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IMPROVED VERSIONS OF FORMS OF PLESSNER'S THEOREM

PETER COLWELL

1. Introduction. With the aid of a theorem about the Julia points of a function meromorphic in the unit disk, this paper strengthens a theorem of K. Meier. As a consequence a stronger form of Plessner's Theorem is seen to hold which contains a theorem of E. F. Collingwood. An additional consequence is a stronger form of Meier's analogue to Plessner's Theorem.

First we set the terminology and notation. If $D = \{z : |z| < 1\}$, $C = \{z : |z| = 1\}$, and W is the Riemann sphere, let $f: D \to W$ be meromorphic. If $\gamma \in C$ and T is a chord in D ending at γ, $C(f, \gamma, T)$ denotes the *chordal cluster set* of f at γ along T; $C(f, \gamma, T)$ is the set of points $w \in W$ for which there exists a sequence $\{z_n\} \subset T$ such that $z_n \to \gamma$ and $f(z_n) \to w$. We let

$$\Pi^*(f, \gamma) = \bigcap C(f, \gamma, T),$$

where the intersection is taken over all chords T at γ.

A *Stolz angle* Δ at γ is a triangular domain in D bounded by two chords in D ending at γ, and the cluster set $C(f, \gamma, \Delta)$ is the set of points $w \in W$ for which there exists a sequence $\{z_n\} \subset \Delta$ such that $z_n \to \gamma$ and $f(z_n) \to w$. A point $\gamma \in C$ is called a *Fatou point* of f if there exists $w \in W$ such that $C(f, \gamma, \Delta) = \{w\}$ for every Stolz angle Δ at γ; we call $\gamma \in C$ a *Plessner point* of f if $C(f, \gamma, \Delta) = W$ for every Stolz angle Δ at γ. $F(f)$ and $I(f)$ will denote the set of Fatou points of f and the set of Plessner points of f, respectively.

A chord T at $\gamma \in C$ is called a *Julia segment* for f if, for every Stolz angle Δ at γ containing T, f assumes every value of W, with at most two exceptions, infinitely often in Δ. If every chord at γ is a Julia segment for f, then γ is called a *Julia point* of f. We let $JS(f)$ be the set of points of C at which f has a Julia segment, and $J(f)$ will denote the set of Julia points of f.

If A is a set of C, "almost every (nearly every) point of A" will mean "every point of A with the exception of a set of linear measure zero (first category) on C."

The results we present in §§ 2, 3 and 4 rest on the following result.

THEOREM 1. *If f is meromorphic in D, then almost every and nearly every point of $JS(f) - J(f)$ lies in $\{\gamma \in C : \Pi^*(f, \gamma) = W\}$.*

For expository reasons we defer the proof of Theorem 1 to § 5.

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2. Meier's Theorem. If, for each $\gamma \in C$, $\Lambda(f, \gamma)$ denotes the set of values on W which f assumes infinitely often in every Stolz angle at γ, Meier's Theorem [**6**, Satz 1] states: if f is meromorphic in D, then almost every point of C lies in one of the three sets: (i) $F(f)$; (ii) $J(f)$; (iii)

$$\{\gamma \in C: \Lambda(f, \gamma) \cup \Pi^*(f, \gamma) = W\}.$$

Theorem 1 permits the following stronger version.

THEOREM 2. *If f is meromorphic in D, then almost every point of C lies in one of the three sets:* (i) $F(f)$; (ii) $J(f)$; (iii) $\{\gamma \in C: \Pi^*(f, \gamma) = W\}$.

Proof. Let E be the set of points on C which lie in none of the sets (i), (ii), (iii), and suppose that E has positive measure on C. Then E contains a set U of positive measure such that $U \subset I(f) - J(f)$, and $\Pi^*(f, \gamma) \neq W$ for each $\gamma \in U$. Thus, if $\gamma \in U$, there is a chord T at γ such that $C(f, \gamma, T) \neq W$. Since $\gamma \in I(f)$, it will be the case that T is a Julia segment for f. (The justification for this fact appears in § 5.) Hence $U \subset JS(f) - J(f)$. Theorem 1 produces a contradiction.

3. Stronger forms of Plessner's Theorem. Plessner's Theorem [**7**; **2**, Theorem 8.2] states: if f is meromorphic in D, then almost every point of C is either a Fatou point or a Plessner point.

In [**1**, Theorem 1], Collingwood applied Meier's Theorem to show that for at least one class of meromorphic functions (Tsuji functions) one can replace "Plessner points" by "Julia points." We use Theorem 2 in place of Meier's Theorem in Collingwood's argument.

THEOREM 3. *Let f be meromorphic in D and suppose that $\Pi^*(f, \gamma) \neq W$ at almost every point of C. Then almost every point of C is either a Fatou point or a Julia point.*

Proof. Suppose not. Then $U = [I(f) - J(f)] \cap \{\gamma \in C: \Pi^*(f, \gamma) \neq W\}$ has positive measure on C. But

$$U \subset [JS(f) - J(f)] \cap \{\gamma \in C: \Pi^*(f, \gamma) \neq W\},$$

and this last set has measure zero by Theorem 1.

The hypothesis in Theorem 3 is relatively mild: for almost every point $\gamma \in C$ there exists a chord T at γ for which $C(f, \gamma, T) \neq W$. This suggests an interesting question for which the methods of this paper are not effective.

Question. Suppose f is meromorphic in D and for almost every point $\gamma \in C$ there exists a *curve* Γ in D ending at γ such that $C(f, \gamma, \Gamma) \neq W$. Must almost every point of C then be either a Fatou point or a Julia point?

4. Meier's analogue of Plessner's Theorem. If $\gamma \in C$, $C(f, \gamma) \neq W$, and $\Pi^*(f, \gamma) = C(f, \gamma)$, we call γ a *Meier point of f*, and we denote by $M(f)$

the set of Meier points of f. In [**6**, Satz 5] Meier proved this result: if f is meromorphic in D, then nearly every point of C is either a Meier point or a Plessner point.

With Theorem 1 we obtain a result bearing the same relation to Meier's analogue as Theorem 3 bears to Plessner's Theorem.

THEOREM 4. *Let f be meromorphic in D and suppose that $\Pi^*(f, \gamma) \neq W$ at nearly every point of C. Then nearly every point of C is either a Meier point or a Julia point.*

Proof. At any Plessner point of f where $\Pi^*(f, \gamma) \neq W, f$ has a Julia segment (cf. details in §5.) Thus

$$I(f) - J(f) - \{\gamma : \Pi^*(f, \gamma) = W\} \subset JS(f) - J(f) - \{\gamma : \Pi^*(f, \gamma) = W\}.$$

By Theorem 1 this last set is of first category; by hypothesis $\{\gamma : \Pi^*(f,\gamma) = W\}$ is of first category. Hence $I(f) - J(f)$ is of first category, and Meier's analogue to Plessner's Theorem implies $M(f) \cup J(f)$ is residual on C.

5. Proof of Theorem 1. For the proof some additional notation and preliminary facts will be helpful.

Let $\gamma \in C$ and $\alpha \in (-\pi/2, \pi/2)$. By $T(\gamma, \alpha)$ we denote the chord at γ making angle α with the radius to γ. If $\beta \in (0, \pi/2 - |\alpha|)$, $\Delta(\gamma, \alpha, \beta)$ will be the Stolz angle at γ symmetric about the chord $T(\gamma, \alpha)$ with vertex angle β. And for $r \in (0, 1)$, we let $\Delta_r(\gamma, \alpha, \beta) = \Delta(\gamma, \alpha, \beta) \cap \{z : |z| > r\}$.

For $z, w \in D$, $\rho(z, w)$ is the hyperbolic distance between z and w.

LEMMA 1. *Let $\alpha \in (-\pi/2, \pi/2)$ and $\beta \in (0, \pi/2 - |\alpha|)$ be fixed, and set $M(\beta) = \tanh^{-1} \{\sin (\beta/2)/[4 + \sin (\beta/2)]\}$. For any $\gamma \in C$, if $z \in T(\gamma, \alpha)$, then*

$$\{w \in D : \rho(w, z) < M(\beta)\} \subset \Delta(\gamma, \alpha, \beta).$$

Proof. From a lemma of P. Lappan [**5**, Lemma 2], if $\rho(w, z) < M(\beta)$, then

$$\begin{aligned} |w - z|/(1 - |z|) &\leqq [2 \tanh M(\beta)]/[1 - \tanh M(\beta)] \\ &= (1/2) \sin (\beta/2) < \sin (\beta/2). \end{aligned}$$

Thus $|w - z| < (1 - |z|) \sin (\beta/2) \leqq |\gamma - z| \sin (\beta/2)$, and $w \in \Delta(\gamma, \alpha, \beta)$.

In [**3**], P. Gauthier defined the concept of a ρ-sequence of points in D. A result of Gauthier [**4**, Theorem 1] contains the following fact.

LEMMA 2. *Let f be meromorphic in D, $\gamma \in C$, and T be a chord at γ. If $C(f, \gamma, T) \neq \bigcap C(f, \gamma, \Delta)$, the intersection being taken over all Stolz angles Δ at γ containing T, then T contains a ρ-sequence for f.*

LEMMA 3. *If f is meromorphic in D, $\gamma \in C$, and T is a chord at γ containing a ρ-sequence for f, then T is a Julia segment for f.*

Proof. Let $T = T(\gamma, \alpha)$ and $\Delta = \Delta(\gamma, \alpha, \beta)$ be any Stolz angle containing T. Suppose $\{z_n\}$ is a ρ-sequence for f on T.

By Lemma 1, for any r, $0 < r < M(\beta)$, and each positive integer n, $\{w \in D: \rho(w, z_n) < r\} \subset \Delta(\gamma, \alpha, \beta)$. Since $\{z_n\}$ is a ρ-sequence for f, [**3**, Theorem 2] implies that for each such r and all n sufficiently large there exist sets $E(r, n)$ and $G(r, n)$ on W, with chordal diameters at most r, such that

$$W - [E(r, n) \cup G(r, n)] \subset f[\{w \in D: \rho(w, z_n) < r\}].$$

Hence T is a Julia segment for f.

(We note that if γ is a Plessner point for f, and $\Pi^*(f, \gamma) \neq W$, Lemmas 2 and 3 imply f has a Julia segment at γ.)

Let $E = JS(f) - J(f)$. Clearly $E \cap F(f) = \emptyset$ and $E \cap M(f) = \emptyset$, so almost every and nearly every point of E is a Plessner point. Let $F = E \cap I(f)$, and $G = F \cap \{\gamma \in : \Pi^*(f, \gamma) \neq W\}$.

For any $\gamma \in G$, since $\gamma \notin J(f)$, there exist rational numbers $\alpha \in (-\pi/2, \pi/2)$ and $\beta \in (0, \pi/2 - |\alpha|)$ such that f omits at least three values of W in $\Delta(\gamma, \alpha, \beta)$. Also for some chord $T(\gamma, \mu)$, $\mu \in (-\pi/2, \pi/2)$, $C(f, \gamma, T(\gamma, \mu)) \neq W$. Since $\gamma \in I(f)$, Lemma 2 implies $T(\gamma, \mu)$ contains a ρ-sequence for f.

Now let $\alpha \in (-\pi/2, \pi/2)$, $\beta \in (0, \pi/2 - |\alpha|)$, and $r \in (0, 1)$ all be rational, and let k be a positive integer. Define the subset $G(\alpha, \beta, r, k)$ of G as follows: $\gamma \in G(\alpha, \beta, r, k)$ if $\gamma \in G$, if the set $W - f[\Delta_r(\gamma, \alpha, \beta)]$ contains at least three points, and if for any two sets A, B on W such that $A \cup B = W - f[\Delta_r(\gamma, \alpha, \beta)]$, either A or B has chordal diameter at least $1/k$. It is not difficult to show that

$$G = \bigcup_{\alpha,\beta,r,k} G(\alpha, \beta, r, k).$$

We wish to show that G is of measure zero and of first category on C.

(i) If G has positive measure on C, then for some choice of α, β, r, k – henceforth fixed – $H = G(\alpha, \beta, r, k)$ has positive measure on C. Let L be a perfect subset of H of positive measure on C.

Form a simply connected domain R in D by taking all the domains $\Delta_r(\gamma, \alpha, \beta/2)$ for $\gamma \in L$, together with $\{z: |z| < r\}$ and appropriate open arcs on $\{z: |z| = r\}$. The boundary of R is a rectifiable Jordan curve Γ with $\Gamma \cap C = L$. At almost every point of L there is a tangent to L which coincides with the tangent to C at that point. Let $\lambda \in L$ be any such point.

Except for the point λ itself, some "last segment" of every chord in D at λ must lie in R. Since $\lambda \in L \subset H$, there exists chord $T(\lambda, \mu)$ containing a ρ-sequence $\{z_n\}$. For n sufficiently large, each $z_n \in R$, and hence there is a corresponding point $\gamma_n \in L$ such that $z_n \in \Delta_r(\gamma_n, \alpha, \beta/2) \subset \Delta_r(\gamma_n, \alpha, \beta)$. Both $z_n \to \lambda$ and $\gamma_n \to \lambda$. From Lemma 1, for all n sufficiently large,

$$\{w \in D: \rho(w, z_n) < M(\beta/2)\} \subset \Delta_r(\gamma_n, \alpha, \beta).$$

Now choose s, $0 < s < \min\{M(\beta/2), 1/k\}$. If we let $\mathscr{D}(z_n, s) = \{w \in D: \rho(z_n, w) < s\}$, since $\{z_n\}$ is a ρ-sequence, we know that for all n sufficiently large $f[\mathscr{D}(z_n, s)]$ must cover all of W except for two sets $A(s, n)$, $B(s, n)$ whose chordal diameters are less than $1/k$. The same is then true for $f[\Delta_r(\gamma_n, \alpha, \beta)]$. But each $\gamma_n \in H$ and we have a contradiction.

(ii) If G is of second category on C, then for some choice of α, β, r, k - henceforth fixed - $H = G(\alpha, \beta, r, k)$ is of second category on C and thus dense in some arc Γ of C. Let Ω be a closed nondegenerate subarc of Γ and R be the domain $\bigcup_{\gamma \in \Omega} \Delta_r(\gamma, \alpha, \beta/2)$. If $L = \Omega \cap H$, let λ be any point of L lying in the interior of Ω. The argument proceeds as in (i) to a contradiction.

Thus G is of measure zero and of first category on C, and Theorem 1 is proved.

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Iowa State University,
Ames, Iowa

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