatorname{ord}_{p^m} b - 1\}$.

It is worth noting that Proposition 3.1 is the full generalisation of Theorem 1.1(ii).

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We require the following lemma.

Lemma 3.2. *Let $b, c \geqslant 2$ be coprime. Then, for any $m \geqslant 1$,*

$$\alpha_{b,c} - \sum_{n=1}^{m} \frac{1}{c^n b^{c^n}} < \frac{1}{b^{c^{m+1}}}.$$

That is, the base-b expansion of $\alpha_{b,c}$ agrees with the b-ary expansion of its mth partial sum up to the c^{m+1}th place.

Proof. Let $m \geqslant 1$ and note that

$$\sum_{n\geqslant m+1} \frac{1}{c^n} = \frac{1}{c^{m+1} - c^m} < 1.$$

Using this fact, we have that

$$\alpha_{b,c} - \sum_{n=1}^{m} \frac{1}{c^n b^{c^n}} = \sum_{n\geqslant m+1} \frac{1}{c^n b^{c^n}} < \frac{1}{b^{c^{m+1}}} \sum_{n\geqslant m+1} \frac{1}{c^n} < \frac{1}{b^{c^{m+1}}},$$

which is the desired result. □

Proof of Proposition 3.1. Let $m \geqslant 1$, $s_m = p^m b^{p^m}$, and define the positive integer r_m by

$$\frac{r_m}{s_m} = \sum_{n=1}^{m} \frac{1}{p^n b^{p^n}}.$$

Then

$$\gcd(r_m, s_m) = \gcd(r_m, p^m b^{p^m}) = \gcd(r_m, pb) = 1.$$

We apply Theorem 2.5 with $b = b$, $r = r_m$, $s = s_m$, $T = b^{p^m}$, and $U = p^m$ to give that the period length of the base-b expansion of r_m/s_m is the order of b modulo p^m, which we will write as

$$\mathrm{period}(r_m/s_m) = \mathrm{ord}_{p^m} b,$$

and the preperiod length of r_m/s_m is p^m, which we will write as

$$\mathrm{preperiod}(r_m/s_m) = p^m.$$

Combining the observations of the previous paragraph with Lemma 3.2 gives that

$$a_{p^m+1} a_{p^m+2} \ldots a_{p^{m+1}} = \underbrace{ww w \cdots w}_{(p \cdot \varphi(p^m)/\mathrm{ord}_{p^m} b) \text{ times}},$$

where $w = q_1 q_2 \cdots q_{\mathrm{ord}_{p^m} b}$ is a word on the alphabet $\{0, 1, \ldots, b\}$ with length $\mathrm{ord}_{p^m} b$. To finish the proof of this proposition, it is enough to appeal to Lemma 3.2 to show that

$$\left(\frac{\sum_{j=0}^{m-1} p^j}{p^m} \right)_b = .\overline{w}$$

where w is as defined in the previous sentence, which follows directly from the definition of $\alpha_{b,p}$. □

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Theorem 1.1 concerns a base-b^2 expansion; we will provide some specialised results for this case only when $b = 2$, in order to specifically prove Theorem 1.1, as the more interesting case for generalisations is the base-b case.

LEMMA 3.3. *Let $b, c \geqslant 2$ be coprime. Then, for any $m \geqslant 1$,*

$$\alpha_{b,c} - \sum_{n=1}^{m} \frac{1}{c^n b^{c^n}} < \frac{1}{(b^2)^{c^{m+1}/2}}.$$

That is, the base-b^2 expansion of $\alpha_{b,c}$ agrees with the base-b^2 expansion of its mth partial sum up to the $\lceil c^{m+1}/2 \rceil$th place.

PROOF. This is a direct consequence of Lemma 3.2. □

PROPOSITION 3.4. *Let p be an odd prime such that 2 is a primitive root of p and p^2. Denote the base-4 expansion of $\alpha_{2,p}$ as*

$$\alpha_{2,p} = \sum_{j \geqslant 1} \frac{1}{p^j 2^{p^j}} = \sum_{k \geqslant 1} \frac{d_k}{4^k},$$

where $d_k \in \{0, 1, \ldots, 3\}$, and write

$$\left(\frac{\sum_{j=0}^{m-1} p^j}{p^m} \right)_4 = .\overline{q_1 q_2 \cdots q_n}\,,$$

where the q_is are determined by the base-4 algorithm, so $n = \mathrm{ord}_{p^m} 4 = \varphi(p^m)/2$. Then $q_i = d_{(p^m+1)/2 + jn + i}$ for each $i \in \{1, \ldots, n\}$ and each $j \in \{0, 1, 2, \ldots, p-1\}$.

PROOF. This proposition follows as a corollary of Proposition 3.1. Indeed, by Proposition 3.1, we have a prefix u of odd length p and words w_m of even length $\varphi(p^m)$ such that

$$(\alpha_{2,p})_2 = .u \underbrace{w_1 w_1 \cdots w_1}_{p \text{ times}} \underbrace{w_2 w_2 \cdots w_2}_{p \text{ times}} \cdots \underbrace{w_m w_m \cdots w_m}_{p \text{ times}} \cdots .$$

Now the word w_m is the minimal repeated word given by the base-2 expansion of $(\sum_{j=0}^{m-1} p^j)/p^m$. But

$$0 < \frac{\sum_{j=0}^{m-1} p^j}{p^m} = \frac{p^m - 1}{p^m(p-1)} < \frac{1}{p-1} \leqslant \frac{1}{2},$$

and so the first letter of w_m, for each m, is necessarily 0. Define the word v_m by $w_m = 0v_m$. Then

$$\begin{aligned}(\alpha_{2,p})_2 &= .u \underbrace{w_1 w_1 \cdots w_1}_{p \text{ times}} \underbrace{w_2 w_2 \cdots w_2}_{p \text{ times}} \cdots \underbrace{w_m w_m \cdots w_m}_{p \text{ times}} \cdots \\ &= .u \underbrace{0v_1 0v_1 \cdots 0v_1}_{p \text{ times}} \underbrace{0v_2 0v_2 \cdots 0v_2}_{p \text{ times}} \cdots \underbrace{0v_m 0v_m \cdots 0v_m}_{p \text{ times}} \cdots \\ &= .u0 \underbrace{v_1 0 v_1 0 \cdots v_1 0}_{p \text{ times}} \underbrace{v_2 0 v_2 0 \cdots v_2 0}_{p \text{ times}} \cdots \underbrace{v_m 0 v_m 0 \cdots v_m 0}_{p \text{ times}} \cdots ,\end{aligned} \tag{3.1}$$

where the word $u0$ is of even length $p + 1$ and the word $v_m 0$ is of even length $\varphi(p^m)$.

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As in the statement of Proposition 3.1, let a_k be the kth letter in the base-2 expansion of $\alpha_{2,p}$, and as in the statement of the current proposition, let d_k be the kth letter in the base-4 expansion of $\alpha_{2,p}$. Then

$$d_k = 2a_{2k-1} + a_{2k}.$$

Using this fact, it is an immediate consequence of (3.1) that there are words U of length $(p + 1)/2$ and W_m of length $\varphi(p^m)/2$ such that

$$(\alpha_{2,p})_4 = .U\, \underbrace{W_1W_1 \cdots W_1}_{p \text{ times}}\, \underbrace{W_2W_2 \cdots W_2}_{p \text{ times}} \cdots \underbrace{W_mW_m \cdots W_m}_{p \text{ times}} \cdots .$$

As in Proposition 3.1, to finish the proof of this proposition, it is enough to apply Lemma 3.3 to show that

$$\left(\frac{\sum_{j=0}^{m-1} p^j}{p^m}\right)_4 = .\overline{W_m},$$

where W_m is as defined in the previous sentence, which follows directly from the definition of $\alpha_{2,p}$. □

4. The Aragon, Bailey, Borwein and Borwein conjectures

In this section, we apply the results of Section 3 to prove Theorems 1.1 and 1.2. As it turns out, the proof of Theorem 1.2 is a bit more straightforward, so we present its proof first.

Proof of Theorem 1.2 For convenience let us write $\omega := e^{\pi i/3}$ and let r_i and q_i be given by the base-3 algorithm for $1/5^n$. Note that, by Proposition 3.1,

$$\sum_{k=1+5^{n+1}}^{1+5^{n+1}+4\cdot 5^n} \omega^{a_k} = \sum_{j=0}^{2} \#\{i \leqslant \varphi(5^{n+1}) : q_i = j\} \cdot \omega^j.$$

Now $\#\{i \leqslant \varphi(5^n) : q_i = j\}$ can be given by looking at where the number 5^n lies modulo 15. Since, for every 15 consecutive numbers, 12 of them are coprime to 5, and these 12 fall into the three equivalence classes modulo 3 with an equal frequency of 4 times each, we need only look at the remainder of 5^n modulo 15. An easy calculation gives that

$$5^n \equiv \begin{cases} 5 \pmod{15} & \text{if } n \text{ is odd,} \\ 10 \pmod{15} & \text{if } n \text{ is even.} \end{cases}$$

This allows us to give that

$$\#\{i \leqslant \varphi(5^n) : r_i \equiv 0 \pmod 3\} = \begin{cases} 4 \cdot \left\lfloor \dfrac{5^n}{15} \right\rfloor + 1 & \text{if } n \text{ is odd,} \\[2ex] 4 \cdot \left\lfloor \dfrac{5^n}{15} \right\rfloor + 3 & \text{if } n \text{ is even,} \end{cases}$$

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$$\#\{i \leqslant \varphi(5^n) : r_i \equiv 1 \pmod 3\} = \begin{cases} 4 \cdot \left\lfloor \dfrac{5^n}{15} \right\rfloor + 2 & \text{if } n \text{ is odd,} \\ 4 \cdot \left\lfloor \dfrac{5^n}{15} \right\rfloor + 3 & \text{if } n \text{ is even,} \end{cases}$$

and

$$\#\{i \leqslant \varphi(5^n) : r_i \equiv 2 \pmod 3\} = \begin{cases} 4 \cdot \left\lfloor \dfrac{5^n}{15} \right\rfloor + 1 & \text{if } n \text{ is odd,} \\ 4 \cdot \left\lfloor \dfrac{5^n}{15} \right\rfloor + 2 & \text{if } n \text{ is even.} \end{cases}$$

Applying Lemma 2.3 to the preceding equalities gives that

$$\#\{i \leqslant \varphi(5^n) : q_i = 0\} = \begin{cases} 4 \cdot \left\lfloor \dfrac{5^n}{15} \right\rfloor + 1 & \text{if } n \text{ is odd,} \\ 4 \cdot \left\lfloor \dfrac{5^n}{15} \right\rfloor + 3 & \text{if } n \text{ is even,} \end{cases}$$

$$\#\{i \leqslant \varphi(5^n) : q_i = 1\} = \begin{cases} 4 \cdot \left\lfloor \dfrac{5^n}{15} \right\rfloor + 2 & \text{if } n \text{ is odd,} \\ 4 \cdot \left\lfloor \dfrac{5^n}{15} \right\rfloor + 2 & \text{if } n \text{ is even,} \end{cases}$$

and

$$\#\{i \leqslant \varphi(5^n) : q_i = 2\} = \begin{cases} 4 \cdot \left\lfloor \dfrac{5^n}{15} \right\rfloor + 1 & \text{if } n \text{ is odd,} \\ 4 \cdot \left\lfloor \dfrac{5^n}{15} \right\rfloor + 3 & \text{if } n \text{ is even.} \end{cases}$$

Thus, since $1 + \omega + \omega^2 = 0$,

$$\begin{aligned} \sum_{k=1+5^{n+1}}^{1+5^{n+1}+4\cdot 5^n} \omega^{a_k} &= \sum_{j=0}^{2} \#\{i \leqslant \varphi(5^{n+1}) : q_i = j\} \cdot \omega^j \\ &= \begin{cases} \omega & \text{if } n+1 \text{ is odd,} \\ -\omega & \text{if } n+1 \text{ is even,} \end{cases} \\ &= (-1)^n \omega, \end{aligned}$$

which proves part (i).

Part (ii) follows directly from Proposition 3.1 with $b = 3$ and $p = 5$. □

Proof of Theorem 1.1. Note that

$$\frac{1}{3^n 2^{3^n}} = \frac{8}{3^n} \cdot \frac{1}{4^{\frac{3}{2}(3^{n-1}+1)}}.$$

Let r_i and q_i be given by the base 4 algorithm for $8/3^n$. We will use the fact that each of these r_i is equivalent to 2 modulo 3. This is easily seen as we have for

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each i that $r_{i-1}4 = q_i 3^n + r_i$, so that, taking this equality modulo 3, we have that $r_{i-1} \equiv r_i \pmod 3$. Recalling that $r_0 = 8$ shows that indeed $r_i \equiv 2 \pmod 3$ for each i.

Since $\mathrm{ord}_{3^n} 4 = 3^{n-1}$, we have, by Proposition 3.4, that

$$\sum_{k=\frac{3}{2}(3^n+1)}^{\frac{3}{2}(3^n+1)+3^n-1} (e^{\pi i/2})^{a_k} = \sum_{j=0}^{3} \#\{i \leqslant \varphi(3^{n+1})/2 : q_i = j\} \cdot (e^{\pi i/2})^j.$$

Now $\#\{i \leqslant 3^n : q_i = j\}$ can be given by looking at where the number 3^n lies modulo 12. Since, for every 12 consecutive numbers, four of them are equivalent to 2 modulo 3, and these four fall into the four distinct equivalence classes modulo 4, we must consider the remainder of 3^n modulo 12. We have that

$$3^n \equiv \begin{cases} 3 \pmod{12} & \text{if } n \text{ is odd,} \\ 9 \pmod{12} & \text{if } n \text{ is even.} \end{cases}$$

Thus

$$\#\{i \leqslant \varphi(3^n)/2 : r_i \equiv 0 \pmod 4\} = \begin{cases} \left\lfloor \frac{3^n}{12} \right\rfloor & \text{if } n \text{ is odd,} \\ \left\lfloor \frac{3^n}{12} \right\rfloor + 1 & \text{if } n \text{ is even,} \end{cases}$$

$$\#\{i \leqslant \varphi(3^n)/2 : r_i \equiv 1 \pmod 4\} = \begin{cases} \left\lfloor \frac{3^n}{12} \right\rfloor & \text{if } n \text{ is odd,} \\ \left\lfloor \frac{3^n}{12} \right\rfloor + 1 & \text{if } n \text{ is even,} \end{cases}$$

$$\#\{i \leqslant \varphi(3^n)/2 : r_i \equiv 2 \pmod 4\} = \begin{cases} \left\lfloor \frac{3^n}{12} \right\rfloor + 1 & \text{if } n \text{ is odd,} \\ \left\lfloor \frac{3^n}{12} \right\rfloor + 1 & \text{if } n \text{ is even,} \end{cases}$$

and

$$\#\{i \leqslant \varphi(3^n)/2 : r_i \equiv 3 \pmod 4\} = \begin{cases} \left\lfloor \frac{3^n}{12} \right\rfloor & \text{if } n \text{ is odd,} \\ \left\lfloor \frac{3^n}{12} \right\rfloor & \text{if } n \text{ is even.} \end{cases}$$

By Lemma 2.3, we have that

$$\#\{i \leqslant \varphi(3^n)/2 : q_i = 0\} = \begin{cases} \left\lfloor \frac{3^n}{12} \right\rfloor & \text{if } n \text{ is odd,} \\ \left\lfloor \frac{3^n}{12} \right\rfloor + 1 & \text{if } n \text{ is even,} \end{cases}$$

$$\#\{i \leqslant \varphi(3^n)/2 : q_i = 1\} = \begin{cases} \left\lfloor \frac{3^n}{12} \right\rfloor & \text{if } n \text{ is odd,} \\ \left\lfloor \frac{3^n}{12} \right\rfloor & \text{if } n \text{ is even,} \end{cases}$$

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$$\#\{i \leqslant \varphi(3^n)/2 : q_i = 2\} = \begin{cases} \left\lfloor \dfrac{3^n}{12} \right\rfloor + 1 & \text{if } n \text{ is odd,} \\ \left\lfloor \dfrac{3^n}{12} \right\rfloor + 1 & \text{if } n \text{ is even,} \end{cases}$$

and

$$\#\{i \leqslant \varphi(3^n)/2 : q_i = 3\} = \begin{cases} \left\lfloor \dfrac{3^n}{12} \right\rfloor & \text{if } n \text{ is odd,} \\ \left\lfloor \dfrac{3^n}{12} \right\rfloor + 1 & \text{if } n \text{ is even.} \end{cases}$$

Since $1 + (e^{\pi i/2}) + (e^{\pi i/2})^2 + (e^{\pi i/2})^3 = 0$, we thus have that

$$\begin{aligned} \sum_{k=\frac{3}{2}(3^n+1)}^{\frac{3}{2}(3^n+1)+3^n-1} (e^{\pi i/2})^{a_k} &= \sum_{j=0}^{3} \#\{i \leqslant \varphi(3^{n+1})/2 : q_i = j\} \cdot (e^{\pi i/2})^j \\ &= \begin{cases} -1 & \text{if } n+1 \text{ is odd,} \\ -i & \text{if } n+1 \text{ is even,} \end{cases} \\ &= -\begin{cases} i & \text{if } n \text{ is odd,} \\ 1 & \text{if } n \text{ is even,} \end{cases} \end{aligned}$$

which proves part (i).

Part (ii) follows directly from Proposition 3.4 with $b = 2$ and $p = 3$. □

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Appendix A. Transcendence of Stoneham numbers

In this appendix, we give details of the transcendence of the Stoneham number $\alpha_{b,c}$ for any choice of integers $b, c \geqslant 2$. In fact, Mahler's method gives much stronger results, which imply this desired conclusion.

We start out by letting $c \geqslant 2$ be an integer and define

$$F_c(x) := \sum_{n \geqslant 1} \frac{x^{c^n}}{c^n}.$$

Notice that $F_c(x)$ satisfies the Mahler functional equation

$$F_c(x^c) = cF_c(x) - x^c. \tag{A.1}$$

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Now suppose that $F_c(x) \in \mathbb{C}(x)$. Then there are polynomials $a(x), b(x) \in \mathbb{C}[x]$ such that

$$F_c(x) - \frac{a(x)}{b(x)} = 0.$$

Since $F_c(x) \in \mathbb{C}[[x]]$ is not a polynomial, we may assume, without loss of generality, that $\gcd(a(x), b(x)) = 1$ and $b(0) \neq 0$ and $b(x) \notin \mathbb{C}$. Sending $x \to x^c$ and applying the functional equation, we thus have that

$$F_c(x) - \frac{a(x)}{b(x)} = 0 = F_c(x^c) - \frac{a(x^c)}{b(x^c)} = F_c(x) - \left(\frac{x^c}{c} + \frac{a(x^c)}{b(x^c)}\right),$$

so that

$$\frac{x^c}{c} + \frac{a(x^c)}{b(x^c)} = \frac{a(x)}{b(x)}. \tag{A.2}$$

Now as functions, the right- and left-hand sides of (A.2) must have the same singularities. But $b(x^c)$ will have more zeros (counting multiplicity if needed) than $b(x)$ unless $b(x)$ is a constant, which is a contradiction. Thus $F_c(x)$ does not represent a rational function. In fact, we can now appeal to the following theorem, to give that $F_c(x)$ is transcendental over $\mathbb{C}(x)$.

Theorem A.1 (Nishioka [6]). *Suppose that $F(x) \in \mathbb{C}[[x]]$ satisfies one of the following for an integer $d > 1$:*

(i) $F(x^d) = \phi(x, F(x))$,
(ii) $F(x) = \phi(x, F(x^d))$,

where $\phi(x, u)$ is a rational function in x, u over $\mathbb{C}$. If $F(x)$ is algebraic over $\mathbb{C}(x)$, then $F(x) \in \mathbb{C}(x)$.

To prove the transcendence of the Stoneham numbers, we appeal to a classical result of Mahler [8], We record it here as taken from Nishioka's monograph [7].

Theorem A.2 (Mahler [8]). *Let $\mathbf{I}$ be the set of algebraic integers over $\mathbb{Q}$, K be an algebraic number field, $\mathbf{I}_K = K \cap \mathbf{I}$, $f(x) \in K[[x]]$ with radius of convergence $R > 0$ satisfying the functional equation for an integer $d > 1$,*

$$f(x^d) = \frac{\sum_{i=0}^m a_i(x) f(x)^i}{\sum_{i=0}^m b_i(x) f(x)^i}, \quad m < d,\ a_i(x), b_i(x) \in \mathbf{I}_K[x],$$

and $\Delta(x) := \mathrm{Res}(A, B)$ be the resultant of $A(u) = \sum_{i=0}^m a_i(x) u^i$ and $B(u) = \sum_{i=0}^m b_i(x) u^i$ as polynomials in u. If $f(x)$ is transcendental over $K(x)$ and ξ is an algebraic number with $0 < |\xi| < \min\{1, R\}$ and $\Delta(\xi^{d^n}) \neq 0$ $(n \geqslant 0)$, then $f(\xi)$ is transcendental.

Since $F_c(x)$ is transcendental over $\mathbb{C}(x)$, $F_c(x)$ satisfies the functional equation (A.1), and $\mathrm{Res}(cu - x^c, 1) \neq 0$ for all x, we have the following corollary to Mahler's theorem.

Corollary A.3. *Let $c \geqslant 2$ be an integer. The number $\sum_{n \geqslant 1} (1/c^n) \xi^{c^n}$ is transcendental for all algebraic numbers ξ with $0 < |\xi| < 1$. In particular, for all $b, c \geqslant 2$, the Stoneham number $\alpha_{b,c}$ is transcendental.*

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