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Flows associated with product type odometers

M. OSIKAWA
College of General Education, Kyushu University, Fukuoka 810, *Japan*

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Abstract. An AC-flow is the associated flow of a product type odometer (PTO). We give examples of AC-flows and compute their L^∞-point-spectra. We also introduce an invariant for isomorphism of aperiodic conservative ergodic non-singular flows which is a closed subset of the unit interval and contains 0 and 1. We give a necessary condition for the associated flow of an approximately finite ergodic group to be finite measure preserving.

0. *Introduction*

In § 1 we define an AC-flow and give examples of AC-flows, one of which has trivial L^∞-point spectrum. In § 2 we introduce an invariant, $\Gamma(\{T_t\})$, for isomorphisms of aperiodic conservative ergodic non-singular flows $\{T_t\}$, which is a closed subset of $[0, 1]$ and contains 0 and 1, and show that if $\{T_t\}$ is finite measure preserving then

$$\Gamma(\{T_t\}) = [0, 1],$$

and that for any closed subset Γ of $[0, 1]$ that contains 0 and 1 there exists an AC-flow $\{T_t\}$ with

$$\Gamma(\{T_t\}) = \Gamma.$$

Applying this invariant to associated flows of approximately finite ergodic groups G, it is realized as a set $\Delta(G)$ relating to the recurrence of the Radon–Nikodym cocycle $(dPg/dP)(\omega)$, $g \in G$. We should note that a PTO induces an infinite tensor product of finite type I von Neumann factor by the group measure space construction.

The following definitions are omitted in the present paper but can be found in [**1**], [**4**]: L^∞-point spectrum; orbit equivalence, (weak equivalence); the associated flow; type II_1; III_λ $(0 < \lambda < 1)$ and III_0; and approximate finiteness.

1. AC-*flow*

Let T be an ergodic non-singular transformation of a lebesgue space (Ω, P) and let $\xi(\omega)$ be measurable positive function on (Ω, P). Define $\xi_T(k, \omega)$, $\omega \in \Omega$, $k = 0, \pm 1, \dots$ by

$$\xi_T(k,\omega) = \begin{cases} \sum_{i=0}^{k-1} \xi(T^i\omega) & k = 1, 2, \dots \\ 0 & k = 0 \\ -\sum_{i=1}^{-k} \xi(T^{-i}\omega) & k = -1, -2, \dots \end{cases} \tag{1}$$

Denote by $\tilde{\Omega}$ the subset $\{(\omega, u); \omega \in \Omega, u \in \mathbb{R}, 0 \le u < \xi(\omega)\}$ of $\Omega \times \mathbb{R}$ and by $\tilde{P}$ the restriction of $P \times du$ to $\tilde{\Omega}$, where du is Lebesgue measure on the real line $\mathbb{R}$. Define $\tilde{T}_t(\omega, u)$ for (ω, u) in $\tilde{\Omega}$ and $-\infty < t < \infty$ by

$$\tilde{T}_t(\omega, u) = (T^k\omega, u + t - \xi_T(k, \omega))$$

if $\xi_T(k, \omega) \le u + t < \xi_T(k+1, \omega)$. Then $\{\tilde{T}_t\}$ is an aperiodic conservative ergodic measurable flow on $(\tilde{\Omega}, \tilde{P})$ and is called *the flow built under the function* $\xi(\omega)$ *with base transformation T.*

If a measurable function $\xi(\omega)$ is positive-integer-valued one can define a transformation $\tilde{T}$ in the same way as above taking $\mathbb{Z}$ instead of $\mathbb{R}$; that is,

$$\tilde{T}^n(\omega, i) = (T^k\omega, i + n - \xi_T(k, \omega))$$

if $\xi_T(k, \omega) \le i + n < \xi_T(k+1, \omega)$; $i, n \in \mathbb{Z}$. $\tilde{T}$ is called the *transformation built under the function* $\xi(\omega)$ *with base transformation T.*

An ergodic countable group of non-singular transformations can be uniquely associated to an egodic measurable flow [**2**]. W. Krieger [**4**] showed that the correspondence gives a one-to-one mapping from orbit equivalence classes of approximately finite ergodic groups of non-singular transformations of type III_0 onto isomorphism classes of aperiodic conservative ergodic measurable flows.

Let n_k, $k = 1, 2, \ldots$ be a sequence of positive integers $(n_k \ge 2)$. For each k we denote by Ω_k the finite set $\{0, 1, \ldots, n_k - 1\}$; by G_k the permutation group on Ω_k; and let P_k be a probability measure on Ω_k such that $P_k(\{j\}) > 0$ for $j = 0, 1, \ldots, n_k - 1$. Let (Ω, P) be the infinite direct product measure space of (Ω_k, P_k), $k = 1, 2, \ldots$. Each G_k may be considered to act on Ω. The transformation group, which we denote by G, on (Ω, P) generated by G_k, $k = 1, 2, \ldots$ is non-singular and ergodic. The group G is called *a product type odometer group* (PTOG). We denote by $A_{k,j}$, for $j = 0, 1, \ldots, n_k - 2$, $k = 1, 2, \ldots$, the set of points ω in Ω such that

$$\omega_i = n_i - 1 \qquad \text{for } i = 1, 2, \ldots, k-1$$
$$\omega_k = j,$$

where ω_k is the kth coordinate of ω. Then $A_{k,j}$, $j = 0, 1, \ldots, n_k - 2$, $k = 1, 2, \ldots$ are disjoint. For ω in $A_{k,j}$, $j = 0, 1, \ldots, n_k - 2$, $k = 1, 2, \ldots$ we denote by $T\omega$ the point in Ω such that

$$(T\omega)_i = 0 \qquad \text{for } i = 1, 2, \ldots, k-1,$$
$$(T\omega)_k = j + 1$$
$$(T\omega)_i = \omega_i \qquad \text{for } i = k+1, k+2, \ldots.$$

Then T is a mapping from $\Omega - \{\bar{\omega}\}$ onto $\Omega - \{\underline{\omega}\}$, where $\bar{\omega}$ and $\underline{\omega}$ are the points such that

$$(\bar{\omega})_k = n_k - 1 \quad \text{and} \quad (\underline{\omega})_k = 0 \qquad \text{for } k = 1, 2, \ldots.$$

Assume that

$$P(\{\bar{\omega}\}) = \prod_{k=1}^{\infty} P_k(\{n_k - 1\}) = 0$$

and

$$P(\{\underline{\omega}\}) = \prod_{k=1}^{\infty} P_k(\{0\}) = 0,$$

then T is a non-singular transformation of (Ω, P) and satisfies

$$\{T^n\omega; n = 0, \pm 1, \ldots\} = \{g\omega; g \in G\} \qquad \text{for a.e. } \omega \text{ in } \Omega. \tag{2}$$

The transformation T is therefore ergodic and is called *a product type odometer* (PTO). For any positive numbers $C_{k,j}$, $j = 0, 1, \ldots, n_k - 2$, $k = 1, 2, \ldots$ we define a positive measurable function $\xi(\omega)$ on $\Omega - \{\bar{\omega}\}$ by

$$\xi(\omega) = C_{k,j}$$

if ω is in $A_{k,j}$, $j = 0, 1, \ldots, n_k - 2$, $k = 1, 2, \ldots$. We will call the flow built under the function $\xi(\omega)$ with base transformation T, *the* AC-*flow generated by* P_k, $k = 1, 2, \ldots$ *and* $C_{k,j}$, $j = 0, 1, \ldots, n_k - 2$, $k = 1, 2, \ldots$ ([**5**]).

If the $C_{k,j}$'s are positive integers we will call the transformation built under the function $\xi(\omega)$ with base transformation T, *an* AC-*transformation.*

In [**5**] we proved the following theorems:

THEOREM 1. *Any* AC-*flow is the associated flow of a* PTO *of type* III_0.

THEOREM 2. *Let* $\{T_t\}$ *be an* AC-*flow generated by* P_k, $k = 1, 2, \ldots$ *and* $C_{k,j}$, $j = 0, 1, \ldots, n_k - 2$, $k = 1, 2, \ldots$. *A real number* $2\pi t$ *is in the* L^∞*-point-spectrum,* Sp $(\{T_t\})$, *if and only if there exists a real sequence* a_k, $k = 1, 2, \ldots$ *such that*

$$\exp\left(2\pi i t \sum_{k=1}^{n} (\xi_k(\omega) - a_k)\right)$$

converges a.e. ω *as* $n \to \infty$, *where* $\xi_k(\omega) = b_k(\omega_k)$ *for* ω *in* $\Omega - \{\bar{\omega}\}$, $k = 1, 2, \ldots$, *and* $b_k(j)$, $j = 0, 1, \ldots, n_k - 1$, $k = 1, 2, \ldots$ *are defined inductively by*

$$b_k(j) = \sum_{m=0}^{j-1} C_{k,m} + j \sum_{i=1}^{k-1} b_i(n_i - 1).$$

The crucial properties used to prove theorem 2 were that $\xi_k(\omega)$, $k = 1, 2, \ldots$ should be a sequence of independent random variables and that we have

$$\xi_G(g, \omega) = \sum_{k=1}^{\infty} (\xi_k(g\omega) - \xi_k(\omega)) \qquad \text{for } \omega \text{ in } \Omega, \text{for all } g \text{ in the PTO } G, \tag{3}$$

where $\xi_G(g, \omega) := \xi_T(i(g, \omega), \omega)$, and $i(g, \omega)$ denotes the integers given by (2) satisfying $g\omega = T^{i(g,\omega)}\omega$, $\omega \in \Omega$.

Examples. In the following examples we put $n_k = 2$, $k = 1, 2, \ldots$, and

$$P_k(\{0\}) = \frac{1}{1+\lambda}, \qquad P_k(\{1\}) = \frac{\lambda}{1+\lambda}, \qquad k = 1, 2, \ldots, \quad \text{for some } 0 < \lambda \leq 1.$$

The odometer transformation T defined by P_k, $k = 1, 2, \ldots$ is of type II_1 if $\lambda = 1$, of type III if $0 < \lambda < 1$. In examples (i)–(iv) below we compute the L^∞-point-spectrum, Sp $(\{T_t\})$, of the AC-flow $\{T_t\}$ generated by particular $C_{k,0}$'s, $k = 1, 2, \ldots$. Note that in each case the L^∞-point-spectrum, Sp $(\{T_t\})$ is the T-set of the corresponding PTO of type III_0 given by theorem 1 ([**2**]).

(i) Let m_k, $k=1,2,\ldots$ be a sequence of positive integers in which every positive integer appears infinitely often. Put

$$C_{4k+1,0}=2-2^{-m_k},\quad C_{4k+2,0}=2+2^{-m_k},\quad C_{4k+3,0}=2+2^{-m_k+1}\quad\text{and}\quad C_{4k+4,0}=2,$$

$k=0,1,\ldots$; then

$$\mathrm{Sp}\,(\{T_t\})=\{0\}.$$

(ii) Put

$$C_{4k+1,0}=1/a,\quad C_{4k+2,0}=3/a,\quad C_{4k+3,0}=4/a\quad\text{and}\quad C_{4k+4,0}=2/a,$$

$k=0,1,\ldots$ for some positive number a; then

$$\mathrm{Sp}\,(\{T_t\})=2\pi a\mathbb{Z}.$$

(iii) Put

$$C_{k,0}=\left(P^k-\sum_{i=1}^{k-1}p^i\right)\Big/a,$$

$k=1,2,\ldots$ for some positive number a and positive integer p; then

$$\mathrm{Sp}\,(\{T_t\})=2\pi a\times\{p\text{-adic rational number}\}.$$

(iv) Put

$$C_{k,0}=M_k-\sum_{i=1}^{k-1}M_i,$$

$k=1,2,\ldots$ where $M_1=2$ and $M_k=2^kM_{k-1}$, $k=2,3,\ldots$; then $\mathrm{Sp}\,(\{T_t\})$ is the set of real numbers

$$2\pi\sum_{k=1}^{\infty}(t_k/M_k)$$

for all sequences of integers t_k, $k=1,2,\ldots$ such that $\sum_{k=1}^{\infty}(t_k/2^k)^2$ converges. The set $\mathrm{Sp}\,(\{T_t\})$ is a nontrivial uncountable subgroup of $\mathbb{R}$.

Proof. (i) We have

$$b_k(0)=0\qquad\text{for } k=1,2,\ldots,$$
$$b_{4k+1}(1)=2^{4k+1}-2^{-m_k},\qquad b_{4k+2}(1)=2^{4k+2},$$
$$b_{4k+3}(1)=2^{4k+3}+2^{-m_k}\quad\text{and}\quad b_{4k+4}(1)=2^{4k+4},$$

$k=0,1,\ldots$. Let $2\pi t$ be in $\mathrm{Sp}\,(\{T_t\})$, then by theorem 2 there is a real sequence a_k, $k=1,2,\ldots$ with $P(E)=1$ where E is the set consisting of all ω in Ω such that

$$\lim_{k\to\infty}\exp\,(2\pi it(\xi_k(\omega)-a_k))=1.$$

Assume first that t is not a 2-adic rational number, then there exists an infinite set of positive integers, N_0, such that for any infinite subset N_0' of N_0,

$$\lim_{k\in N_0',k\to\infty}\exp\,(2\pi it\times 2^{2k})\neq 1.$$

By the Borel–Cantelli lemma there exists a point ω^1 in the set E and an infinite subset N_1 of N_0 such that

$$(\omega^1)_{2k}=1\qquad\text{for } k \text{ in } N_1.$$

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Then we have

$$\lim_{k\in N_1, k\to\infty} \exp(2\pi i t(2^{2k}-a_{2k}))=1.$$

Again by the Borel–Cantelli lemma there exists a point ω^2 in E and an infinite subset N_2 of N_1 such that

$$(\omega^2)_{2k}=0 \qquad \text{for } k \text{ in } N_2.$$

Then we have

$$\lim_{k\in N_2, k\to\infty} \exp(2\pi i t(-a_{2k}))=1.$$

This contradicts our assumption that

$$\lim_{k\in N_2, k\to\infty} \exp(2\pi i t\times 2^{2k})\neq 1.$$

Assume next that t is a 2-adic rational non-zero number; then from the property of the sequence m_k, $k=1,2,\dots$ and by the Borel–Cantelli lemma there exists a point ω^3 in E and an infinite set N_3 of positive integers such that

$$(\omega^3)_{4k+3}=1 \qquad \text{for } k \text{ in } N_3$$

and such that $\exp(2\pi i t\times 2^{-m_k})$ is a constant and not 1 for k in N_3. The Borel–Cantelli lemma implies that there exists a point ω^4 in E and an infinite subset N_4 of N_3 such that

$$(\omega^4)_{4k+3}=0 \qquad \text{for k in } N_4.$$

From $(\omega^3)_{4k+3}=1$ and $(\omega^4)_{4k+3}=0$ for k in N_4 and from the fact that ω^3 and ω^4 are in the set E we have the contradiction

$$\lim_{k\in N_4, k\to\infty} \exp(2\pi i t\times 2^{-m_k})=1.$$

Hence we have $\mathrm{Sp}(\{T_t\})=\{0\}$.

(ii) We have

$$b_k(0)=0, \qquad k=1,2,\dots,$$
$$b_{4k+1}(1)=(2^{4k+1}-1)/a, \qquad b_{4k+2}(1)=2^{4k+2}/a,$$
$$b_{4k+3}(1)=(2^{4k+3}+1)/a \quad \text{and} \quad b_{4k+4}(1)=2^{4k+4}/a,$$

$k=0,1,\dots$. One can prove (ii) using the same method as for (i).

(iii) We have

$$b_k(0)=0 \quad \text{and} \quad b_k(1)=p^k/a,$$

$k=1,2,\dots$. We proved (iii) for the case $p=2$ in [**2**]. A similar proof can be constructed for the general case.

(iv) We have

$$b_k(0)=0 \quad \text{and} \quad b_k(1)=M_k,$$

$k=1,2,\dots$. This case was proved in [**5**]. □

Remark. Take $a=1$ in example (ii) and consider the AC-transformation instead of the AC-flow, then its L^∞-point spectrum is trivial.

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2. *An invariant of ergodic flows*

Let T be an ergodic non-singular transformation on a Lebesgue space (Ω, P) and $\xi(\omega)$ be a measurable positive function on Ω. For a set A with $P(A)>0$ we denote by $k_A(i,\omega)$ the ith return time to A starting from ω in A, and by T_A the induced transformation of T on A, that is,

$$T_A^i\omega = T^{k_A(i,\omega)}\omega \qquad \text{for } \omega \text{ in } A, \quad i=1,2,\ldots.$$

Put

$$\xi_A(\omega)=\sum_{j=1}^{k_A(1,\omega)-1}\xi(T^j\omega) \qquad \text{for } \omega \text{ in } A$$

and

$$\xi_{T,A}(i,\omega)=\sum_{j=0}^{i-1}\xi_A(T_A^j\omega)=\xi_T(k_A(i,\omega),\omega) \qquad \text{for } \omega \text{ in } A, \quad i=1,2,\ldots.$$

We denote by $\Gamma(T,\xi)$ the set of numbers a in the unit interval $[0,1]$ such that for any set A with $P(A)>0$ the following condition (*) holds:

(*) for any $\varepsilon, r>0$ there exists a positive number $s>r$ and positive integers i, j with

$$P(\{\omega\in A; |a-(1/s)\xi_{T,A}(i,\omega)|<\varepsilon, \quad |1-(1/s)\xi_{T,A}(j,\omega)|<\varepsilon\})>0.$$

LEMMA 1. (1) *$\Gamma(T,\xi)$ is a closed subset of* $[0,1]$ *that contains* 0 *and* 1.

(2) $\Gamma(T,\xi)=\Gamma(T_A,\xi_A)$, *for any set A with $P(A)>0$.*

(3) *If flows built under functions $\xi(\omega)$ and $\xi'(\omega')$ with base transformations T and T' respectively are isomorphic,*

$$\Gamma(T,\xi)=\Gamma(T',\xi').$$

Proof. (1) is obvious.

(2) $\Gamma(T,\xi)\subset\Gamma(T_A,\xi_A)$ is obvious. To prove the converse let B be a set with $P(B)>0$. From the ergodicity of T there exists a subset C of A with $P(C)>0$ and a non-negative integer k such that $C, TC,\ldots, T^kC$ are disjoint and such that T^kC is a subset of B. We have

$$\xi_{T,T^kC}(i,T^k\omega)=\xi_{T,C}(i,\omega)-\sum_{j=0}^{k-1}\xi(T^j\omega)+\sum_{j=k_C(i,\omega)}^{k_C(i,\omega)+k-1}\xi(T^j\omega)$$

for ω in C and $i=1,2,\ldots.$ We may assume $\sum_{j=0}^{k-1}\xi(T^j\omega)$ is bounded for ω in C. Then if (*) holds for the set C it holds for T^kC, and so for B. This means that

$$\Gamma(T_A,\xi_A)\subset\Gamma(T,\xi).$$

Thus we have proved (2).

(3) From the assumption there exist subsets A and A' with $P(A)>0$, $P'(A')>0$, and a non-singular mapping ϕ from A onto A' such that

$$T'_{A'}\phi\omega=\phi T_A\omega \qquad \text{for } \omega \text{ in } A$$

and such that

$$\xi_A(\omega)-\xi'_{A'}(\phi\omega)=\eta(T_A\omega)-\eta(\omega) \qquad \text{for } \omega \text{ in } A,$$

for some measurable function $\eta(\omega)$. Considering subsets of A on which $\eta(\omega)$ is bounded we have

$$\Gamma(T_A, \xi_A) = \Gamma(T_A, \xi'_{A'}(\phi)) = \Gamma(T'_{A'}, \xi'_{A'}).$$

By (2) we have $\Gamma(T, \xi) = \Gamma(T', \xi')$. □

Denote by $\Gamma(\{T_t\})$ the set $\Gamma(T, \xi)$ where $\{T_t\}$ is the flow built under the function $\xi(\omega)$ with base transformation T, and note that every measurable aperiodic conservative ergodic flow is isomorphic to a flow built under a function with a base transformation (Ambrose–Kakutani–Kubo–Krengel). Then lemma 1 (3) says that $\Gamma(\{T_t\})$ is an invariant for isomorphism of aperiodic conservative ergodic flows.

THEOREM 3. *If* $\{T_t\}$ *is finite measure preserving then* $\Gamma(\{T_t\}) = [0, 1]$.

Proof. By the pointwise ergodic theorem, for any set A and any $\varepsilon > 0$ there exists an integer N such that

$$P(\{\omega \in A; |(\xi_{T,A}(n, \omega)/n) - L| < (\varepsilon/2)L \quad \text{for } n \geq N\}) > 0,$$

where $L = \int_A \xi_A(\omega)\, dP(\omega)/P(A) > 0$. For $0 \leq a \leq 1$ there exists $i, j \geq N$ with

$$\left| a - \frac{i}{j} \right| < \varepsilon/2.$$

Then for ω in A such that

$$|(\xi_{T,A}(n, \omega)/n) - L| < (\varepsilon/2)L \qquad \text{for } n > N$$

we have

$$|(1/s)\xi_{T,A}(i, \omega) - a| < \varepsilon \quad \text{and} \quad |(1/s)\xi_{T,A}(j, \omega) - 1| < \varepsilon,$$

where $s = jL$. This means that a is in $\Gamma(\{T_t\})$ and we have

$$\Gamma(\{T_t\}) = [0, 1].$$ □

THEOREM 4. *For any closed subset* Γ *of* $[0, 1]$ *that contains* 0 *and* 1 *there exists an* AC-*flow* $\{T_t\}$ *with* $\Gamma(\{T_t\}) = \Gamma$.

Proof. We first prove the case of $\Gamma \neq \{0, 1\}$. Let Γ_0 be a countable dense subset of Γ that contains neither 0 nor 1. Let $C(k)$, $k = 1, 2, \ldots$ be a sequence of numbers in Γ_0 such that every element of Γ_0 appears infinitely often, and let $S(k)$, $k = 1, 2, \ldots$ be a sequence of positive numbers such that

$$\min\{C(k)S(k), (1 - C(k))S(k)\} > k \sum_{i=1}^{k-1} S(i)$$

for $k = 1, 2, \ldots$. Let P_k, $k = 1, 2, \ldots$ be measures on the 3-point set $\{0, 1, 2\}$ such that

$$P_k(\{0\}) = \frac{1}{1 + \lambda + \eta}, \quad P_k(\{1\}) = \frac{\lambda}{1 + \lambda + \eta}, \quad P_k(\{2\}) = \frac{\eta}{1 + \lambda + \eta},$$

$k = 1, 2, \ldots$, for some positive numbers λ and η, and let

$$C_{k,0} = C(k)S(k) - \sum_{i=1}^{k-1} S(i)$$

and

$$C_{k,1} = (1 - C(k))S(k) - \sum_{i=1}^{k-1} S(i),$$

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$k = 1, 2, \ldots$. Consider the AC-flow generated by P_k, $k = 1, 2, \ldots$ and $C_{k,j}$, $j = 0, 1$, $k = 1, 2, \ldots$; then we have

$$b_k(0) = 0, \quad b_k(1) = C(k)S(k) \quad \text{and} \quad b_k(2) = S(k),$$

$k = 1, 2, \ldots$. Let a be a number in Γ_0 and let A be a subset with $P(A) > 0$. One can choose a positive integer I and an element u in $\prod_1^I \{0, 1, 2\}$ such that

$$P(A \cap Z_u) > \left(1 - \frac{\delta}{3}\right) P(Z_u),$$

where Z_u is the cylinder set determined by u, and

$$\delta = \min\left\{\frac{1}{1+\lambda+\eta}, \frac{\lambda}{1+\lambda+\eta}, \frac{\eta}{1+\lambda+\eta}\right\}.$$

For any $\varepsilon > 0$ and $r > 0$ a positive integer J can be chosen so that

$$C(J) = a, \quad J > I, \quad S(J) > r \quad \text{and} \quad \frac{1}{J} < \varepsilon.$$

For each $j = 0, 1, 2$ one can choose an element v_j in $\prod_{I+1}^J \{0, 1, 2\}$ such that $(v_j)_J = j$ and

$$P(A \cap Z_u \cap Z_{v_j}) > (1 - \tfrac{1}{3}) P(Z_u) P(Z_{v_j}).$$

Let f and g be elements of the PTO G such that

$$f(Z_u \cap Z_{v_0}) = Z_u \cap Z_{v_1}, \qquad g(Z_u \cap Z_{v_0}) = Z_u \cap Z_{v_2}$$

and

$$(f\omega)_k = (g\omega)_k = \omega_k$$

for $k \geq J + 1$ and ω in $Z_u \cap Z_{v_0}$. Since the Radon–Nikodym densities of f and g are constant on $Z_u \cap Z_{v_0}$ we have

$$P((A \cap Z_u \cap Z_{v_0}) \cap f^{-1}(A \cap Z_u \cap Z_{v_1}) \cap g^{-1}(A \cap Z_u \cap Z_{v_2})) > 0.$$

From (3) we have

$$\xi_G(f, \omega) = b_J(1) - b_J(2) + \sum_{i=I+1}^{J-1} (b_i((v_1)_i) - b_i((v_0)_i))$$

and

$$\xi_G(g, \omega) = b_J(2) - b_J(0) + \sum_{i=I+1}^{J-1} (b_i((v_2)_i) - b_i((v_0)_i))$$

for ω in $Z_u \cap Z_{v_0}$, and hence,

$$|a - (1/S(J))\xi_G(f, \omega)| \leq \sum_{i=1}^{J-1} S(i)/S(J) < 1/J < \varepsilon$$

and

$$|1 - (1/S(J))\xi_G(g, \omega)| \leq \sum_{i=1}^{J-1} S(i)/S(J) < 1/J < \varepsilon$$

for ω in

$$(A \cap Z_u \cap Z_{v_0}) \cap f^{-1}(A \cap Z_u \cap Z_{v_1}) \cap g^{-1}(A \cap Z_u \cap Z_{v_2}).$$

This means that

$$P(\{\omega \in A; f\omega \in A, g\omega \in A, |a - (1/S(J))\xi_G(f, \omega)| < \varepsilon,$$
$$|1 - (1/S(J))\xi_G(g, \omega)| < \varepsilon\}) > 0.$$

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Hence a is in $\Gamma(\{T_t\})$ and we have proved

$$\Gamma \subset \Gamma(\{T_t\}).$$

Next let a be a number of $\Gamma(\{T_t\})$, then there exists a sequence $(s_n, f_n, g_n, \omega_n)$, $n = 1, 2, \ldots$ of positive numbers s_n, elements f_n, g_n in G and points ω_n in Ω such that

$$\lim_{n\to\infty} s_n = \infty, \quad \lim_{n\to\infty} \frac{1}{s_n} \xi_G(f_n, \omega_n) = a \quad \text{and} \quad \lim_{n\to\infty} \frac{1}{s_n} \xi_G(g_n, \omega_n) = 1.$$

We may assume that

$$\xi_G(f_n, \omega_n) \le \xi_G(g_n, \omega_n),$$

$n = 1, 2, \ldots$. For each integer n let $m(n)$ be the maximum coordinate of ω_n that is changed by g_n; then at least one of the following cases holds for infinitely many n:

(a) $(\omega_n)_{m(n)} = 0, \quad (f_n\omega_n)_{m(n)} = 2, \quad (g_n\omega_n)_{m(n)} = 2$

(b) $(\omega_n)_{m(n)} = 0, \quad (f_n\omega_n)_{m(n)} = 1, \quad (g_n\omega_n)_{m(n)} = 2$

(c) $(\omega_n)_{m(n)} = 0, \quad (f_n\omega_n)_{m(n)} = 0, \quad (g_n\omega_n)_{m(n)} = 2$

(d) $(\omega_n)_{m(n)} = 0, \quad (f_n\omega_n)_{m(n)} = 1, \quad (g_n\omega_n)_{m(n)} = 1$

(e) $(\omega_n)_{m(n)} = 1, \quad (f_n\omega_n)_{m(n)} = 2, \quad (g_n\omega_n)_{m(n)} = 1$

(f) $(\omega_n)_{m(n)} = 1, \quad (f_n\omega_n)_{m(n)} = 2, \quad (g_n\omega_n)_{m(n)} = 2$

(g) $(\omega_n)_{m(n)} = 1, \quad (f_n\omega_n)_{m(n)} = 1, \quad (g_n\omega_n)_{m(n)} = 2.$

In case (b) we have

$$|\xi_G(g_n, \omega_n) - S(m(n))| < \sum_{i=1}^{m(n)-1} S(i)$$

and

$$|\xi_G(f_n, \omega_n) - C(m(n))S(m(n))| < \sum_{i=1}^{m(n)-1} S(i)$$

for infinitely many n. We have

$$\lim_{n\to\infty} \frac{\xi_G(g_n, \omega_n)}{S(m(n))} = 1, \quad \lim_{n\to\infty} \frac{S(m(n))}{S_n} = 1 \quad \text{and} \quad \lim_{n\to\infty} C(m(n)) = a.$$

Hence, a is a limit point of Γ_0. In the same way as above we have $a = 1$ in cases (a), (d) and (f), and $a = 0$ in cases (c), (e) and (g). We have proved $\Gamma(\{T_t\}) \subset \Gamma$ and therefore

$$\Gamma(\{T_t\}) = \Gamma.$$

We next prove the case of $\Gamma = \{0, 1\}$. Let $S(k)$, $k = 1, 2, \ldots$ be a sequence of positive numbers such that

$$S(k) > k \sum_{i=1}^{k-1} S(i)$$

for $k = 1, 2, \ldots$ and let P_k, $k = 1, 2, \ldots$ be measures on the 2-point set $\{0, 1\}$ such that

$$P_k(\{0\}) = \frac{1}{1+\lambda}, \qquad P_k(\{1\}) = \frac{\lambda}{1+\lambda},$$

$k=1,2,\ldots$ for some positive number λ, and

$$C_{k,0}=S(k)-\sum_{i=1}^{k-1} S(i),$$

$k=1,2,\ldots$. Consider the AC-flow generated by P_k, $k=1,2,\ldots$ and $C_{k,0}$, $k=1,2,\ldots$, then we have $b_k(0)=0$, $b_k(1)=S(k)$, $k=1,2,\ldots$ and we can show in the same way as above that $\Gamma(\{T_t\})=\{0,1\}$. □

For an ergodic non-singular transformation T let us denote by $\Gamma(T)$ the set $\Gamma(T,1)$, then a number $a\in[0,1]$ is in $\Gamma(T)$ if and only if for any set A with $P(A)>0$ and $\varepsilon, r>0$ there exists a positive number $s>r$ and positive integers i, j with

$$P\left(\left\{\omega\in A;\ \left|a-\frac{1}{s}k_A(i,\omega)\right|<\varepsilon,\quad \left|1-\frac{1}{s}k_A(j,\omega)\right|<\varepsilon\right\}\right)>0,$$

where $k_A(i,\omega)$ is the ith return time of T on A. The set $\Gamma(T)$ is an invariant for isomorphism of ergodic non-singular transformations. Similar results to theorems 3 and 4 can be obtained in this setting.

Let G be an ergodic countable group of non-singular type *III* transformations on a Lebesgue space (Ω, P). We denote by $\Delta(G)$ the set of numbers $a\in[0,1]$ such that for any subset A with $P(A)>0$ and $\varepsilon, r>0$ there exists a positive number $s>r$ and elements f, g in G with

$$P\left(\left\{\omega\in A; f\omega\in A, g\omega\in A, \left|a-\frac{1}{s}\log\frac{dPf}{dp}(\omega)\right|<\varepsilon, \left|1-\frac{1}{s}\log\frac{dPg}{dP}(\omega)\right|<\varepsilon\right\}\right)>0.$$

$\Delta(G)$ is a closed subset of $[0,1]$ that contains 0 and 1, does not depend on the measure P, and hence is an invariant for orbit equivalence of ergodic countable groups of non-singular transformations of type III.

THEOREM 5. *Let G be an approximately finite group of type III_0 and let $\{T_t\}$ be its associated flow, then $\Delta(G)=\Gamma(\{T_t\})$.*

Proof. Let S be an ergodic m-measure preserving transformation of an infinite σ-finite Lebesgue space (W, m), and denote by N[S] the set of non-singular transformations R such that

$$R\ \mathrm{Orb}_S(w)=\mathrm{Orb}_S(Rw)\qquad \text{for a.e. } w.$$

By a result of W. Krieger [4] there exists an ergodic non-singular transformation U of a Lebesgue space (Y,ν) and for each y in Y an element $R(y)$ in $N[S]$ such that the mapping $(y,w)\to(Y,R(y)w)$ is measurable and such that G is orbit equivalent to the group generated by $\tilde{U}_R$ and $\tilde{S}$, where

$$\tilde{U}_R(y,w)=(Uy,R(y)w)\quad\text{and}\quad \tilde{S}(y,w)=(y,Sw)$$

for (y,w) in $Y\times W$. Put

$$\xi(y)=\log\frac{d\nu U}{d\nu}(y)+\log\frac{dmR(y)}{dm}(w),$$

(which does not depend on w), then the associated flow $\{T_t\}$ of the group is isomorphic to the flow built under the function $\xi(y)$ with base transformation U. One can easily see that

$$\Delta(G)=\Gamma(U,\xi)=\Gamma(\{T_t\}).$$ □

By theorem 3 and theorem 5 we have:

COROLLARY. *If the associated flow of an approximately finite group G of type III_0 is finite measure preserving, then* $\Delta(G) = [0, 1]$.

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