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WEIERSTRASS ELLIPTIC DIFFERENCE EQUATIONS

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The Weierstrass elliptic function satisfies a nonlinear first order and a nonlinear second order differential equation. It is shown that these differential equations can be discretized in such a way that the solutions of the resulting difference equations exactly coincide with the corresponding values of the elliptic function.

1. Introduction.

In a sequence of recent papers [2]-[5], it has been shown that, in choosing a difference equation (ΔE) approximation to a differential equation (DE), theoretical advantage can be obtained by exploiting a wider range of approximations than is customary.

For the simple linear second order DE

$$w''(z) + w(z) = 0 \tag{1.1}$$

it was shown [3] that it is 'best' to use the expression

$$w''(z) \approx (w_{n+1} - 2w_n + w_{n-1})/[4\sin^2(h/2)] \tag{1.2}$$

rather than the usual

$$w''(z) \approx (w_{n+1} - 2w_n + w_{n-1})/h^2 \tag{1.3}$$

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for the approximation to the derivative.

For the nonlinear Verhulst *DE*

(1.4) $$w'(z) = w(z) - w(z)^2$$

the approximation

(1.5) $$w'(z) \approx (w_{n+1} - w_n)/(e^h - 1)$$

is the best to use [4], and with $w(z)^2$ replaced by $w_n w_{n+1}$.

For the nonlinear Duffing equation [2],

(1.6) $$w''(z) + aw(z) + bw(z)^3 = 0 ,$$

the cubic term $w(z)^3$ is best replaced by $\frac{1}{2}bw_n^2(w_{n+1} + w_{n-1})$.

It is the purpose of the present paper to extend this approach to the important *DE*'s satisfied by the Weierstrass elliptic function $p(z)$ [1], namely the first order nonlinear *DE* of the second degree

(1.7) $$p'(z)^2 = 4p(z)^3 - g_2 p(z) - g_3$$

and the consequent second order nonlinear *DE* of the first degree

(1.8) $$p''(z) = 6p(z)^2 - \frac{1}{2} g_2 .$$

The function $p(z)$ is an even function of z, $p(z) - z^{-2}$ is analytic at $z = 0$ and equal to 0 at $z = 0$, and the constants g_2 and g_3 are the so-called invariants.

Sinple ΔE approximations to (1.7) and (1.8) are

(1.9) $$(p_{n+1} - p_n)^2/h^2 = 4p_n^3 - g_2 p_n - g_3$$

and

(1.10) $$(p_{n+1} - 2p_n + p_{n-1})/h^2 = 6p_n^2 - \frac{1}{2} g_2 ,$$

which are correct to $O(h)$. It will be shown that modifications of these can be made so that they are the 'best' ΔE's possible in the sense that their solutions coincide exactly with the values of $p(z)$. More precisely, for the points

(1.11) $$z = a + nh$$

where a is an initial value and h a constant stepsize (not necessarily 'small'), then the solution p_n of the ΔE's will have the property that

(1.12) $$p_n = p(a + nh).$$

2. First Order Difference Equation.

The best ΔE approximation to the first order DE (1.7) is obtained by using the addition formula for $p(z)$ to derive a difference equation for p_n as defined by (1.12).

If

(2.1) $$p(h) = k$$

then the addition formula [1] for $p(z)$ can be written

(2.2) $$p_{n+1} = \frac{1}{4}\left[\frac{p'(z) - p'(h)}{p_n - k}\right]^2 - p_n - k\ .$$

Solving for $p'(z)$, namely

(2.3) $$p'(z) = p'(h) \pm 2(p_n - k)(p_{n+1} + p_n + k)^{\frac{1}{2}}$$

and squaring gives

(2.4) $$-4(p_n - k)p_{n+1} + 4kp_n + 8k^2 - g_2 = \pm 4p'(h)(p_{n+1} + p_n + k)^{\frac{1}{2}}$$

where use has been made of (1.7). Squaring and using (1.7) again yields the required best first order nonlinear ΔE of the second degree

$$\begin{aligned}(p_n - k)^2(p_{n+1})^2 - [4kp_n\ \tfrac{1}{2}(k+p_n) - g_2\ \tfrac{1}{2}(k+p_n) - g_3]p_{n+1}\\ + [(kp_n + \tfrac{1}{4}\,g_2)^2 + g_3(k+p_n)] = 0\ .\end{aligned} \tag{2.5}$$

To recognize this ΔE as an approximation for small h to the DE (1.1) requires rearranging (2.5) to the form

$$\begin{aligned}(p_{n+1} - p_n)^2 k = 4p_n p_{n+1}\ \tfrac{1}{2}(p_n + p_{n+1}) - g_2\ \tfrac{1}{2}(p_n + p_{n+1}) - g_3\\ - k^{-1}[(p_n p_{n+1} + \tfrac{1}{4}\,g_2)^2 + g_3(p_n + p_{n+1})]\end{aligned} \tag{2.6}$$

which is just (2.5) with k and p_{n+1} interchanged.

For small h ,

(2.7) $$k = p(h) = h^{-2} + O(h^2)$$

so that

(2.8) $$(p_{n+1} - p_n)k^{\frac{1}{2}} = p'(z) + O(h)$$

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and to $O(h)$, (2.6) becomes

$$p'(z)^2 = 4p(z)^3 - g_2 p(z) - g_3$$

as required.

It is interesting to note that in the difference approximation to the first derivative, the denominator used is not h but $[p(h)]^{-1/2}$ which is $O(h)$. The replacements

$$p(z)^3 \to p_n p_{n+1} \tfrac{1}{2}(p_n + p_{n+1}) \tag{2.9}$$

$$p(z) \to \tfrac{1}{2}(p_n + p_{n+1}) \tag{2.10}$$

are not unexpected, being similar to the results previously obtained with the Duffing equation [2].

While (2.6) is valid for any h , the simpler result

$$(p_{n+1} - p_n)^2/h^2 = 4p_n p_{n+1} \tfrac{1}{2}(p_n + p_{n+1}) - g_2 \tfrac{1}{2}(p_n + p_{n+1}) - g_3 \tag{2.11}$$

is an approximation to $O(h^2)$ to the DE (1.7), while

$$\begin{aligned}(p_{n+1} - p_n)^2/h^2 = {} & 4p_n p_{n+1} \tfrac{1}{2}(p_n + p_{n+1}) - g_2 \tfrac{1}{2}(p_n + p_{n+1}) - g_3 \\ & - h^2[(p_n p_{n+1} + \tfrac{1}{4} g_2)^2 + g_3(p_n + p_{n+1})]\end{aligned} \tag{2.12}$$

is an approximation to $O(h^4)$.

3. Second Order Difference Equation.

The best ΔE approximation to the second order DE (1.8) can be derived by differencing (2.6), in which, for convenience, n is replaced by $n - 1$. From the simple identities

$$\Delta(p_n - p_{n-1})^2 = (p_{n+1} - p_{n-1})(p_{n+1} - 2p_n + p_{n-1}) \tag{3.1}$$

$$\Delta[p_{n-1}p_n(p_{n-1} + p_n)] = (p_{n+1} - p_{n-1})p_n(p_{n+1} + p_n + p_{n-1}) \tag{3.2}$$

$$\Delta(p_{n-1} + p_n) = p_{n+1} - p_{n-1} \tag{3.3}$$

$$\Delta(p_{n-1}p_n + \tfrac{1}{4} g_2)^2 = (p_{n+1} - p_{n-1})[p_n^2(p_{n+1} + p_{n-1}) + \tfrac{1}{2} g_2 p_n] \tag{3.4}$$

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follows the required best second order ΔE:

$$(p_{n+1} - 2p_n + p_{n-1})\,k = 2p_n(p_{n+1} + p_n + p_{n-1}) - \tfrac{1}{2}\,g_2$$

$$- k^{-1}[p_n^2(p_{n+1} + p_{n-1}) + \tfrac{1}{2}\,g_2 p_n + g_3] \;. \tag{3.5}$$

For small h, this is seen to be an approximation to the DE (1.8) since (2.7) gives

$$(p_{n+1} - 2p_n + p_{n-1})\,k = p''(z) + O(h) \tag{3.6}$$

and

$$2p_n(p_{n+1} + p_n + p_{n-1}) = 6\,p'(z)^2 + O(h) \;. \tag{3.7}$$

While (3.5) is valid for any h, the simpler ΔE

$$(p_{n+1} - 2p_n + p_{n-1})/h^2 = 2p_n(p_{n+1} + p_n + p_{n-1}) - \tfrac{1}{2}\,g_2 \tag{3.8}$$

is an approximation to $O(h^2)$ to the DE (1.8), while

$$(p_{n+1} - 2p_n + p_{n-1})/h^2 = 2p_n(p_{n+1} + p_n + p_{n-1}) - \tfrac{1}{2}\,g_2$$

$$- h^2[p_n^2(p_{n+1} + p_{n-1}) + \tfrac{1}{2}\,g_2 p_n + g_3] \tag{3.9}$$

is an approximation to $O(h^4)$.

4. Numerical Results.

Although the main purpose of this paper is not the numerical analysis of nonlinear DE's it is interesting to illustrate the above theory with some numerical results.

The example taken is the second order DE (1.8) and its approximating ΔE's (1.10), (3.8), (3.9) which can be written respectively as

$$p_{n+1} = 2p_n - p_{n-1} + h^2(6p_n^2 - \tfrac{1}{2}\,g_2) \tag{4.1}$$

$$p_{n+1} = [2p_n - p_{n-1} + h^2(2p_n^2 + 2p_n p_{n-1} - \tfrac{1}{2}\,g_2)]/(1 - 2h^2 p_n)$$

$$p_{n+1} = [2p_n - p_{n-1} + h^2(2p_n^2 + 2p_n p_{n-1} - \tfrac{1}{2}\,g_2)$$

$$- h^4(p_n^2 p_{n-1} + \tfrac{1}{2}\,g_2 p_n + g_3)]/(1 - h^2 p_n)^2 \;. \tag{4.3}$$

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With chosen values of

$$g_2 = 8.124218, \; g_3 = 4.443052, \; h = 0.05$$

and exact values for $p(0.50)$ and $p(0.55)$, the results after 4 and 9 iterations of the ΔE's gave the following:

z	exact	(4.1)	(4.2)	(4.3)
0.50	4.1124			
0.55	3.4449			
0.75	2.0684	2.051	2.074	2.0684
1.00	1.6451	1.566	1.671	1.6451

The ΔE's prove to be a simple and convenient method for generating values of $p(z)$.

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