

HYPERGEOMETRIC RATIONAL APPROXIMATIONS TO $\zeta(4)$

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(Received 15 June 2019; first published online 3 February 2020)

Abstract We give a new hypergeometric construction of rational approximations to $\zeta(4)$, which absorbs the earlier one from 2003 based on Bailey’s ${}_9F_8$ hypergeometric integrals. With the novel ingredients we are able to gain better control of the arithmetic and produce a record irrationality measure for $\zeta(4)$.

Keywords: irrationality measure; π^4 ; hypergeometric function; hypergeometric integral; rational approximation

2010 *Mathematics subject classification:* Primary 11J82
Secondary 11Y60; 33C20; 33C60

1. Introduction

Apéry’s proof [1, 6, 18] of the irrationality of $\zeta(3)$ in the 1970s sparked research in arithmetic on the values of Riemann’s zeta function $\zeta(s)$ at integers $s \geq 2$. Some particular representatives of this development include [4, 8, 9, 13], and the story culminated in a remarkable arithmetic method [14, 15] of Rhin and Viola to produce sharp irrationality measures for $\zeta(2)$ and $\zeta(3)$ using groups of transformations of rational approximations to the quantities. In spite of hopes to (promptly) extend Apéry’s success to $\zeta(5)$ and other zeta values, the next achievement in this direction [3, 16] materialized only in the 2000s in the work of Ball and Rivoal. The latter result helped to unify different-looking approaches for arithmetic investigations of zeta values $\zeta(s)$ and related constants under a ‘hypergeometric’ umbrella, with some particular highlights given in [19, 20] by one of these authors. The hypergeometric machinery has proven to be useful in further arithmetic applications; see, for example, [7, 11, 12, 22] for more recent achievements.

The quantity $\zeta(4)$, though known to be irrational and even transcendental, remains a natural target for testing the hypergeometry. Apéry-type approximations to the number were discovered and rediscovered on several occasions [5, 17, 19], but they were not good enough to draw conclusions about its irrationality. In [19], a general construction of rational approximations to $\zeta(4)$ is proposed, which makes use of very-well-poised

hypergeometric integrals and a group of their transformations; this leads to an estimate for the irrationality exponent of the number in question provided that a certain ‘denominator conjecture’ for the rational approximations is valid. The conjecture appears to be difficult enough, with its only special case established in [10] but insufficient for arithmetic applications. This case is usually dubbed ‘most symmetric’, because the group of transformations acts trivially on the corresponding approximations.

The principal goal of this work is to recast the rational approximations to $\zeta(4)$ from [19] in a different (but still hypergeometric) form and obtain, by these means, better control of the arithmetic of their coefficients. In this way, we are able to produce the estimate

$$\mu(\zeta(4)) \leq 12.51085940\dots$$

for the irrationality exponent of the zeta value, which is better than the conjectural one given in [19]. This is not surprising, as we do not attempt to prove the denominator conjecture from [19] but instead investigate the arithmetic of approximations from a different hypergeometric family.

The plan of our exposition below is as follows. In §2 we give a Barnes-type double integral for rational approximations to $\zeta(4)$ and then, in §3, work out the particular ‘most symmetric’ case of this integral, which clearly illustrates arithmetic features of the new representation of the approximations. We recall general settings from [19] in §4 and embed the approximations into a 12-parametric family of hypergeometric-type sums that are further discussed in greater detail in §5. Furthermore, §6 reviews (and recovers) the permutation group related to the linear forms in 1 and $\zeta(4)$ from a special subfamily of the approximations constructed. Finally, we investigate arithmetic aspects of the general rational approximations in §7 and produce a calculation that leads to the new bound for $\mu(\zeta(4))$ in §8.

In the text below, we intentionally avoid producing claims (in the form of propositions and lemmas), to give our exposition the nature of storytelling rather than traditional mathematical writing.

2. Integral representations

For $k \geq 2$ even, fix a *generic* set of complex parameters

$$\mathbf{h} = (h_0, h_{-1}; h_1, h_2, \dots, h_k)$$

satisfying the conditions

$$\max\{0, \operatorname{Re}(h_0 - h_{-1})\} < \operatorname{Re} h_j < \frac{1}{2} \operatorname{Re} h_0 \quad \text{for } j = 1, \dots, k,$$

and define as in [21] the very-well-poised hypergeometric integrals

$$\begin{aligned} F'_k(\mathbf{h}) &= F'_k(h_0, h_{-1}; h_1, h_2, \dots, h_k) \\ &= \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} (h_0 + 2t) \frac{\prod_{j=-1}^k \Gamma(h_j + t) \cdot \Gamma(h_{-1} - h_0 - t) \Gamma(-t)}{\prod_{j=1}^k \Gamma(1 + h_0 - h_j + t)} dt. \end{aligned}$$

By Bailey’s integral analogue of Dougall’s theorem [2, § 6.6],

$$F'_2(h_0, h_{-1}; h_1, h_2) = \frac{\Gamma(h_{-1})\Gamma(h_1)\Gamma(h_2)\Gamma(h_1 + h_{-1} - h_0)\Gamma(h_2 + h_{-1} - h_0)}{\Gamma(1 + h_0 - h_1 - h_2)\Gamma(h_1 + h_2 + h_{-1} - h_0)}.$$

Substituting this into the iteration

$$\begin{aligned} &F'_{k+2}(h_0, h_{-1}; h_1, \dots, h_{k-1}, h_k, h_{k+1}, h_{k+2}) \\ &= \frac{1}{\Gamma(1 + h_0 - h_k - h_{k+1})\Gamma(1 + h_0 - h_k - h_{k+2})\Gamma(1 + h_0 - h_{k+1} - h_{k+2})} \\ &\quad \times \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \Gamma(h_k + s)\Gamma(h_{k+1} + s)\Gamma(h_{k+2} + s) \\ &\quad \times \Gamma(1 + h_0 - h_k - h_{k+1} - h_{k+2} - s) \cdot F'_k(h_0, h_{-1}; -s, h_1, \dots, h_{k-1}) \, ds \end{aligned}$$

obtained in [21, § 3] (which is itself a corollary of Barnes’s second lemma [2, § 6.2]), we deduce that

$$\begin{aligned} &F'_4(h_0, h_{-1}; h_1, h_2, h_3, h_4) \\ &= \frac{\Gamma(h_{-1})\Gamma(h_1)\Gamma(h_1 + h_{-1} - h_0)}{\Gamma(1 + h_0 - h_2 - h_3)\Gamma(1 + h_0 - h_2 - h_4)\Gamma(1 + h_0 - h_3 - h_4)} \\ &\quad \times \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \Gamma(h_2 + s)\Gamma(h_3 + s)\Gamma(h_4 + s)\Gamma(1 + h_0 - h_2 - h_3 - h_4 - s) \\ &\quad \times \frac{\Gamma(h_{-1} - h_0 - s)\Gamma(-s)}{\Gamma(1 + h_0 - h_1 + s)\Gamma(h_1 + h_{-1} - h_0 - s)} \, ds \end{aligned}$$

and

$$\begin{aligned} &F'_6(h_0, h_{-1}; h_1, h_2, h_3, h_4, h_5, h_6) \\ &= \frac{1}{\Gamma(1 + h_0 - h_4 - h_5)\Gamma(1 + h_0 - h_4 - h_6)\Gamma(1 + h_0 - h_5 - h_6)} \\ &\quad \times \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \Gamma(h_4 + t)\Gamma(h_5 + t)\Gamma(h_6 + t) \\ &\quad \times \Gamma(1 + h_0 - h_4 - h_5 - h_6 - t) \cdot F'_4(h_0, h_{-1}; -t, h_1, h_2, h_3) \, dt \\ &= \frac{1}{\Gamma(1 + h_0 - h_4 - h_5)\Gamma(1 + h_0 - h_4 - h_6)\Gamma(1 + h_0 - h_5 - h_6)} \\ &\quad \times \frac{\Gamma(h_{-1})}{\Gamma(1 + h_0 - h_1 - h_2)\Gamma(1 + h_0 - h_1 - h_3)\Gamma(1 + h_0 - h_2 - h_3)} \\ &\quad \times \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \Gamma(h_4 + t)\Gamma(h_5 + t)\Gamma(h_6 + t)\Gamma(1 + h_0 - h_4 - h_5 - h_6 - t) \end{aligned}$$

$$\begin{aligned} &\times \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \Gamma(h_1 + s)\Gamma(h_2 + s)\Gamma(h_3 + s)\Gamma(1 + h_0 - h_1 - h_2 - h_3 - s) \\ &\times \frac{\Gamma(h_{-1} - h_0 - t)\Gamma(-t)\Gamma(h_{-1} - h_0 - s)\Gamma(-s)}{\Gamma(1 + h_0 + s + t)\Gamma(h_{-1} - h_0 - s - t)} ds dt. \end{aligned}$$

Furthermore, if

$$h_{-1} - h_0 \in \mathbb{Z}, \quad h_0 - h_1 - h_2 - h_3 \in \mathbb{Z} \quad \text{and} \quad h_0 - h_4 - h_5 - h_6 \in \mathbb{Z},$$

the latter can be given as

$$\begin{aligned} &F'_6(h_0, h_{-1}; h_1, h_2, h_3, h_4, h_5, h_6) \\ &= \frac{(-1)^{(h_{-1}-h_0)+(h_0-h_1-h_2-h_3)+(h_0-h_4-h_5-h_6)}\Gamma(h_{-1})}{\Gamma(1 + h_0 - h_1 - h_2)\Gamma(1 + h_0 - h_1 - h_3)\Gamma(1 + h_0 - h_2 - h_3)} \\ &\quad \times \frac{1}{\Gamma(1 + h_0 - h_4 - h_5)\Gamma(1 + h_0 - h_4 - h_6)\Gamma(1 + h_0 - h_5 - h_6)} \\ &\quad \times \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \frac{\Gamma(h_4 + t)\Gamma(h_5 + t)\Gamma(h_6 + t)}{\Gamma(1 + t)\Gamma(1 + h_0 - h_{-1} + t)\Gamma(h_4 + h_5 + h_6 - h_0 + t)} \left(\frac{\pi}{\sin \pi t}\right)^3 \\ &\quad \times \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \frac{\Gamma(h_1 + s)\Gamma(h_2 + s)\Gamma(h_3 + s)}{\Gamma(1 + s)\Gamma(1 + h_0 - h_{-1} + s)\Gamma(h_1 + h_2 + h_3 - h_0 + s)} \left(\frac{\pi}{\sin \pi s}\right)^3 \\ &\quad \times \frac{\Gamma(1 + h_0 - h_{-1} + s + t)}{\Gamma(1 + h_0 + s + t)} \frac{\sin \pi(s + t)}{\pi} ds dt. \end{aligned} \tag{1}$$

3. The most symmetric case

Equation (1) has an interesting structure. For example, in the most symmetric case it implies

$$\begin{aligned} F_6^{\text{sym}}(n) &= F'_6(3n + 2, 3n + 2; n + 1, \dots, n + 1) = \frac{1}{2\pi i} \int_{c_1 - i\infty}^{c_1 + i\infty} \left(\frac{(t + 1)_n}{n!}\right)^3 \left(\frac{\pi}{\sin \pi t}\right)^3 \\ &\quad \times \frac{1}{2\pi i} \int_{c_2 - i\infty}^{c_2 + i\infty} \left(\frac{(s + 1)_n}{n!}\right)^3 \left(\frac{\pi}{\sin \pi s}\right)^3 \frac{(3n + 1)!}{(s + t + 1)_{3n+2}} \frac{\sin \pi(s + t)}{\pi} ds dt. \end{aligned}$$

Notice that the function

$$\frac{(3n + 1)!}{(s + t + 1)_{3n+2}} \frac{\sin \pi(s + t)}{\pi}$$

is entire in both its variables, while the poles of

$$\left(\frac{(s + 1)_n}{n!}\right)^3 \left(\frac{\pi}{\sin \pi s}\right)^3$$

in a right half-plane are at $s = 0, 1, 2, \dots$, and the latter function is analytic in the strip $-(n + 1) < \text{Re } s < 0$. There is a similar structure for

$$\left(\frac{(t + 1)_n}{n!}\right)^3 \left(\frac{\pi}{\sin \pi t}\right)^3.$$

This implies that one can take $c_1, c_2 \in \mathbb{R}$ to be any in the range $-(n + 1) < c_1, c_2 < 0$; we choose $c_1 = c_2 = c - n$ with $c = -1/3$ for our discussion below.

Now write

$$\sin \pi(s + t) = \sin \pi s \cos \pi t + \cos \pi s \sin \pi t,$$

so that the integral is split into the integration

$$\begin{aligned} \frac{1}{2} F_6^{\text{sym}}(n) &= \frac{1}{2\pi i} \int_{c-n-i\infty}^{c-n+i\infty} \left(\frac{(t+1)_n}{n!}\right)^3 \left(\frac{\pi}{\sin \pi t}\right)^3 \cos \pi t \\ &\quad \times \frac{1}{2\pi i} \int_{c-n-i\infty}^{c-n+i\infty} \left(\frac{(s+1)_n}{n!}\right)^3 \frac{(3n+1)!}{(s+t+1)_{3n+2}} \left(\frac{\pi}{\sin \pi s}\right)^2 ds dt \end{aligned} \tag{2}$$

(twice, because of the symmetry $s \leftrightarrow t$).

We first deal with the internal integral in (2). The rational integrand is decomposed into the sum of partial fractions:

$$\left(\frac{(s+1)_n}{n!}\right)^3 \frac{(3n+1)!}{(s+t+1)_{3n+2}} = \sum_{k=1}^{3n+2} \frac{A_k(t)}{s+t+k},$$

where $A_k(t) = (-1)^{k-1} \binom{3n+1}{k-1} \left(\frac{(-t-k+1)_n}{n!}\right)^3$ for $k = 1, 2, \dots, 3n+2$.

Then

$$\begin{aligned} H_n(t) &= \frac{1}{2\pi i} \int_{c-n-i\infty}^{c-n+i\infty} \left(\frac{(s+1)_n}{n!}\right)^3 \frac{(3n+1)!}{(s+t+1)_{3n+2}} \left(\frac{\pi}{\sin \pi s}\right)^2 ds \\ &= - \sum_{\nu=-n}^{\infty} \frac{\partial}{\partial s} \left(\left(\frac{(s+1)_n}{n!}\right)^3 \frac{(3n+1)!}{(s+t+1)_{3n+2}} \right) \Big|_{s=\nu} \end{aligned}$$

(take ν_0 any from the interval $-n \leq \nu_0 \leq 0$)

$$\begin{aligned} &= - \sum_{\nu=\nu_0}^{\infty} \frac{\partial}{\partial s} \left(\left(\frac{(s+1)_n}{n!}\right)^3 \frac{(3n+1)!}{(s+t+1)_{3n+2}} \right) \Big|_{s=\nu} \\ &= \sum_{\nu=\nu_0}^{\infty} \sum_{k=1}^{3n+2} \frac{A_k(t)}{(\nu+t+k)^2} = \sum_{k=1}^{3n+2} A_k(t) \sum_{\nu=\nu_0}^{\infty} \frac{1}{(\nu+t+k)^2} \\ &= \sum_{k=1}^{3n+2} A_k(t) \left(\sum_{l=1}^{\infty} - \sum_{l=1}^{k-1} \right) \frac{1}{(t+l+\nu_0)^2} = - \sum_{k=1}^{3n+2} A_k(t) \sum_{l=1}^{k-1} \frac{1}{(t+l+\nu_0)^2}, \end{aligned}$$

because

$$\sum_{k=1}^{3n+2} A_k(t) = 0$$

by the residue sum theorem. The choices $\nu_0 = 0$ and $\nu_0 = -n$ lead to the equality

$$-\sum_{k=1}^{3n+2} A_k(t) \sum_{l=1}^{k-1} \frac{1}{(t+l)^2} = H_n(t) = -\sum_{k=1}^{3n+2} A_k(t) \sum_{l=1}^{k-1} \frac{1}{(t+l-n)^2}; \tag{3}$$

since $A_k(t)$ are polynomials, the two representations imply that the only poles of $H_n(t)$ are located at the integers

$$\begin{aligned} & \{-1, -2, \dots, -3n, -(3n+1)\} \cap \{-(2n+1), -2n, \dots, n-2, n-1\} \\ &= \{-1, -2, \dots, -2n, -(2n+1)\}. \end{aligned}$$

Furthermore, the function

$$\tilde{H}_n(t) = \left(\frac{(t+1)_n}{n!}\right)^3 H_n(t)$$

has poles only at $t = -(n+1), -(n+2), \dots, -(2n+1)$ and vanishes at $t = -1, -2, \dots, -n$. Moreover, $\tilde{H}_n(t)$ is in fact a rational function of degree at most -2 (so that it has the zero residue at infinity); indeed, it is the sum of rational functions

$$\left(\frac{(t+1)_n}{n!}\right)^3 \frac{\partial}{\partial s} \left(\left(\frac{(s+1)_n}{n!}\right)^3 \frac{(3n+1)!}{(s+t+1)_{3n+2}} \right) \Big|_{s=\nu}, \text{ where } \nu = 0, 1, 2, \dots,$$

each of degree at most -2 (in t). This means that we have a partial-fraction decomposition

$$\tilde{H}_n(t) = \sum_{j=n+1}^{2n+1} \left(\frac{B_j}{(t+j)^2} + \frac{C_j}{t+j} \right)$$

with $\sum_{j=n+1}^{2n+1} C_j = 0$. With the help of the following consequence of formula (3),

$$H_n(t) = -\sum_{l=1}^{3n+1} \frac{1}{(t+l-n)^2} \sum_{k=l+1}^{3n+2} A_k(t) = -\sum_{j=-n+1}^{2n+1} \frac{1}{(t+j)^2} \sum_{k=j+n+1}^{3n+2} A_k(t),$$

we find that

$$\begin{aligned} B_j &= \tilde{H}_n(t)(t+j)^2 \Big|_{t=-j} = -\left(\frac{(-j+1)_n}{n!}\right)^3 \sum_{k=j+n+1}^{3n+2} A_k(-j) \\ &= \left(\frac{(-j+1)_n}{n!}\right)^3 \sum_{k=j+n+1}^{3n+2} (-1)^k \binom{3n+1}{k-1} \left(\frac{(j-k+1)_n}{n!}\right)^3 \end{aligned}$$

and similarly

$$\begin{aligned}
 C_j &= \frac{\partial}{\partial t} (\tilde{H}_n(t)(t+j)^2) \Big|_{t=-j} \\
 &= \frac{\partial}{\partial t} \left(\frac{(-t+1)_n}{n!} \right)^3 \Big|_{t=-j} \cdot \sum_{k=j+n+1}^{3n+2} (-1)^k \binom{3n+1}{k-1} \left(\frac{(j-k+1)_n}{n!} \right)^3 \\
 &\quad + \left(\frac{(-j+1)_n}{n!} \right)^3 \sum_{k=j+n+1}^{3n+2} (-1)^k \binom{3n+1}{k-1} \cdot \frac{\partial}{\partial t} \left(\frac{(-t-k+1)_n}{n!} \right)^3 \Big|_{t=-j}.
 \end{aligned}$$

Note that

$$\frac{(-j+1)_n}{n!} \in \mathbb{Z}, \quad \frac{(j-k+1)_n}{n!} \in \mathbb{Z}$$

and

$$d_n \cdot \frac{\partial}{\partial t} \left(\frac{(-t+1)_n}{n!} \right) \Big|_{t=-j} \in \mathbb{Z}, \quad d_n \cdot \frac{\partial}{\partial t} \left(\frac{(-t-k+1)_n}{n!} \right) \Big|_{t=-j} \in \mathbb{Z}$$

for all $j, k \in \mathbb{Z}$ by the standard arithmetic properties of integer-valued polynomials [22, Lemma 4], where d_n denotes the least common multiple of $1, 2, \dots, n$. Furthermore, each term of the sums for B_j and C_j has a factor of the form

$$\frac{(-j+1)_n}{n!} \binom{3n+1}{k-1} \frac{(j-k+1)_n}{n!} = \binom{3n+1}{k-1} \binom{j-1}{n} \binom{k-j-1}{n},$$

and these quantities are all divisible by the greatest common divisor Φ_n of numbers

$$\binom{3n+1}{a+b+1} \binom{a}{n} \binom{b}{n}, \quad \text{where } a, b \in \mathbb{Z}$$

(there are only finitely many non-zero products on the list). Thus,

$$\Phi_n^{-1} B_j \in \mathbb{Z} \quad \text{and} \quad \Phi_n^{-1} d_n C_j \in \mathbb{Z} \quad \text{for } j = n+1, \dots, 2n+1.$$

Now

$$\frac{1}{2} F_6^{\text{sym}}(n) = \frac{1}{2\pi i} \int_{c-n-i\infty}^{c-n+i\infty} \tilde{H}_n(t) \left(\frac{\pi}{\sin \pi t} \right)^3 \cos \pi t \, dt.$$

Since

$$\left(\frac{\pi}{\sin \pi t} \right)^3 \cos \pi t = \frac{1}{(t-\nu)^3} + O(t-\nu) \quad \text{as } t \rightarrow \nu \in \mathbb{Z},$$

we have

$$\begin{aligned} \frac{1}{2}F_6^{\text{sym}}(n) &= \sum_{\nu=-n}^{\infty} \operatorname{Res}_{t=\nu} \tilde{H}_n(t) \left(\frac{\pi}{\sin \pi t}\right)^3 \cos \pi t = \frac{1}{2} \sum_{\nu=-n}^{\infty} \left. \frac{\partial^2 \tilde{H}_n(t)}{\partial t^2} \right|_{t=\nu} \\ &= \frac{1}{2} \sum_{\nu=-n}^{\infty} \sum_{j=n+1}^{2n+1} \left(\frac{6B_j}{(\nu+j)^4} + \frac{2C_j}{(\nu+j)^3} \right) \\ &= 3 \sum_{j=n+1}^{2n+1} B_j \sum_{\nu=-n}^{\infty} \frac{1}{(\nu+j)^4} + \sum_{j=n+1}^{2n+1} C_j \sum_{\nu=-n}^{\infty} \frac{1}{(\nu+j)^3} \\ &= 3 \sum_{j=n+1}^{2n+1} B_j \cdot \zeta(4) - \left(3 \sum_{j=n+1}^{2n+1} B_j \sum_{l=1}^{j-n-1} \frac{1}{l^4} + \sum_{j=n+1}^{2n+1} C_j \sum_{l=1}^{j-n-1} \frac{1}{l^3} \right). \end{aligned}$$

This implies that

$$\frac{1}{2}\Phi_n^{-1}d_n^4F_6^{\text{sym}}(n) \in \mathbb{Z}\zeta(4) + \mathbb{Z}.$$

In § 7 we reveal details of the computation of Φ_n (and its asymptotics as $n \rightarrow \infty$); we show that Φ_n is divisible by the product over primes

$$\prod_{\substack{p > \sqrt{3n} \\ 2/3 \leq \{n/p\} < 1}} p. \tag{4}$$

This corresponds to the ‘denominator conjecture’ from [19]; for the most symmetric case in this section, it was established earlier in [10] using different hypergeometric techniques.

4. Old approximations to $\zeta(4)$

We now concentrate on a specific setting of § 2, where $k = 6$ and the parameters

$$\mathbf{h} = (h_0, h_{-1}; h_1, h_2, h_3, h_4, h_5, h_6)$$

are positive integers satisfying the conditions

$$h_0 - h_{-1} < h_j < \frac{1}{2}h_0 \quad \text{for } j = 1, 2, 3, 4, 5, 6.$$

Define the rational function

$$\begin{aligned} R(t) = R(\mathbf{h}; t) &= \gamma(\mathbf{h})(h_0 + 2t) \frac{\prod_{j=-1}^6 \Gamma(h_j + t)}{\prod_{j=-1}^6 \Gamma(1 + h_0 - h_j + t)} \\ &= (h_0 + 2t) \frac{(t+1)_{h_1-1}}{(h_1-1)!} \frac{(t+1+h_0-h_2)_{h_2-1}}{(h_2-1)!} \\ &\quad \times \frac{(t+1+h_0-h_5)_{h_5+h_{-1}-h_0-1}}{(h_5+h_{-1}-h_0-1)!} \frac{(t+1+h_0-h_{-1})_{h_6+h_{-1}-h_0-1}}{(h_6+h_{-1}-h_0-1)!} \end{aligned}$$

$$\begin{aligned} &\times \frac{(h_0 - h_2 - h_4)!}{(t + h_2)_{h_0 - h_2 - h_4 + 1}} \frac{(h_0 - h_1 - h_3)!}{(t + h_3)_{h_0 - h_1 - h_3 + 1}} \\ &\times \frac{(h_0 - h_4 - h_6)!}{(t + h_4)_{h_0 - h_4 - h_6 + 1}} \frac{(h_0 - h_3 - h_5)!}{(t + h_5)_{h_0 - h_3 - h_5 + 1}} \end{aligned}$$

with

$$\gamma(\mathbf{h}) = \frac{(h_0 - h_2 - h_4)!(h_0 - h_1 - h_3)!(h_0 - h_4 - h_6)!(h_0 - h_3 - h_5)!}{(h_1 - 1)!(h_2 - 1)!(h_5 + h_{-1} - h_0 - 1)!(h_6 + h_{-1} - h_0 - 1)!}.$$

Then

$$F(\mathbf{h}) = \gamma(\mathbf{h})F'_6(\mathbf{h}) = - \sum_{t=t_0}^{\infty} \frac{d}{dt} R(\mathbf{h}; t) \in \mathbb{Q} + \mathbb{Q}\zeta(4)$$

$$\text{with any } t_0 \in \mathbb{Z}, \quad 1 - \min_{1 \leq j \leq 6} \{h_j\} \leq t_0 \leq 1 - \max\{0, h_0 - h_{-1}\},$$

is essentially the very-well-poised hypergeometric integral given in [19]. Notice, however, that the arithmetic normalization factor $\gamma(\mathbf{h})$ differs slightly from the one used in [19]. Rearranging the order of parameters in (1) we obtain

$$\begin{aligned} F(\mathbf{h}) &= \frac{(-1)^{h_{-1}+h_0+h_1+\dots+h_6} (h_{-1} - 1)! \gamma(\mathbf{h})}{(h_0 - h_1 - h_3)!(h_0 - h_1 - h_5)!(h_0 - h_3 - h_5)!} \\ &\times \frac{1}{(h_0 - h_2 - h_4)!(h_0 - h_2 - h_6)!(h_0 - h_4 - h_6)!} \\ &\times \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \frac{\Gamma(h_2 + t)\Gamma(h_4 + t)\Gamma(h_6 + t)}{\Gamma(1 + t)\Gamma(1 + h_0 - h_{-1} + t)\Gamma(h_2 + h_4 + h_6 - h_0 + t)} \left(\frac{\pi}{\sin \pi t}\right)^3 \\ &\times \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \frac{\Gamma(h_1 + s)\Gamma(h_3 + s)\Gamma(h_5 + s)}{\Gamma(1 + s)\Gamma(1 + h_0 - h_{-1} + s)\Gamma(h_1 + h_3 + h_5 - h_0 + s)} \left(\frac{\pi}{\sin \pi s}\right)^3 \\ &\times \frac{\Gamma(1 + h_0 - h_{-1} + s + t)}{\Gamma(1 + h_0 + s + t)} \frac{\sin \pi(s + t)}{\pi} ds dt \\ &= (-1)^{h_{-1}+\dots+h_6} \\ &\times \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \frac{(t + 1)_{h_2 - 1}}{(h_2 - 1)!} \frac{(t + h_2 + h_4 + h_6 - h_0)_{h_0 - h_2 - h_6}}{(h_0 - h_2 - h_6)!} \\ &\times \frac{(t + 1 + h_0 - h_{-1})_{h_6 + h_{-1} - h_0 - 1}}{(h_6 + h_{-1} - h_0 - 1)!} \left(\frac{\pi}{\sin \pi t}\right)^3 \\ &\times \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \frac{(s + 1)_{h_1 - 1}}{(h_1 - 1)!} \frac{(s + h_1 + h_3 + h_5 - h_0)_{h_0 - h_1 - h_5}}{(h_0 - h_1 - h_5)!} \\ &\times \frac{(s + 1 + h_0 - h_{-1})_{h_5 + h_{-1} - h_0 - 1}}{(h_5 + h_{-1} - h_0 - 1)!} \left(\frac{\pi}{\sin \pi s}\right)^3 \\ &\times \frac{(h_{-1} - 1)!}{(t + s + 1 + h_0 - h_{-1})_{h_{-1}}} \frac{\sin \pi(s + t)}{\pi} ds dt. \tag{5} \end{aligned}$$

The double integral we arrive at belongs to a more general (12-parametric) family, which we discuss in the next section.

5. General approximations to $\zeta(4)$

The integral in (5) is a special case of

$$\begin{aligned}
 G(\mathbf{a}, \mathbf{b}) &= \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \frac{(t+b_2)_{a_2-b_2}}{(a_2-b_2)!} \frac{(t+b_4)_{a_4-b_4}}{(a_4-b_4)!} \frac{(t+b_6)_{a_6-b_6}}{(a_6-b_6)!} \left(\frac{\pi}{\sin \pi t}\right)^3 \\
 &\quad \times \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \frac{(s+b_1)_{a_1-b_1}}{(a_1-b_1)!} \frac{(s+b_3)_{a_3-b_3}}{(a_3-b_3)!} \frac{(s+b_5)_{a_5-b_5}}{(a_5-b_5)!} \left(\frac{\pi}{\sin \pi s}\right)^3 \\
 &\quad \times \frac{(b_0-a_0-1)!}{(t+s+a_0)_{b_0-a_0}} \frac{\sin \pi(s+t)}{\pi} ds dt,
 \end{aligned} \tag{6}$$

where the integral parameters

$$\mathbf{a} = (a_0; a_1, a_2, a_3, a_4, a_5, a_6) \quad \text{and} \quad \mathbf{b} = (b_0; b_1, b_2, b_3, b_4, b_5, b_6) \tag{7}$$

are subject to the conditions

$$\begin{aligned}
 b_0 - a_0 - 2 &\geq (a_1 + a_3 + a_5) - (b_1 + b_3 + b_5), \\
 b_0 - a_0 - 2 &\geq (a_2 + a_4 + a_6) - (b_2 + b_4 + b_6),
 \end{aligned} \tag{8}$$

and

$$\max\{b_1, b_3, b_5\} \leq \min\{a_1, a_3, a_5\}, \quad \max\{b_2, b_4, b_6\} \leq \min\{a_2, a_4, a_6\}.$$

Note that simultaneous shifts of a_0, a_1, a_3, a_5 and b_0, b_1, b_3, b_5 by the same integer do not affect $G(\mathbf{a}, \mathbf{b})$; the same is true for simultaneous shifts of a_0, a_2, a_4, a_6 and b_0, b_2, b_4, b_6 . (In particular, the shifts by given $1 - b_1$ and $1 - b_2$, respectively, allow us to assume that $b_1 = b_2 = 1$.) The latter two symmetries potentially leave 12 out of 14 parameters (7) independent. Furthermore, we choose

$$\mathbf{a}^* = (a_0; a_1^*, a_2^*, a_3^*, a_4^*, a_5^*, a_6^*) \quad \text{and} \quad \mathbf{b}^* = (b_0; b_1^*, b_2^*, b_3^*, b_4^*, b_5^*, b_6^*) \tag{9}$$

to be a reordering of the parameters (7) (so that (9) and (7) coincide as multisets) such that

$$a_1^* \leq a_3^* \leq a_5^*, \quad b_1^* \leq b_3^* \leq b_5^* \quad \text{and} \quad a_2^* \leq a_4^* \leq a_6^*, \quad b_2^* \leq b_4^* \leq b_6^*.$$

Additionally, we assume

$$a_0 + 1 \geq b_3^* + b_4^*. \tag{10}$$

Similar to the most symmetric case in §3, we may choose the integration paths in (6) to be the vertical lines $\{c_1 + iy : y \in \mathbb{R}\}$ for s and $\{c_2 + iy : y \in \mathbb{R}\}$ for t , with

$$-a_1^* < c_1 < 1 - b_5^*, \quad -a_2^* < c_2 < 1 - b_6^*,$$

and we take $c_1 = 1/3 - a_1^*$ and $c_2 = 1/3 - a_2^*$. Also, the rational function in s and t at the integrand in (6) has degree at most -2 both in s and in t , and the functions

$$\frac{1}{\sin \pi s}, \quad \frac{\cos \pi s}{(\sin \pi s)^2} \quad \text{and} \quad \frac{1}{\sin \pi t}$$

are bounded in their respective integration domains. By

$$\sin \pi(s + t) = \sin \pi s \cos \pi t + \cos \pi s \sin \pi t,$$

the integral $G(\mathbf{a}, \mathbf{b})$ is split into two absolutely convergent integrals, and, after interchanging the order of integrations in s and in t in the second integral, we obtain

$$\begin{aligned} G(\mathbf{a}, \mathbf{b}) &= \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \frac{(t + b_2)_{a_2 - b_2}}{(a_2 - b_2)!} \frac{(t + b_4)_{a_4 - b_4}}{(a_4 - b_4)!} \frac{(t + b_6)_{a_6 - b_6}}{(a_6 - b_6)!} \left(\frac{\pi}{\sin \pi t}\right)^3 \cos \pi t \\ &\times \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \frac{(s + b_1)_{a_1 - b_1}}{(a_1 - b_1)!} \frac{(s + b_3)_{a_3 - b_3}}{(a_3 - b_3)!} \frac{(s + b_5)_{a_5 - b_5}}{(a_5 - b_5)!} \left(\frac{\pi}{\sin \pi s}\right)^2 \\ &\times \frac{(b_0 - a_0 - 1)!}{(t + s + a_0)_{b_0 - a_0}} \, ds \, dt \\ &+ \text{a similar integral with } a_j, b_j \text{ changed to } a_{7-j}, b_{7-j} \text{ for } j = 1, \dots, 6. \end{aligned} \tag{11}$$

As already seen in the most symmetric case, the integral

$$\begin{aligned} H(t) &= H(a_0, a_1, a_3, a_5; b_0, b_1, b_3, b_5; t) \\ &= \frac{1}{2\pi i} \int_{c_1 - i\infty}^{c_1 + i\infty} \frac{(s + b_1)_{a_1 - b_1}}{(a_1 - b_1)!} \frac{(s + b_3)_{a_3 - b_3}}{(a_3 - b_3)!} \frac{(s + b_5)_{a_5 - b_5}}{(a_5 - b_5)!} \\ &\times \frac{(b_0 - a_0 - 1)!}{(t + s + a_0)_{b_0 - a_0}} \left(\frac{\pi}{\sin \pi s}\right)^2 \, ds \end{aligned} \tag{12}$$

is a rational function in t , and we may even vary c_1 in the interval $-a_3^* < c_1 < 1 - b_3^*$, because a power of $\sin \pi s$ is dropped in the denominator of (12) with respect to the integral (6). In executing this, we do not have to take care of possible poles coming from $(t + s + a_0)_{b_0 - a_0}$, because it never vanishes if t is chosen in an appropriate region of the complex plane, and two rational functions that coincide in such a region must coincide everywhere.

Explicitly, we have

$$\frac{(s + b_1)_{a_1 - b_1}}{(a_1 - b_1)!} \frac{(s + b_3)_{a_3 - b_3}}{(a_3 - b_3)!} \frac{(s + b_5)_{a_5 - b_5}}{(a_5 - b_5)!} \frac{(b_0 - a_0 - 1)!}{(t + s + a_0)_{b_0 - a_0}} = \sum_{k=a_0}^{b_0 - 1} \frac{A_k(t)}{t + s + k},$$

where

$$\begin{aligned} A_k(t) &= (-1)^{k+a_0} \binom{b_0 - a_0 - 1}{k - a_0} \frac{(-t - k + b_1)_{a_1 - b_1}}{(a_1 - b_1)!} \\ &\times \frac{(-t - k + b_3)_{a_3 - b_3}}{(a_3 - b_3)!} \frac{(-t - k + b_5)_{a_5 - b_5}}{(a_5 - b_5)!} \quad \text{for } k = a_0, \dots, b_0 - 1 \end{aligned} \tag{13}$$

satisfy $\sum_{k=a_0}^{b_0-1} A_k(t) = 0$. Then

$$\begin{aligned}
 H(t) &= - \sum_{\nu=\nu_0}^{\infty} \frac{\partial}{\partial s} \sum_{k=a_0}^{b_0-1} \frac{A_k(t)}{t+s+k} \Big|_{s=\nu} = \sum_{k=a_0}^{b_0-1} A_k(t) \sum_{\nu=\nu_0}^{\infty} \frac{1}{(\nu+t+k)^2} \\
 &= - \sum_{k=a_0}^{b_0-1} A_k(t) \sum_{l=a_0}^{k-1} \frac{1}{(t+l+\nu_0)^2},
 \end{aligned}
 \tag{14}$$

where ν_0 is any integer in the interval $1 - a_3^* \leq \nu_0 \leq 1 - b_3^*$. Since all $A_k(t)$ are polynomials, the poles of function (14) are only possible at

$$t = a_3^* - b_0 + 1, a_3^* - b_0 + 2, \dots, b_3^* - a_0 - 1.$$

For a similar reason, with ν_0 in the larger interval $1 - a_5^* \leq \nu_0 \leq 1 - b_1^*$, the function

$$I(t) = \sum_{\nu=\nu_0}^{\infty} \sum_{k=a_0}^{b_0-1} \frac{A_k(t)}{t+s+k} \Big|_{s=\nu} = - \sum_{k=a_0}^{b_0-1} A_k(t) \sum_{l=a_0}^{k-1} \frac{1}{t+l+\nu_0}$$

has poles possible only at

$$t = a_5^* - b_0 + 1, a_5^* - b_0 + 2, \dots, b_1^* - a_0 - 1.$$

Since

$$H(t)(t+l+\nu_0)^2 \Big|_{t=-l-\nu_0} = \sum_{k=l+1}^{b_0} A_k(-l-\nu_0) = I(t)(t+l+\nu_0) \Big|_{t=-l-\nu_0}$$

when $1 - a_3^* \leq \nu_0 \leq 1 - b_3^*$, it follows that the set of double poles of $H(t)$ coincides with the set of simple poles of $I(t)$ and therefore is also contained at integers in $[a_5^* - b_0 + 1, b_1^* - a_0 - 1]$; however, $H(t)$ may still possess simple poles at integers in $[a_3^* - b_0 + 1, b_3^* - a_0 - 1]$. Arguing as in § 3, we arrive at the partial-fraction decomposition

$$\begin{aligned}
 \tilde{H}(t) = \tilde{H}(\mathbf{a}, \mathbf{b}; t) &= \frac{(t+b_2)_{a_2-b_2}}{(a_2-b_2)!} \frac{(t+b_4)_{a_4-b_4}}{(a_4-b_4)!} \frac{(t+b_6)_{a_6-b_6}}{(a_6-b_6)!} H(t) \\
 &= \sum_{j=1+a_0-b_1^*}^{b_0-a_5^*-1} \frac{B_j}{(t+j)^2} + \sum_{j=1+a_0-b_3^*}^{b_0-a_3^*-1} \frac{C_j}{t+j},
 \end{aligned}
 \tag{15}$$

because the rational function $\tilde{H}(t)$ has degree at most -2 by (8). Noticing that the expression

$$\frac{(t+b_2)_{a_2-b_2}}{(a_2-b_2)!} \frac{(t+b_4)_{a_4-b_4}}{(a_4-b_4)!} \frac{(t+b_6)_{a_6-b_6}}{(a_6-b_6)!}$$

has at least simple zeroes at $t = 1 - a_6^*, \dots, -b_2^*$ and at least double zeroes at $t = 1 - a_4^*, 2 - a_4^*, \dots, -b_4^*$, and taking into account condition (10), we find that $\tilde{H}(t)$ does not have

poles in the half-plane $\text{Re } t > c_2$; hence, the expansion (15) ‘shortens’ to

$$\tilde{H}(t) = \sum_{j=a_2^*}^{b_0-a_5^*-1} \frac{B_j}{(t+j)^2} + \sum_{j=a_2^*}^{b_0-a_3^*-1} \frac{C_j}{t+j}.$$

In fact, the second sum is over the interval $\max\{a_4^*, 1 + a_0 - b_3^*\} \leq j \leq b_0 - a_3^* - 1$, while the first one is over $\max\{a_6^*, 1 + a_0 - b_1^*\} \leq j \leq b_0 - a_5^* - 1$ and may be even empty if the interval is empty. With the explicit expressions (13) and (14) (used, for example, with $\nu_0 = 1 - a_3^*$) in mind, we conclude that the coefficients

$$B_j = \tilde{H}(t)(t+j)^2 \quad \text{and} \quad C_j = \left. \frac{\partial}{\partial t}(\tilde{H}(t)(t+j)^2) \right|_{t=-j} \quad \text{for } j \in \mathbb{Z}$$

satisfy

$$B_j \in \mathbb{Z}, \quad d_m C_j \in \mathbb{Z}$$

with $m = \max\{a_1 - b_1, a_2 - b_2, a_3 - b_3, a_4 - b_4, a_5 - b_5, a_6 - b_6\}$,

but also

$$\begin{aligned} \text{ord}_p B_j, \text{ord}_p(d_m C_j) \geq & \min_{j,k \in \mathbb{Z}} \left(\left\lfloor \frac{b_0 - a_0 - 1}{p} \right\rfloor - \left\lfloor \frac{k - a_0}{p} \right\rfloor - \left\lfloor \frac{b_0 - k - 1}{p} \right\rfloor \right) \\ & + \sum_{r \in \{2,4,6\}} \left(\left\lfloor \frac{j - b_r}{p} \right\rfloor - \left\lfloor \frac{j - a_r}{p} \right\rfloor - \left\lfloor \frac{a_r - b_r}{p} \right\rfloor \right) \\ & + \sum_{r \in \{1,3,5\}} \left(\left\lfloor \frac{k - j - b_r}{p} \right\rfloor - \left\lfloor \frac{k - j - a_r}{p} \right\rfloor - \left\lfloor \frac{a_r - b_r}{p} \right\rfloor \right) \end{aligned}$$

for primes $p > \sqrt{b_0 - a_0}$ (see [20, Lemmas 17, 18]). Furthermore,

$$\begin{aligned} & \frac{1}{2\pi i} \int_{c_2-i\infty}^{c_2+i\infty} \tilde{H}(t) \left(\frac{\pi}{\sin \pi t} \right)^3 \cos \pi t \, dt \\ &= \sum_{\nu=1-a_2^*}^{\infty} \left(\sum_{j=a_2^*}^{b_0-a_5^*-1} \frac{3B_j}{(\nu+j)^4} + \sum_{j=a_2^*}^{b_0-a_3^*-1} \frac{C_j}{(\nu+j)^3} \right) \\ &= 3\zeta(4) \sum_{j=\max\{a_6^*, 1+a_0-b_1^*\}}^{b_0-a_5^*-1} B_j - \left(3 \sum_{j=a_2^*}^{b_0-a_5^*-1} B_j \sum_{l=1}^{j-a_2^*} \frac{1}{l^4} + \sum_{j=a_2^*}^{b_0-a_3^*-1} C_j \sum_{l=1}^{j-a_2^*} \frac{1}{l^3} \right), \end{aligned}$$

where $\sum_j C_j = 0$ is implemented. Proceeding in the same way for the second double integral in (11), we conclude that

$$G(\mathbf{a}, \mathbf{b}) = B(\mathbf{a}, \mathbf{b})\zeta(4) - C(\mathbf{a}, \mathbf{b}), \quad \text{where } B \in \mathbb{Z}, \quad d_{m_1}^3 d_{m_2} C \in \mathbb{Z} \tag{16}$$

with $m_1 = \max\{b_0 - a_2^* - a_3^* - 1, b_0 - a_1^* - a_4^* - 1\}$,

$m_2 = \max\{b_0 - a_2^* - a_5^* - 1, b_0 - a_1^* - a_6^* - 1, a_1 - b_1, \dots, a_6 - b_6\}$,

and

$$\begin{aligned} \text{ord}_p B, 4 + \text{ord}_p C \geq & \min_{j,l \in \mathbb{Z}} \left(\left\lfloor \frac{b_0 - a_0 - 1}{p} \right\rfloor - \left\lfloor \frac{j + l - a_0}{p} \right\rfloor - \left\lfloor \frac{b_0 - j - l - 1}{p} \right\rfloor \right) \\ & + \sum_{r \in \{2,4,6\}} \left(\left\lfloor \frac{j - b_r}{p} \right\rfloor - \left\lfloor \frac{j - a_r}{p} \right\rfloor - \left\lfloor \frac{a_r - b_r}{p} \right\rfloor \right) \\ & + \sum_{r \in \{1,3,5\}} \left(\left\lfloor \frac{l - b_r}{p} \right\rfloor - \left\lfloor \frac{l - a_r}{p} \right\rfloor - \left\lfloor \frac{a_r - b_r}{p} \right\rfloor \right) \end{aligned} \tag{17}$$

for primes $p > \sqrt{b_0 - a_0 - 2}$.

Finally, we remark that condition (10) is conventional (and happens to hold in our applications, even in the form of equality $b_3^* + b_4^* = a_0 - 1$) but can potentially be dropped without significant arithmetic losses. For example, if $b_1^* + b_2^* > a_0 - 1$ then the partial-fraction decomposition (15) translates into

$$\tilde{H}(t) = \sum_{j=1+a_0-b_1^*}^{b_2^*-1} \frac{B_j}{(t+j)^2} + \sum_{j=a_6^*}^{b_0-a_5^*-1} \frac{B_j}{(t+j)^2} + \sum_{j=1+a_0-b_3^*}^{b_4^*-1} \frac{C_j}{t+j} + \sum_{j=a_4^*}^{b_0-a_3^*-1} \frac{C_j}{t+j},$$

so that there are poles of $\tilde{H}(t)$ to the right of the contour $\text{Re } t = c_2$. The corresponding residues of the integrand are

$$\begin{aligned} \text{Res}_{t=-j} \tilde{H}(t) \left(\frac{\pi}{\sin \pi t} \right)^3 \cos \pi t &= D_j - 6\zeta(4) B_j \\ \text{with } D_j &= \frac{1}{24} \frac{\partial^4}{\partial t^4} (\tilde{H}(t)(t+j)^2) \Big|_{t=-j} = \frac{1}{2} \frac{\partial^2}{\partial t^2} \left(\tilde{H}(t) - \frac{B_j}{(t+j)^2} - \frac{C_j}{t+j} \right) \Big|_{t=-j}, \end{aligned}$$

where j is an integer in the interval $1 + a_0 - b_3^* \leq j \leq b_4^* - 1$ and we use the expansion

$$\left(\frac{\pi}{\sin \pi t} \right)^3 \cos \pi t = \frac{1}{(t+j)^3} - 6\zeta(4)(t+j) + O((t+j)^3) \quad \text{as } t \rightarrow -j.$$

Proceeding as above we deduce that

$$\begin{aligned} \frac{1}{2\pi i} \int_{c_2-i\infty}^{c_2+i\infty} \tilde{H}(t) \left(\frac{\pi}{\sin \pi t} \right)^3 \cos \pi t dt &= \sum_{\nu=1-a_2^*}^{\infty} \text{Res}_{t=\nu} \tilde{H}(t) \left(\frac{\pi}{\sin \pi t} \right)^3 \cos \pi t \\ &= -6\zeta(4) \sum_{j=1+a_0-b_1^*}^{b_2^*-1} B_j + 3 \sum_{j=1+a_0-b_1^*}^{b_2^*-1} B_j \sum_{\substack{l=j+1-a_2^* \\ l \neq 0}}^{\infty} \frac{1}{l^4} + 3 \sum_{j=a_6^*}^{b_0-a_5^*-1} B_j \sum_{l=j+1-a_2^*}^{\infty} \frac{1}{l^4} \\ &+ \sum_{j=1+a_0-b_3^*}^{b_4^*-1} C_j \sum_{\substack{l=j+1-a_2^* \\ l \neq 0}}^{\infty} \frac{1}{l^3} + \sum_{j=a_4^*}^{b_0-a_3^*-1} C_j \sum_{l=j+1-a_2^*}^{\infty} \frac{1}{l^3}, \end{aligned}$$

which again can be seen to be a linear form in $\mathbb{Z}\zeta(4) + \mathbb{Q}$.

6. The group structure for $\zeta(4)$

Following [19], to any set of parameters \mathbf{h} from § 4 we assign the 27-element multiset of non-negative integers

$$\begin{aligned} e_{0j} &= h_j - 1, & \bar{e}_{0j} &= h_j + h_{-1} - h_0 - 1 & \text{for } 1 \leq j \leq 6, \\ e_{jk} &= h_0 - h_j - h_k & \text{for } 1 \leq j < k \leq 6, \end{aligned} \tag{18}$$

and set $H(\mathbf{e}) = F(\mathbf{h})$ for the quantity defined in that section. By construction,

$$\gamma(\mathbf{h})^{-1}F(\mathbf{h}) = \frac{e_{01}!e_{02}!\bar{e}_{05}!\bar{e}_{06}!}{e_{13}!e_{24}!e_{35}!e_{46}!}H(\mathbf{e})$$

is invariant under any permutation of the parameters h_1, h_2, \dots, h_6 (which we can view as the ‘ \mathbf{h} -trivial’ action). Clearly, any such permutation induces the corresponding permutation of the parameter set (18).

On the other hand, it can be seen from (6) that the quantity

$$\prod_{j=1}^6 (a_j - b_j)! \cdot G(\mathbf{a}, \mathbf{b})$$

does not change when the parameters in either collection a_1, a_3, a_5 or a_2, a_4, a_6 permute; we can regard such permutations as ‘ \mathbf{a} -trivial’. (The same effect is produced by ‘ \mathbf{b} -trivial’ permutations, when we change the order in b_1, b_3, b_5 or b_2, b_4, b_6 .) We can also add to the list the ‘trivial’ involution

$$i: a_j \leftrightarrow a_{7-j}, \quad b_j \leftrightarrow b_{7-j} \quad \text{for } j = 1, \dots, 6,$$

which reflects the symmetry $s \leftrightarrow t$ of the double integral (6). In addition, we recall that $G(\mathbf{a}, \mathbf{b})$ is left unchanged by the simultaneous shifts of a_0, a_1, a_3, a_5 and b_0, b_1, b_3, b_5 (or of a_0, a_2, a_4, a_6 and b_0, b_2, b_4, b_6 , respectively) by the same integer. We regard the action of all these transformations (permutations, shifts and involution) and their compositions as the ‘ (\mathbf{a}, \mathbf{b}) -trivial’ action.

By setting

$$\begin{aligned} a_0 &= 1 + h_0 - h_{-1}, & a_j &= h_j & \text{for } j = 1, \dots, 6, \\ b_0 &= 1 + h_0, & b_1 &= b_2 = 1, & b_5 &= b_6 = 1 + h_0 - h_{-1}, \\ b_3 &= h_1 + h_3 + h_5 - h_0, & b_4 &= h_2 + h_4 + h_6 - h_0, \end{aligned} \tag{19}$$

we have $F(\mathbf{h}) = G(\mathbf{a}, \mathbf{b})$. If we request that the condition

$$h_{-1} = 2 + 3h_0 - (h_1 + h_2 + h_3 + h_4 + h_5 + h_6) \tag{20}$$

holds, then the shift of h_0, h_1, h_3, h_5 by $1 + h_0 - h_1 - h_3 - h_5$, that is, the transformation

$$\begin{aligned} \mathbf{b}_{135}: \mathbf{h} \mapsto & (1 + 2h_0 - h_1 - h_3 - h_5, h_{-1}; 1 + h_0 - h_3 - h_5, h_2, \\ & 1 + h_0 - h_1 - h_5, h_4, 1 + h_0 - h_1 - h_3, h_6), \end{aligned}$$

induces the composition of the shift of a_0, a_1, a_3, a_5 and b_0, b_1, b_3, b_5 by $1 + h_0 - h_1 - h_3 - h_5$ and the permutation $(b_1 \ b_3)(b_4 \ b_6)$. Therefore, \mathbf{b}_{135} , which also induces

the permutation

$$\mathbf{b} = (e_{01} e_{35})(e_{03} e_{15})(e_{05} e_{13})(\bar{e}_{02} e_{46})(\bar{e}_{04} e_{26})(\bar{e}_{06} e_{24})$$

on the parameter set (18), is an (\mathbf{a}, \mathbf{b}) -trivial transformation. As a consequence, the quantity

$$\prod_{j=1}^6 (a_j - b_j)! \cdot G(\mathbf{a}, \mathbf{b}) = e_{01}!e_{02}!\bar{e}_{05}!\bar{e}_{06}!e_{15}!e_{26}!H(\mathbf{e})$$

does not change by the action of the permutation \mathbf{b} . We remark that (20) is a very natural condition for the application of the (\mathbf{a}, \mathbf{b}) -trivial action to $F(\mathbf{h})$. Indeed, by (19) we have $b_1 - b_5 = b_2 - b_6$, and (20) is equivalent to $b_1 - b_3 = b_4 - b_6$ or to $b_3 - b_5 = b_2 - b_4$.

Taking the multiset

$$\mathcal{E} = \{e_{03}, e_{04}, e_{05}, e_{06}, \bar{e}_{01}, \bar{e}_{02}, \bar{e}_{03}, \bar{e}_{04}, e_{13}, e_{24}, e_{35}, e_{46}\}$$

we conclude that the quantity

$$\frac{H(\mathbf{e})}{\prod_{e \in \mathcal{E}} e!} = \frac{1}{\prod_{j=1}^6 e_{0j}!\bar{e}_{0j}!} \cdot \frac{e_{01}!e_{02}!\bar{e}_{05}!\bar{e}_{06}!}{e_{13}!e_{24}!e_{35}!e_{46}!} H(\mathbf{e})$$

is invariant under the \mathbf{h} -trivial permutations and also, by

$$\frac{H(\mathbf{e})}{\prod_{e \in \mathcal{E}} e!} = \frac{\prod_{j=1}^6 (a_j - b_j)! \cdot G(\mathbf{a}, \mathbf{b})}{e_{13}!e_{15}!e_{35}!e_{24}!e_{26}!e_{46}! \prod_{j=1}^6 e_{0j}!\bar{e}_{0j}!},$$

under the permutation \mathbf{b} . The permutation group of the multiset (18), which is generated by all \mathbf{h} -trivial permutations and the permutation \mathbf{b} , coincides with the group \mathfrak{G} (of order 51840) considered in [19]. (Note that the group contains the above involution \mathbf{i} as well.) By these means, we also recover the invariance of the quantity

$$\frac{H(\mathbf{e})}{\Pi(\mathbf{e})}, \text{ where } \Pi(\mathbf{e}) = e_{03}!e_{04}!e_{05}!e_{06}!\bar{e}_{01}!\bar{e}_{02}!\bar{e}_{03}!\bar{e}_{04}!e_{13}!e_{24}!e_{35}!e_{46}!,$$

under the action of \mathfrak{G} and corresponding to the arithmetic normalization of $H(\mathbf{e}) = F(\mathbf{h}) = G(\mathbf{a}, \mathbf{b})$ in § 4.

Because our access to the arithmetic of coefficients of linear forms $H(\mathbf{e}) \in \mathbb{Z}\zeta(4) + \mathbb{Q}$ is through their $G(\mathbf{a}, \mathbf{b})$ -representation, we are interested in collecting a set of representatives which are distinct modulo (\mathbf{a}, \mathbf{b}) -trivial transformations. For a generic set of integral parameters \mathbf{h} subject to (20), such a set of representatives contains 120 different elements. Indeed, by (19) and (20), the subgroup of all the (\mathbf{a}, \mathbf{b}) -trivial permutations in \mathfrak{G} contains $3!^3 \cdot 2! = 432$ elements and is generated by:

- the \mathbf{a} - and \mathbf{h} -trivial permutations $(h_1 h_3)$ and $(h_3 h_5)$;
- the \mathbf{a} - and \mathbf{h} -trivial permutations $(h_2 h_4)$ and $(h_4 h_6)$;
- the \mathbf{b} -trivial permutation $(b_1 b_3)(b_4 b_6)$ (that is, by \mathbf{b}_{135}); and
- the involution \mathbf{i} (that is, by $(h_1 h_6)(h_2 h_5)(h_3 h_4)$).

This subgroup also contains $(b_2\ b_4)(b_3\ b_5)$ (namely, $\mathfrak{b}_{246} = i\mathfrak{b}_{135}i$) and is isomorphic to $\mathfrak{S}_3^3 \times \mathfrak{S}_2$. Now, the group \mathfrak{G} is generated by $(h_1\ h_3)$, $(h_3\ h_5)$, $(h_2\ h_4)$, $(h_4\ h_6)$, \mathfrak{b}_{135} and $(h_3\ h_4)$. Note that $(h_1\ h_3)$, $(h_3\ h_5)$, $(h_2\ h_4)$ and $(h_4\ h_6)$ commute with \mathfrak{b}_{135} , while $(h_3\ h_4)$ acts on (\mathbf{a}, \mathbf{b}) by $a_3 \leftrightarrow a_4$, $b_3 \mapsto b_3 + a_4 - a_3$, $b_4 \mapsto b_4 + a_3 - a_4$ (and leaves a_i, b_i unchanged for $i \neq 3, 4$). Hence there are exactly $|\mathfrak{G}|/432 = 120$ elements in \mathfrak{G} that are distinct modulo the (\mathbf{a}, \mathbf{b}) -trivial subgroup, each for any simultaneous choice of a subset $\{a_1, a_3, a_5\}$ (or $\{a_2, a_4, a_6\}$) of $\{h_1, \dots, h_6\}$ (among all $\binom{6}{3} = 20$ such subsets) and a permutation in the \mathbf{b} -trivial subgroup (of $3! = 6$ elements) generated by \mathfrak{b}_{135} and \mathfrak{b}_{246} .

7. Arithmetic of linear forms

In order to compute the minimum on the right-hand side of (17), we distinguish between two different situations: (a) $j + l - a_0$ is coprime with p ; and (b) $j + l - a_0$ is divisible by p . In case (a), we get $\lfloor (j + l - a_0)/p \rfloor = \lfloor (j + l - a_0 - 1)/p \rfloor$, so that the minimum in (17) is greater than or equal to

$$\begin{aligned} \Omega_1(\mathbf{a}, \mathbf{b}; p) &= \min_{j, l \in \mathbb{Z}} \left(\left\lfloor \frac{b_0 - a_0 - 2}{p} \right\rfloor - \left\lfloor \frac{j + l - a_0 - 1}{p} \right\rfloor - \left\lfloor \frac{b_0 - j - l - 1}{p} \right\rfloor \right. \\ &\quad + \sum_{r \in \{2, 4, 6\}} \left(\left\lfloor \frac{j - b_r}{p} \right\rfloor - \left\lfloor \frac{j - a_r}{p} \right\rfloor - \left\lfloor \frac{a_r - b_r}{p} \right\rfloor \right) \\ &\quad \left. + \sum_{r \in \{1, 3, 5\}} \left(\left\lfloor \frac{l - b_r}{p} \right\rfloor - \left\lfloor \frac{l - a_r}{p} \right\rfloor - \left\lfloor \frac{a_r - b_r}{p} \right\rfloor \right) \right). \end{aligned} \tag{21}$$

In case (b), we have $l = -j + a_0 + \mu p$ for some $\mu \in \mathbb{Z}$, and

$$\begin{aligned} &\left\lfloor \frac{b_0 - a_0 - 1}{p} \right\rfloor - \left\lfloor \frac{j + l - a_0}{p} \right\rfloor - \left\lfloor \frac{b_0 - j - l - 1}{p} \right\rfloor \\ &\quad + \sum_{r \in \{2, 4, 6\}} \left(\left\lfloor \frac{j - b_r}{p} \right\rfloor - \left\lfloor \frac{j - a_r}{p} \right\rfloor - \left\lfloor \frac{a_r - b_r}{p} \right\rfloor \right) \\ &\quad + \sum_{r \in \{1, 3, 5\}} \left(\left\lfloor \frac{l - b_r}{p} \right\rfloor - \left\lfloor \frac{l - a_r}{p} \right\rfloor - \left\lfloor \frac{a_r - b_r}{p} \right\rfloor \right) \\ &= \sum_{r \in \{2, 4, 6\}} \left(\left\lfloor \frac{j - b_r}{p} \right\rfloor - \left\lfloor \frac{j - a_r}{p} \right\rfloor - \left\lfloor \frac{a_r - b_r}{p} \right\rfloor \right) \\ &\quad + \sum_{r \in \{1, 3, 5\}} \left(\left\lfloor \frac{-j + a_0 - b_r}{p} \right\rfloor - \left\lfloor \frac{-j + a_0 - a_r}{p} \right\rfloor - \left\lfloor \frac{a_r - b_r}{p} \right\rfloor \right) \\ &= \sum_{r \in \{2, 4, 6\}} \left(\left\lfloor \frac{j - b_r}{p} \right\rfloor - \left\lfloor \frac{j - a_r}{p} \right\rfloor - \left\lfloor \frac{a_r - b_r}{p} \right\rfloor \right) \\ &\quad + \sum_{r \in \{1, 3, 5\}} \left(\left\lfloor \frac{j + a_r - a_0 - 1}{p} \right\rfloor - \left\lfloor \frac{j + b_r - a_0 - 1}{p} \right\rfloor - \left\lfloor \frac{a_r - b_r}{p} \right\rfloor \right) \end{aligned}$$

for primes $p > \sqrt{b_0 - a_0 - 2}$, using the property $\lfloor \alpha + \mu \rfloor = \lfloor \alpha \rfloor + \mu$ together with the passage

$$\left\{ \frac{a-1}{p} \right\} + \left\{ \frac{-a}{p} \right\} = \frac{p-1}{p}, \text{ where } a \in \mathbb{Z},$$

for the fractional part $\{\alpha\} = \alpha - \lfloor \alpha \rfloor$ of a number. This means that in case (b) the minimum in (17) is equal to

$$\begin{aligned} \Omega_2(\mathbf{a}, \mathbf{b}; p) = \min_{j \in \mathbb{Z}} & \left(\sum_{r \in \{2,4,6\}} \left(\left\lfloor \frac{j-b_r}{p} \right\rfloor - \left\lfloor \frac{j-a_r}{p} \right\rfloor - \left\lfloor \frac{a_r-b_r}{p} \right\rfloor \right) \right. \\ & \left. + \sum_{r \in \{1,3,5\}} \left(\left\lfloor \frac{j+a_r-a_0-1}{p} \right\rfloor - \left\lfloor \frac{j+b_r-a_0-1}{p} \right\rfloor - \left\lfloor \frac{a_r-b_r}{p} \right\rfloor \right) \right). \end{aligned} \tag{22}$$

Combining the two cases, we conclude that

$$\text{ord}_p B(\mathbf{a}, \mathbf{b}), \text{ord}_p d_{m_1}^3 d_{m_2} C(\mathbf{a}, \mathbf{b}) \geq \min\{\Omega_1(\mathbf{a}, \mathbf{b}; p), \Omega_2(\mathbf{a}, \mathbf{b}; p)\}$$

for primes $p > \sqrt{b_0 - a_0 - 2}$, where the quantities Ω_1 and Ω_2 are defined in (21) and (22).

When we choose

$$\begin{aligned} a_j &= \alpha_j n + 1 \quad \text{for } j = 0, 1, \dots, 6, \\ b_0 &= \beta_0 n + 3 \quad \text{and} \quad b_j = \beta_j n + 1 \quad \text{for } j = 1, \dots, 6, \end{aligned} \tag{23}$$

for some positive set of integer directions (α, β) , then computing Ω_1, Ω_2 reduces to the computation of the minima $\omega_1^*(x)$ and $\omega_2^*(x)$ of functions

$$\begin{aligned} \omega_1(x, y, z) &= \lfloor (\beta_0 - \alpha_0)x \rfloor - \lfloor y + z - \alpha_0 x \rfloor - \lfloor \beta_0 x - (y + z) \rfloor \\ &+ \sum_{r \in \{2,4,6\}} (\lfloor y - \beta_r x \rfloor - \lfloor y - \alpha_r x \rfloor - \lfloor (\alpha_r - \beta_r)x \rfloor) \\ &+ \sum_{r \in \{1,3,5\}} (\lfloor z - \beta_r x \rfloor - \lfloor z - \alpha_r x \rfloor - \lfloor (\alpha_r - \beta_r)x \rfloor) \end{aligned}$$

and

$$\begin{aligned} \omega_2(x, y) &= \sum_{r \in \{2,4,6\}} (\lfloor y - \beta_r x \rfloor - \lfloor y - \alpha_r x \rfloor - \lfloor (\alpha_r - \beta_r)x \rfloor) \\ &+ \sum_{r \in \{1,3,5\}} (\lfloor y + (\alpha_r - \alpha_0)x \rfloor - \lfloor y + (\beta_r - \alpha_0)x \rfloor - \lfloor (\alpha_r - \beta_r)x \rfloor) \end{aligned}$$

over y, z and over y , respectively. Indeed,

$$\Omega_1(\mathbf{a}, \mathbf{b}; p) = \omega_1 \left(\frac{n}{p}, \frac{j-1}{p}, \frac{l-1}{p} \right) \quad \text{and} \quad \Omega_2(\mathbf{a}, \mathbf{b}; p) = \omega_2 \left(\frac{n}{p}, \frac{j-1}{p} \right)$$

for the settings above. This means that

$$\text{ord}_p B(\mathbf{a}, \mathbf{b}), 4 + \text{ord}_p C(\mathbf{a}, \mathbf{b}) \geq \min \left\{ \omega_1^* \left(\frac{n}{p} \right), \omega_2^* \left(\frac{n}{p} \right) \right\} \tag{24}$$

for primes $p > \sqrt{(\beta_0 - \alpha_0)n}$.

Notice that the functions ω_1 and ω_2 (and hence their minima) are 1-periodic in each variable, so it is sufficient to compute them on the interval $[0, 1)$. In the most symmetric case

$$\alpha_0 = \beta_1 = \dots = \beta_6 = 0, \quad \alpha_1 = \dots = \alpha_6 = 1 \quad \text{and} \quad \beta_0 = 3,$$

we already get (by dropping the four non-negative terms in both ω_1 and ω_2)

$$\begin{aligned} & \lfloor 3x \rfloor - \lfloor y + z \rfloor - \lfloor 3x - (y + z) \rfloor + (\lfloor y \rfloor - \lfloor y - x \rfloor - \lfloor x \rfloor) + (\lfloor z \rfloor - \lfloor z - x \rfloor - \lfloor x \rfloor) \\ &= \lfloor 3x \rfloor - \lfloor y + z \rfloor - \lfloor 3x - (y + z) \rfloor - \lfloor y - x \rfloor - \lfloor z - x \rfloor \geq 1 \quad \text{for } x \in \left[\frac{2}{3}, 1\right) \end{aligned}$$

and

$$\begin{aligned} & (\lfloor y \rfloor - \lfloor y - x \rfloor - \lfloor x \rfloor) + (\lfloor y + x \rfloor - \lfloor y \rfloor - \lfloor x \rfloor) \\ &= \lfloor y + x \rfloor - \lfloor y - x \rfloor \geq 1 \quad \text{for } x \in \left[\frac{1}{2}, 1\right). \end{aligned}$$

Indeed, when $x \in \left[\frac{2}{3}, 1\right)$, the first inequality follows from

$$\lfloor 3x \rfloor - \lfloor y + z \rfloor - \lfloor 3x - (y + z) \rfloor \geq 0 \quad \text{and} \quad \lfloor y - x \rfloor + \lfloor z - x \rfloor \leq -1$$

if either $y < x$ or $z < x$; otherwise, $\frac{2}{3} \leq x \leq y < 1$ and $\frac{2}{3} \leq x \leq z < 1$ imply

$$\begin{aligned} \lfloor 3x \rfloor &= 2, \quad \lfloor y + z \rfloor = 1, \quad \lfloor 3x - (y + z) \rfloor = \lfloor x - (y - x) - (z - x) \rfloor = 0 \\ \text{and} \quad \lfloor y - x \rfloor &= \lfloor z - x \rfloor = 0. \end{aligned}$$

A proof of the second inequality, when $x \in \left[\frac{1}{2}, 1\right)$, makes use of $\lfloor y - x \rfloor = -1$, $\lfloor y + x \rfloor \geq 0$ if $0 \leq y < \frac{1}{2}$, and $\lfloor y + x \rfloor = 1$, $\lfloor y - x \rfloor \leq 0$ if $\frac{1}{2} \leq y < 1$. The two inequalities together mean that the quantity Φ_n from §3 is divisible by (4).

It looks quite plausible (although we do not possess any proof of this) that we always have $\omega_2^*(x) \geq \omega_1^*(x)$ except for possibly finitely many rational points on the interval $[0, 1)$. (Notice that $\omega_1(x, y, \alpha_0x - y)$ coincides with $\omega_2(x, y)$ apart from finitely many rational lines crossing the square $[0, 1)^2$.)

Now assume that the linear forms $G(\mathbf{a}, \mathbf{b})$ originate from the forms $F(\mathbf{h})$ of §4, and condition (20) written as

$$2\beta_0 + \alpha_0 = \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6$$

holds. In this case we can scale all the parameters in (18) to discuss the set en instead, where

$$\begin{aligned} e_{0j} &= \alpha_j, \quad \bar{e}_{0j} = \alpha_j - \alpha_0 \quad \text{for } 1 \leq j \leq 6, \\ e_{jk} &= \beta_0 - \alpha_j - \alpha_k \quad \text{for } 1 \leq j < k \leq 6, \end{aligned}$$

and record the related quantities by $H(en) = B(en)\zeta(4) - C(en) \in \mathbb{Z}\zeta(4) + \mathbb{Q}$, where $n = 0, 1, 2, \dots$. The discussion above (see (24)) implies that

$$\text{ord}_p B(en), \quad 4 + \text{ord}_p C(en) \geq \omega^* \left(\mathbf{e}; \frac{n}{p} \right) \quad \text{for primes } p > \sqrt{(\beta_0 - \alpha_0)n},$$

where

$$\omega^*(\mathbf{e}; x) = \min\{\omega_1^*(\mathbf{e}; x), \omega_2^*(\mathbf{e}; x)\};$$

hence also

$$\text{ord}_p B(\mathbf{g}en), 4 + \text{ord}_p C(\mathbf{g}en) \geq \omega^*\left(\mathbf{g}e; \frac{n}{p}\right) \quad \text{for primes } p > \sqrt{(\beta_0 - \alpha_0)n} \text{ and } \mathbf{g} \in \mathfrak{G},$$

where $\mathbf{g}e$ denotes the image of the multiset \mathbf{e} under the action of $\mathbf{g} \in \mathfrak{G}$. At the same time,

$$\frac{B(en)}{\Pi(en)} = \frac{B(\mathbf{g}en)}{\Pi(\mathbf{g}en)} \quad \text{and} \quad \frac{C(en)}{\Pi(en)} = \frac{C(\mathbf{g}en)}{\Pi(\mathbf{g}en)}$$

for all $\mathbf{g} \in \mathfrak{G}$, in view of the invariance of $H(en)/\Pi(en)$ under the action of \mathfrak{G} (and of the irrationality of $\zeta(4)$). This implies that

$$\begin{aligned} \text{ord}_p B(en), 4 + \text{ord}_p C(en) &\geq \text{ord}_p \left(\frac{\Pi(en)}{\Pi(\mathbf{g}en)} \omega^*\left(\mathbf{g}e; \frac{n}{p}\right) \right) \\ &= \sum_{\mathbf{e} \in \mathcal{E}} \left(\left\lfloor \frac{en}{p} \right\rfloor - \left\lfloor \frac{\mathbf{g}en}{p} \right\rfloor \right) + \omega^*\left(\mathbf{g}e; \frac{n}{p}\right) \end{aligned}$$

for primes $p > \sqrt{(\beta_0 - \alpha_0)n}$ and all $\mathbf{g} \in \mathfrak{G}$, hence

$$\text{ord}_p B(en), 4 + \text{ord}_p C(en) \geq \omega\left(e; \frac{n}{p}\right) \quad \text{for primes } p > \sqrt{(\beta_0 - \alpha_0)n},$$

where

$$\omega(\mathbf{e}; x) = \max_{\mathbf{g} \in \mathfrak{G}} \left(\sum_{\mathbf{e} \in \mathcal{E}} (\lfloor ex \rfloor - \lfloor \mathbf{g}ex \rfloor) + \omega^*(\mathbf{g}e; x) \right). \tag{25}$$

The maximum can be restricted to distinct representatives modulo the group of (\mathbf{a}, \mathbf{b}) -trivial permutations.

8. One concrete example of an irrationality measure for $\zeta(4)$

In the notation of §4 we take

$$h_0 = \eta_0 n + 2, \quad h_{-1} = \eta_{-1} n + 2, \quad h_1 = \eta_1 n + 1, \dots, h_6 = \eta_6 n + 1$$

with

$$\boldsymbol{\eta} = (\eta_0, \eta_{-1}; \eta_1, \dots, \eta_6) = (68, 57; 22, 23, 24, 25, 26, 27).$$

If we set $F_n = F(\mathbf{h}) = G(\mathbf{a}, \mathbf{b}) = u_n \zeta(4) - v_n$ then the asymptotics of F_n and u_n as $n \rightarrow \infty$ can be computed with the help of [19, Proposition 1] (adapted here to address a slightly

different normalization of $F(\mathbf{h})$:

$$C_0 = - \lim_{n \rightarrow \infty} \frac{\log |F_n|}{n} = 36.47011287\dots \quad \text{and} \quad C_1 = \lim_{n \rightarrow \infty} \frac{\log |u_n|}{n} = 106.34774225\dots$$

The above choice of \mathbf{h} translates the form $F_n = F(\mathbf{h})$ from (5) into $G(\mathbf{a}, \mathbf{b})$ from (6) with the parameters (23) as follows:

$$\alpha = (11; 22, 23, 24, 25, 26, 27), \quad \beta = (68; 0, 0, 4, 7, 11, 11). \tag{26}$$

The denominator of $v_n = C(\mathbf{a}, \mathbf{b})$ in (16) is $d_{21n}^3 d_{23n}$. The following table lists 31 out of 120 representatives under the action of group \mathfrak{G} on (26) modulo the trivial (\mathbf{a}, \mathbf{b}) -action, only those that contribute to the computation of the corresponding function $\omega(x) = \omega(\mathbf{e}; x)$ in (25):

1	(68; 22, 23, 24, 25, 26, 27)	11	(67; 21, 22, 23, 25, 27, 26)	22	(65; 19, 21, 22, 23, 26, 27)
2	(68; 22, 23, 24, 25, 27, 26)	12	(67; 21, 22, 23, 26, 25, 27)	23	(65; 19, 21, 22, 23, 27, 26)
3	(68; 22, 23, 24, 26, 25, 27)	13	(67; 21, 22, 25, 23, 26, 27)	24	(65; 19, 21, 23, 22, 26, 27)
4	(68; 22, 23, 25, 24, 26, 27)	14	(66; 20, 21, 23, 24, 26, 27)	25	(65; 19, 21, 23, 22, 27, 26)
5	(68; 22, 23, 25, 24, 27, 26)	15	(66; 20, 21, 23, 24, 27, 26)	26	(65; 19, 21, 26, 22, 27, 23)
6	(68; 22, 23, 26, 24, 27, 25)	16	(66; 20, 21, 23, 26, 24, 27)	27	(65; 19, 22, 21, 23, 26, 27)
7	(68; 22, 24, 23, 25, 26, 27)	17	(66; 20, 21, 24, 23, 27, 26)	28	(65; 19, 23, 20, 24, 27, 25)
8	(68; 22, 24, 23, 25, 27, 26)	18	(65; 19, 20, 23, 24, 25, 27)	29	(64; 18, 19, 23, 25, 24, 26)
9	(68; 22, 25, 23, 26, 24, 27)	19	(65; 19, 20, 23, 24, 27, 25)	30	(64; 19, 20, 21, 22, 27, 26)
10	(67; 21, 22, 23, 25, 26, 27)	20	(65; 19, 20, 23, 25, 24, 27)	31	(64; 19, 21, 20, 22, 26, 27)
		21	(65; 19, 20, 24, 23, 27, 25)		

Here we give the representatives in the format $(\beta_0; \alpha_1, \dots, \alpha_6) = (\eta_0; \eta_1, \dots, \eta_6)$; all other parameters are completely determined by the data.

Then

$$\begin{aligned} \omega(x) = 0 & \quad \text{if } x \in [0, \frac{2}{57}] \cup [\frac{1}{15}, \frac{2}{27}] \cup [\frac{7}{16}, \frac{25}{57}] \cup [\frac{11}{23}, \frac{28}{57}] \cup [\frac{8}{15}, \frac{7}{13}] \cup [\frac{22}{23}, \frac{56}{57}], \\ \omega(x) = 1 & \quad \text{if } x \in [\frac{2}{57}, \frac{1}{27}]^{(1)} \cup [\frac{1}{22}, \frac{1}{21}]^{(1)} \cup [\frac{1}{17}, \frac{1}{16}]^{(1)} \cup [\frac{1}{16}, \frac{1}{15}]^{(3)} \cup [\frac{2}{27}, \frac{1}{13}]^{(1)} \\ & \quad \cup [\frac{2}{15}, \frac{3}{22}]^{(1)} \cup [\frac{3}{22}, \frac{8}{57}]^{(4)} \cup [\frac{1}{7}, \frac{4}{27}]^{(1)} \cup [\frac{5}{23}, \frac{2}{9}]^{(1)} \cup [\frac{5}{21}, \frac{14}{57}]^{(1)} \\ & \quad \cup [\frac{4}{15}, \frac{3}{11}]^{(1)} \cup [\frac{2}{7}, \frac{7}{24}]^{(1)} \cup [\frac{5}{17}, \frac{8}{27}]^{(1)} \cup [\frac{7}{20}, \frac{20}{57}]^{(2)} \cup [\frac{6}{17}, \frac{5}{14}]^{(1)} \\ & \quad \cup [\frac{2}{5}, \frac{23}{57}]^{(2)} \cup [\frac{10}{23}, \frac{7}{16}]^{(1)} \cup [\frac{25}{57}, \frac{11}{25}]^{(1)} \cup [\frac{10}{21}, \frac{11}{23}]^{(1)} \cup [\frac{28}{57}, \frac{1}{2}]^{(1)} \\ & \quad \cup [\frac{9}{17}, \frac{8}{15}]^{(1)} \cup [\frac{7}{13}, \frac{31}{57}]^{(1)} \cup [\frac{14}{23}, \frac{11}{18}]^{(1)} \cup [\frac{11}{18}, \frac{35}{57}]^{(4)} \cup [\frac{12}{17}, \frac{41}{57}]^{(1)} \\ & \quad \cup [\frac{17}{23}, \frac{3}{4}]^{(1)} \cup [\frac{13}{17}, \frac{10}{13}]^{(1)} \cup [\frac{4}{5}, \frac{46}{57}]^{(1)} \cup [\frac{19}{23}, \frac{5}{6}]^{(1)} \cup [\frac{20}{23}, \frac{7}{8}]^{(1)} \\ & \quad \cup [\frac{7}{8}, \frac{50}{57}]^{(2)} \cup [\frac{19}{21}, \frac{10}{11}]^{(1)} \cup [\frac{10}{11}, \frac{52}{57}]^{(7)} \cup [\frac{21}{23}, \frac{11}{12}]^{(1)} \cup [\frac{20}{21}, \frac{21}{22}]^{(1)} \\ & \quad \cup [\frac{21}{22}, \frac{22}{23}]^{(7)} \cup [\frac{56}{57}, 1]^{(1)}, \\ \omega(x) = 2 & \quad \text{if } x \in [\frac{1}{27}, \frac{1}{26}]^{(1)} \cup [\frac{1}{23}, \frac{1}{22}]^{(1)} \cup [\frac{1}{19}, \frac{1}{18}]^{(1)} \cup [\frac{1}{18}, \frac{1}{17}]^{(2)} \cup [\frac{1}{13}, \frac{2}{25}]^{(1)} \\ & \quad \cup [\frac{2}{19}, \frac{1}{9}]^{(1)} \cup [\frac{1}{9}, \frac{3}{26}]^{(2)} \cup [\frac{2}{17}, \frac{7}{57}]^{(1)} \cup [\frac{3}{23}, \frac{2}{15}]^{(3)} \cup [\frac{8}{57}, \frac{1}{7}]^{(4)} \\ & \quad \cup [\frac{4}{27}, \frac{3}{20}]^{(20)} \cup [\frac{3}{20}, \frac{2}{13}]^{(3)} \cup [\frac{3}{19}, \frac{1}{6}]^{(1)} \cup [\frac{4}{23}, \frac{10}{57}]^{(11)} \cup [\frac{3}{17}, \frac{2}{11}]^{(1)} \end{aligned}$$

$$\begin{aligned}
 & \cup \left[\frac{2}{11}, \frac{5}{27} \right]^{(5)} \cup \left[\frac{4}{21}, \frac{11}{57} \right]^{(6)} \cup \left[\frac{11}{57}, \frac{1}{5} \right]^{(1)} \cup \left[\frac{1}{5}, \frac{5}{24} \right]^{(3)} \cup \left[\frac{4}{19}, \frac{5}{23} \right]^{(1)} \\
 & \cup \left[\frac{2}{9}, \frac{5}{22} \right]^{(2)} \cup \left[\frac{5}{22}, \frac{13}{57} \right]^{(4)} \cup \left[\frac{13}{57}, \frac{3}{13} \right]^{(2)} \cup \left[\frac{4}{17}, \frac{5}{21} \right]^{(4)} \cup \left[\frac{14}{57}, \frac{1}{4} \right]^{(1)} \\
 & \cup \left[\frac{5}{19}, \frac{4}{15} \right]^{(3)} \cup \left[\frac{3}{11}, \frac{5}{18} \right]^{(14)} \cup \left[\frac{5}{18}, \frac{16}{57} \right]^{(4)} \cup \left[\frac{16}{57}, \frac{2}{7} \right]^{(2)} \cup \left[\frac{7}{24}, \frac{5}{17} \right]^{(2)} \\
 & \cup \left[\frac{8}{27}, \frac{17}{57} \right]^{(1)} \cup \left[\frac{7}{23}, \frac{4}{13} \right]^{(3)} \cup \left[\frac{6}{19}, \frac{7}{22} \right]^{(1)} \cup \left[\frac{7}{22}, \frac{8}{25} \right]^{(4)} \cup \left[\frac{8}{25}, \frac{9}{26} \right]^{(1)} \\
 & \cup \left[\frac{8}{23}, \frac{7}{20} \right]^{(21)} \cup \left[\frac{20}{57}, \frac{6}{17} \right]^{(2)} \cup \left[\frac{5}{14}, \frac{4}{11} \right]^{(3)} \cup \left[\frac{7}{19}, \frac{3}{8} \right]^{(1)} \cup \left[\frac{8}{21}, \frac{5}{13} \right]^{(1)} \\
 & \cup \left[\frac{9}{23}, \frac{2}{5} \right]^{(1)} \cup \left[\frac{23}{57}, \frac{11}{27} \right]^{(2)} \cup \left[\frac{7}{17}, \frac{5}{12} \right]^{(3)} \cup \left[\frac{8}{19}, \frac{10}{23} \right]^{(1)} \cup \left[\frac{11}{25}, \frac{4}{9} \right]^{(10)} \\
 & \cup \left[\frac{4}{9}, \frac{26}{57} \right]^{(2)} \cup \left[\frac{9}{19}, \frac{10}{21} \right]^{(4)} \cup \left[\frac{1}{2}, \frac{29}{57} \right]^{(2)} \cup \left[\frac{10}{19}, \frac{9}{17} \right]^{(2)} \cup \left[\frac{31}{57}, \frac{6}{11} \right]^{(1)} \\
 & \cup \left[\frac{6}{11}, \frac{11}{20} \right]^{(10)} \cup \left[\frac{11}{20}, \frac{5}{9} \right]^{(5)} \cup \left[\frac{4}{7}, \frac{15}{26} \right]^{(2)} \cup \left[\frac{11}{19}, \frac{7}{12} \right]^{(1)} \cup \left[\frac{10}{17}, \frac{13}{22} \right]^{(1)} \\
 & \cup \left[\frac{13}{22}, \frac{16}{27} \right]^{(6)} \cup \left[\frac{16}{27}, \frac{34}{57} \right]^{(1)} \cup \left[\frac{3}{5}, \frac{14}{23} \right]^{(1)} \cup \left[\frac{35}{57}, \frac{8}{13} \right]^{(4)} \cup \left[\frac{13}{21}, \frac{17}{27} \right]^{(2)} \\
 & \cup \left[\frac{12}{19}, \frac{7}{11} \right]^{(1)} \cup \left[\frac{7}{11}, \frac{16}{25} \right]^{(4)} \cup \left[\frac{11}{17}, \frac{37}{57} \right]^{(1)} \cup \left[\frac{15}{23}, \frac{17}{26} \right]^{(3)} \cup \left[\frac{17}{26}, \frac{17}{25} \right]^{(1)} \\
 & \cup \left[\frac{13}{19}, \frac{11}{16} \right]^{(1)} \cup \left[\frac{11}{16}, \frac{9}{13} \right]^{(4)} \cup \left[\frac{16}{23}, \frac{7}{10} \right]^{(14)} \cup \left[\frac{7}{10}, \frac{40}{57} \right]^{(4)} \cup \left[\frac{40}{57}, \frac{12}{17} \right]^{(2)} \\
 & \cup \left[\frac{41}{57}, \frac{18}{25} \right]^{(1)} \cup \left[\frac{14}{19}, \frac{17}{23} \right]^{(7)} \cup \left[\frac{3}{4}, \frac{43}{57} \right]^{(4)} \cup \left[\frac{43}{57}, \frac{19}{25} \right]^{(2)} \cup \left[\frac{16}{21}, \frac{13}{17} \right]^{(2)} \\
 & \cup \left[\frac{10}{13}, \frac{44}{57} \right]^{(1)} \cup \left[\frac{18}{23}, \frac{11}{14} \right]^{(3)} \cup \left[\frac{15}{19}, \frac{4}{5} \right]^{(1)} \cup \left[\frac{46}{57}, \frac{21}{26} \right]^{(1)} \cup \left[\frac{17}{21}, \frac{13}{16} \right]^{(1)} \\
 & \cup \left[\frac{13}{16}, \frac{9}{11} \right]^{(2)} \cup \left[\frac{14}{17}, \frac{47}{57} \right]^{(8)} \cup \left[\frac{47}{57}, \frac{19}{23} \right]^{(1)} \cup \left[\frac{5}{6}, \frac{21}{25} \right]^{(3)} \cup \left[\frac{16}{19}, \frac{11}{13} \right]^{(1)} \\
 & \cup \left[\frac{13}{15}, \frac{20}{23} \right]^{(7)} \cup \left[\frac{50}{57}, \frac{22}{25} \right]^{(2)} \cup \left[\frac{15}{17}, \frac{8}{9} \right]^{(16)} \cup \left[\frac{17}{19}, \frac{9}{10} \right]^{(1)} \cup \left[\frac{9}{10}, \frac{19}{21} \right]^{(3)} \\
 & \cup \left[\frac{52}{57}, \frac{21}{23} \right]^{(7)} \cup \left[\frac{11}{12}, \frac{23}{25} \right]^{(27)} \cup \left[\frac{23}{25}, \frac{12}{13} \right]^{(1)} \cup \left[\frac{18}{19}, \frac{19}{20} \right]^{(1)} \cup \left[\frac{19}{20}, \frac{20}{21} \right]^{(3)}, \\
 \omega(x) = 3 \quad & \text{if } x \in \left[\frac{1}{26}, \frac{1}{25} \right]^{(1)} \cup \left[\frac{1}{21}, \frac{1}{20} \right]^{(19)} \cup \left[\frac{1}{20}, \frac{1}{19} \right]^{(23)} \cup \left[\frac{2}{25}, \frac{1}{12} \right]^{(14)} \cup \left[\frac{2}{23}, \frac{5}{57} \right]^{(26)} \\
 & \cup \left[\frac{5}{57}, \frac{1}{11} \right]^{(13)} \cup \left[\frac{2}{21}, \frac{1}{10} \right]^{(19)} \cup \left[\frac{1}{10}, \frac{2}{19} \right]^{(23)} \cup \left[\frac{3}{26}, \frac{2}{17} \right]^{(2)} \cup \left[\frac{7}{57}, \frac{1}{8} \right]^{(1)} \\
 & \cup \left[\frac{1}{8}, \frac{3}{23} \right]^{(3)} \cup \left[\frac{2}{13}, \frac{3}{19} \right]^{(19)} \cup \left[\frac{1}{6}, \frac{4}{23} \right]^{(7)} \cup \left[\frac{10}{57}, \frac{3}{17} \right]^{(11)} \cup \left[\frac{5}{27}, \frac{3}{16} \right]^{(30)} \\
 & \cup \left[\frac{3}{16}, \frac{4}{21} \right]^{(14)} \cup \left[\frac{5}{24}, \frac{4}{19} \right]^{(23)} \cup \left[\frac{3}{13}, \frac{4}{17} \right]^{(4)} \cup \left[\frac{1}{4}, \frac{7}{27} \right]^{(2)} \cup \left[\frac{6}{23}, \frac{5}{19} \right]^{(23)} \\
 & \cup \left[\frac{17}{57}, \frac{3}{10} \right]^{(1)} \cup \left[\frac{3}{10}, \frac{7}{23} \right]^{(3)} \cup \left[\frac{4}{13}, \frac{5}{16} \right]^{(19)} \cup \left[\frac{5}{16}, \frac{6}{19} \right]^{(21)} \cup \left[\frac{9}{26}, \frac{8}{23} \right]^{(19)} \\
 & \cup \left[\frac{4}{11}, \frac{7}{19} \right]^{(19)} \cup \left[\frac{3}{8}, \frac{8}{21} \right]^{(10)} \cup \left[\frac{5}{13}, \frac{22}{57} \right]^{(14)} \cup \left[\frac{22}{57}, \frac{7}{18} \right]^{(1)} \cup \left[\frac{7}{18}, \frac{9}{23} \right]^{(2)} \\
 & \cup \left[\frac{11}{27}, \frac{9}{22} \right]^{(2)} \cup \left[\frac{9}{22}, \frac{7}{17} \right]^{(15)} \cup \left[\frac{5}{12}, \frac{8}{19} \right]^{(23)} \cup \left[\frac{26}{57}, \frac{11}{24} \right]^{(2)} \cup \left[\frac{8}{17}, \frac{9}{19} \right]^{(19)} \\
 & \cup \left[\frac{29}{57}, \frac{14}{27} \right]^{(2)} \cup \left[\frac{11}{21}, \frac{10}{19} \right]^{(21)} \cup \left[\frac{5}{9}, \frac{32}{57} \right]^{(5)} \cup \left[\frac{13}{23}, \frac{4}{7} \right]^{(17)} \cup \left[\frac{15}{26}, \frac{11}{19} \right]^{(19)} \\
 & \cup \left[\frac{7}{12}, \frac{10}{17} \right]^{(10)} \cup \left[\frac{34}{57}, \frac{3}{5} \right]^{(1)} \cup \left[\frac{8}{13}, \frac{13}{21} \right]^{(15)} \cup \left[\frac{17}{27}, \frac{12}{19} \right]^{(21)} \cup \left[\frac{16}{25}, \frac{9}{14} \right]^{(14)} \\
 & \cup \left[\frac{9}{14}, \frac{11}{17} \right]^{(10)} \cup \left[\frac{37}{57}, \frac{13}{20} \right]^{(1)} \cup \left[\frac{13}{20}, \frac{15}{23} \right]^{(8)} \cup \left[\frac{17}{25}, \frac{15}{22} \right]^{(24)} \cup \left[\frac{15}{22}, \frac{13}{19} \right]^{(19)}
 \end{aligned}$$

$$\begin{aligned}
 & \cup \left[\frac{9}{13}, \frac{16}{23} \right]^{(14)} \cup \left[\frac{18}{25}, \frac{13}{18} \right]^{(10)} \cup \left[\frac{13}{18}, \frac{8}{11} \right]^{(2)} \cup \left[\frac{8}{11}, \frac{19}{26} \right]^{(6)} \cup \left[\frac{11}{15}, \frac{14}{19} \right]^{(19)} \\
 & \cup \left[\frac{19}{25}, \frac{16}{21} \right]^{(4)} \cup \left[\frac{44}{57}, \frac{17}{22} \right]^{(1)} \cup \left[\frac{17}{22}, \frac{7}{9} \right]^{(8)} \cup \left[\frac{7}{9}, \frac{18}{23} \right]^{(3)} \cup \left[\frac{11}{14}, \frac{15}{19} \right]^{(19)} \\
 & \cup \left[\frac{21}{26}, \frac{17}{21} \right]^{(12)} \cup \left[\frac{9}{11}, \frac{14}{17} \right]^{(14)} \cup \left[\frac{21}{25}, \frac{16}{19} \right]^{(23)} \cup \left[\frac{11}{13}, \frac{17}{20} \right]^{(14)} \cup \left[\frac{17}{20}, \frac{6}{7} \right]^{(3)} \\
 & \cup \left[\frac{6}{7}, \frac{49}{57} \right]^{(9)} \cup \left[\frac{49}{57}, \frac{19}{22} \right]^{(1)} \cup \left[\frac{19}{22}, \frac{13}{15} \right]^{(7)} \cup \left[\frac{22}{25}, \frac{15}{17} \right]^{(14)} \cup \left[\frac{8}{9}, \frac{17}{19} \right]^{(19)} \\
 & \cup \left[\frac{12}{13}, \frac{13}{14} \right]^{(14)} \cup \left[\frac{16}{17}, \frac{17}{18} \right]^{(23)} \cup \left[\frac{17}{18}, \frac{18}{19} \right]^{(19)}, \\
 \omega(x) = 4 \quad & \text{if } x \in \left[\frac{1}{25}, \frac{1}{24} \right]^{(14)} \cup \left[\frac{1}{24}, \frac{1}{23} \right]^{(10)} \cup \left[\frac{1}{12}, \frac{2}{23} \right]^{(24)} \cup \left[\frac{1}{11}, \frac{2}{21} \right]^{(31)} \cup \left[\frac{7}{27}, \frac{6}{23} \right]^{(25)} \\
 & \cup \left[\frac{11}{24}, \frac{6}{13} \right]^{(11)} \cup \left[\frac{7}{15}, \frac{8}{17} \right]^{(18)} \cup \left[\frac{14}{27}, \frac{13}{25} \right]^{(11)} \cup \left[\frac{12}{23}, \frac{11}{21} \right]^{(22)} \cup \left[\frac{32}{57}, \frac{9}{16} \right]^{(5)} \\
 & \cup \left[\frac{9}{16}, \frac{13}{23} \right]^{(12)} \cup \left[\frac{19}{26}, \frac{11}{15} \right]^{(19)} \cup \left[\frac{13}{14}, \frac{53}{57} \right]^{(19)} \cup \left[\frac{14}{15}, \frac{15}{16} \right]^{(28)} \cup \left[\frac{15}{16}, \frac{16}{17} \right]^{(29)}, \\
 \omega(x) = 5 \quad & \text{if } x \in \left[\frac{6}{13}, \frac{7}{15} \right]^{(18)} \cup \left[\frac{13}{25}, \frac{12}{23} \right]^{(22)} \cup \left[\frac{53}{57}, \frac{14}{15} \right]^{(19)},
 \end{aligned}$$

where the notation $[a, b]^{(N)}$ means that the maximum in (25) is attained on the N th representative. Denoting

$$\Phi_n = \prod_{p > \sqrt{57n}} p^{\omega(n/p)},$$

we conclude that $\Phi_n^{-1}u_n \in \mathbb{Z}$ and $\Phi_n^{-1}d_{21n}^3d_{23n}v_n \in \mathbb{Z}$; in other words,

$$\Phi_n^{-1}d_{21n}^3d_{23n}F_n \in \mathbb{Z}\zeta(4) + \mathbb{Z} \quad \text{for } n = 1, 2, \dots$$

At the same time, the asymptotics of $\Phi_n^{-1}d_{21n}^3d_{23n}$ are controlled by the prime number theorem:

$$C_2 = \lim_{n \rightarrow \infty} \frac{\log(\Phi_n^{-1}d_{21n}^3d_{23n})}{n} = 3 \cdot 21 + 23 - \int_0^1 \omega(x) d\psi(x) = 25.05460171 \dots,$$

where $\psi(x)$ is the logarithmic derivative of the gamma function. Now [19, Proposition 3] implies that the irrationality exponent of $\zeta(4)$ is bounded above by

$$\frac{C_0 + C_1}{C_0 - C_2} = 12.51085940 \dots$$

Finally, we point out that the general family of rational approximations to $\zeta(4)$ from § 5 is only exploited here when it is linked to the old approximations reviewed in § 4. The main reason for this is easy access to the asymptotic behaviour of the corresponding forms $G(\mathbf{a}, \mathbf{b})$ and their coefficients $B(\mathbf{a}, \mathbf{b})$. One may hope to gain better control of general approximations from § 5 by covering analytic aspects of the 12-parametric family there; however, this will not necessarily have (significantly) better arithmetic consequences.

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