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Almost Sure Global Well-posedness for the Fractional Cubic Schrödinger Equation on the Torus

Seckin Demirbas

Abstract. In a previous paper, we proved that the 1-d periodic fractional Schrödinger equation with cubic nonlinearity is locally well-posed in H^s for $s > 1 - \alpha/2$ and globally well-posed for $s > 10\alpha - 1/12$. In this paper we define an invariant probability measure μ on H^s for $s < \alpha - 1/2$, so that for any $\epsilon > 0$ there is a set $\Omega \subset H^s$ such that $\mu(\Omega^c) < \epsilon$ and the equation is globally well-posed for initial data in Ω. We see that this fills the gap between the local well-posedness and the global well-posedness range in an almost sure sense for $\frac{1-\alpha}{2} < \alpha - \frac{1}{2}$, *i.e.*, $\alpha > \frac{2}{3}$ in an almost sure sense.

1 Introduction

We consider the cubic periodic fractional Schrödinger equation

$$\begin{cases} iu_t + (-\Delta)^\alpha u = \gamma |u|^2 u, \quad x \in [0, 2\pi], \quad t \in \mathbb{R}, \\ u(x, 0) = u_0(x) \in H^s([0, 2\pi]), \end{cases} \tag{1.1}$$

where $\alpha \in (1/2, 1)$ and $\gamma = \pm 1$. The equation is called focusing for $\gamma = 1$ and defocusing for $\gamma = -1$

On a real line, this equation arises as a model in the theory of fractional quantum mechanics; see [15]. In [14], Kirkpatrick, Lenzmann, and Staffilani derived it as a continuum limit of a model for the interaction of quantum particles on lattice points. Allowing the nearest point interaction gives the usual cubic Schrödinger equation, whereas allowing long range interactions gives rise to the fractional Schrödinger equations with paremeter α.

For $\alpha = 1$, Bourgain [1] proved periodic Strichartz estimates and showed L^2 local and global well-posedness for the cubic Schrödinger equation. In [6], Burq, Gerard, and Tzvetkov noted that this result is sharp, since the solution operator is not uniformly continuous on H^s for $s < 0$.

The fractional Schrödinger equation on the real line was recently studied in [9]. For $\alpha \in (1/2, 1)$, the equation is less dispersive, so one would not expect to be able to get local well-posedness on L^2 level. Indeed, they proved that there is local well-posedness on H^s for $s \geq \frac{1-\alpha}{2}$. They also showed that the solution operator fails to be uniformly continuous in time for $s < \frac{1-\alpha}{2}$.

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In [12], we proved that the periodic fractional equation is locally well-posed in H^s, for $s > \frac{1-\alpha}{2}$ using direct $X^{s,b}$ estimates. Further, we proved a Strichartz estimate of the form

$$\|e^{it(-\Delta)^\alpha} f\|_{L^4_{t\in\mathbb{R}} L^4_{x\in\mathbb{T}}} \lesssim \|f\|_{H^s},$$

for $s > \frac{1-\alpha}{4}$, which also gives local well-posedness for $s > \frac{1-\alpha}{2}$, using the methods in [8, 11].

Moreover, we proved in [12] that the defocusing equation is globally well-posed for $s > \frac{10\alpha+1}{12}$, using Bourgain's high-low frequency decomposition introduced in [2]. This method uses the decomposition of the equation into the evolutions of the high and the low frequencies of the initial data. Since the low frequency part is smooth, its evolution is global due to the conservation of the energy. But the same cannot be said for the high frequency part. To overcome this problem we showed that the nonlinear evolution of the high frequency part is smoother than the initial data. We should mention that for $\alpha = 1$, it coincides with the smoothing estimate for the NLS that was recently obtained in [13].

After obtaining these local and global well-posedness results, the natural question that arises is how much we can push the global well-posedness range. For example, the cubic periodic Schrödinger equation ($\alpha = 1$) in 1-d is locally well-posed in L^2 (see [1]), and with the mass conservation, we know that the equation is globally well-posed. That is, conservation law on the local well-posedness level may give rise to global well-posedness. But then one can ask whether we can show that the equation is globally well-posed whenever it is locally well-posed. Although there are no conservation laws on the local well-posedness level, it is not trivial that the statement is true; we can still make sense of the question in a different way. The idea relies on the intuition that the set of "bad" initial data, where the solutions of the equation with those initial data, may have arbitrarily large norm, should be negligible. This approach of looking at the problem in an "almost sure" sense originated from the work of Lebowitz, Rose, and Spear [16]. They were trying to understand the general behavior of a system containing a large number of particles by looking at the values of the observables by taking averages over certain probability distributions containing only a few parameters instead of looking at the individual initial value problems. With this in mind, they constructed probability measures on Sobolev spaces and proved some basic properties of these measures.

Later, Bourgain [3] proved that the Schrödinger equation with power nonlinearity,

$$\begin{cases} iu_t - \Delta u = -|u|^{p-2}u, & x \in [0, 2\pi], \quad t \in \mathbb{R}, \\ u(x,0) = u_0(x) \in H^s([0,2\pi]), \end{cases}$$

where $4 < p \le 6$ is locally well-posed in H^s with $s > 0$. But for $0 < s < 1$ there is no conservation law that would easily allow us to extend the local solutions to global ones. He used the idea of Lebowitz, Rose, and Spear to construct a probability measure, also known as the Gibbs measure, on H^s for $s < \frac{1}{2}$, which is invariant under the solution flow. Then he showed that for any $\epsilon > 0$, there is global in time H^s norm bounds on the solutions with the initial data in H^s up to a set of measure less than ϵ; *i.e.,* the equation is almost surely globally well-posed in H^s for $0 < s < \frac{1}{2}$.

The idea of the Gibbs measures and almost sure global well-posedness have been used to prove similar results for different equations by [4, 5, 7, 10, 17, 18, 20–22] and many others.

Our main result here is the explicit construction of Gibbs measure for 1-d fractional periodic cubic Schrödinger equation and the proof of almost sure global well-posedness. More precisely, we define an invariant probability measure μ on H^s, for $s < \alpha - \frac{1}{2}$ such that for any $\epsilon > 0$ we can find a set $\Omega \subset H^s$ satisfying $\mu(\Omega^c) < \epsilon$ and the solution to the equation (1.1) exists globally for all initial data in Ω.

For that, we are going to truncate equation (1.1) and use the idea of invariant measures on finite dimensional Hamiltonian systems. Namely, if we look at the equation

$$\begin{cases} iu_t^N + (-\Delta)^\alpha u^N = \gamma P_N |u^N|^2 u^N, \\ u^N(x,0) = P_N u_0(x), \end{cases} \tag{1.2}$$

where P_N is the projection operator onto the first N frequencies, we see that (1.2) is a finite dimensional Hamiltonian system, with the Hamiltonian

$$H_N(u)(t) = \frac{1}{2}\sum_{n\le N}\Big||n|^\alpha \widehat{u_n(t)}\Big|^2 + \frac{\gamma}{4}\int_{\mathbb{T}}\Big|\sum_{n\le N} e^{inx}\widehat{u_n(t)}\Big|^4.$$

By Liouville's theorem, we know that the Lebesgue measure $\prod_{|n|\le N} d\widehat{u_n}$ is invariant under the Hamiltonian flow. Thus, by the conservation of the Hamiltonian and the invariance of the Lebesgue measure under the flow, we see that the finite measure,

$$d\mu_N = e^{-H_N(u)}\prod_{|n|\le N} d\widehat{u_n},$$

is invariant under the solution operator; call it $S(t)$.

We see that equation (1.1) is an infinite-dimensional Hamiltonian system on the Fourier side with the Hamiltonian

$$H(u(t)) = \frac{1}{2}\sum_n\Big||n|^\alpha \widehat{u_n(t)}\Big|^2 + \frac{\gamma}{4}\int_{\mathbb{T}}\Big|\sum_n e^{inx}\widehat{u_n(t)}\Big|^4 = H(u_0).$$

We then define the limiting measure μ on H^s as

$$d\mu = e^{-H(u)}\prod_n d\widehat{u_n} = e^{-\frac{1}{2}\sum_n\left||n|^\alpha\widehat{u_n(t)}\right|^2 - \frac{\gamma}{4}\int_{\mathbb{T}}|\sum_n e^{inx}\widehat{u_n(t)}|^4}\prod_n d\widehat{u_n}$$

and show that the measure μ is indeed the weak limit of μ_N.

To construct this measure μ on appropriate H^s spaces, we use the theory of Gaussian measures on Hilbert spaces following Zhidkov's arguments in [23], and first define

$$dw = e^{-\frac{1}{2}\sum_n\left||n|^\alpha\widehat{u_n(t)}\right|^2}\prod_n d\widehat{u_n}.$$

Then we show that the measure μ is absolutely continuous with respect to the Gaussian measure w under certain conditions and finish the proof of almost sure global well-posedness by constructing the set $\Omega \subset H^s$ as stated above. For the second part, we will mainly use Bourgain's arguments in [3].

2 Notation and Preliminaries

Recall that for $s \geq 0$, $H^s(\mathbb{T})$ is defined as a subspace of L^2 via the norm

$$\|f\|_{H^s(\mathbb{T})} := \sqrt{\sum_{k\in\mathbb{Z}} \langle k\rangle^{2s} |\widehat{f}(k)|^2},$$

where $\langle k\rangle := (1+k^2)^{1/2}$ and $\widehat{f}(k) = \frac{1}{2\pi}\int_0^{2\pi} f(x)e^{-ikx}dx$ are the Fourier coefficients of f. We use $(\,\cdot\,)^+$ to denote $(\,\cdot\,)^\epsilon$ for all $\epsilon > 0$ with implicit constants depending on ϵ.

We denote the linear propogator of the equation as $e^{-it(-\Delta)^\alpha}$, which is defined on the Fourier side as

$$(\widehat{e^{-it(-\Delta)^\alpha} f})(n) = e^{it|n|^{2\alpha}} \widehat{f}(n),$$

and $|\nabla|^\alpha$ is defined as $(\widehat{|\nabla|^\alpha f})(n) = |n|^\alpha \widehat{f}(n)$.

When we say equation (1.1) is locally well-posed in H^s, we mean that there exist a time $T_{LWP} = T_{LWP}(\|u_0\|_{H^s})$ such that the solution exists and is unique in $X^{s,b}_{T_{LWP}} \subset C([0,T_{LWP}), H^s)$ and depends continuously on the initial data. We say that the equation is globally well-posed when T_{LWP} can be taken arbitrarily large. Here, $X^{s,b}$ denote the Bourgain spaces, which are defined via the restriction in time, of the norm,

$$\|u\|_{X^{s,b}} \doteq \|e^{it(-\Delta)^\alpha} u\|_{H^b_t(\mathbb{R})H^s_x(\mathbb{T})} = \|\langle \tau - |n|^{2\alpha}\rangle^b \langle n\rangle^s \widehat{u}(n,\tau)\|_{L^2_\tau l^2_{(m,n)}}$$

and $\langle x\rangle = (1+|x|^2)^{1/2}$

By Duhamel's Principle, we know that the smooth solutions of (1.1) satisfy the integral equation

$$u(t,x) = e^{-it(-\Delta)^\alpha} u_0(x) - i\gamma \int_0^t e^{-i(t-\tau)(-\Delta)^\alpha} |u|^2 u(\tau,x)\,d\tau.$$

We note that along with the Hamiltonian conservation, the equation enjoys mass conservation, namely,

$$M(u)(t) = \int_{\mathbb{T}} |u(t,x)|^2 = M(u)(0).$$

3 Almost Sure Global Well-posedness

The main result of this paper is the following theorem.

Theorem 3.1 *For $\frac{1-\alpha}{2} < s < \alpha - \frac{1}{2}$ and $\epsilon > 0$, there exists an invariant probability measure μ on H^s such that equation (1.1) is globally well-posed for any initial data $u_0 \in \Omega \subset H^s$ such that $\mu(\Omega^c) < \epsilon$ with*

$$\|u(t)\|_{H^s} \lesssim \Big(\log\Big(\frac{1+|t|}{\epsilon}\Big)\Big)^{s+}.$$

As we mentioned above, in the proof of this theorem, we first define the finite dimensional measures μ_N, which are invariant under the solution operator of the truncated equation (1.2), and we define μ as the weak limit of these measures. But then we have to show how equation (1.1) and the truncated equation (1.2) are related, so we have the following lemma.

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Lemma 3.2 *Let $A \in \mathbb{R}$ and $u_0 \in H^s$ be such that $\|u_0\|_{H^s} < A$, and assume that the solution, u_N, of (1.2) satisfies $\|u_N(t)\|_{H^s} < A$ for $t \leq T$. Then equation (1.1) is well-posed in $[0,T]$, and moreover, for any $\frac{1-\alpha}{2} < s' < s$, we have*

$$\|u(t) - u_N(t)\|_{H^{s'}} \leq e^{C_1(1+A)^{C_2}T} N^{s'-s},$$

where C_1 and C_2 independent of s.

Proof We have that

$$u(t) - u_N(t) = \\ e^{-it(-\Delta)^\alpha}(u_0 - P_N u_0) + i\int_0^t e^{-i(t-\tau)(-\Delta)^\alpha}\Big(|u|^2 u(\tau) - P_N(|u^N|^2 u^N)(\tau)\Big)\, d\tau,$$

and, taking the $L^\infty([0,T];H^{s'})$ norms of both sides for $b > \frac{1}{2}$, since $X^{s',b} \subset L^\infty([0,T],H^{s'})$ for $b > \frac{1}{2}$, we get

$$\begin{aligned}
&\|u - u_N\|_{L^\infty([0,T],H^{s'})} \\
&\leq \big\| u_0 - P_N u_0 \big\|_{H^{s'}} + \Big\| \int_0^t e^{-i(t-\tau)(-\Delta)^\alpha}\Big(|u|^2 u(\tau) - P_N(|u^N|^2 u^N)(\tau)\Big)\, d\tau \Big\|_{X^{s',b}} \\
&\leq \|u_0 - P_N u_0\|_{H^{s'}} + (T_{LWP})^{1-b-b'} \big\| |u|^2 u - P_N |u^N|^2 u^N \big\|_{X^{s',b'}} \\
&\leq (T_{LWP})^{1-b-b'}\Big(\big\| |u|^2 u - P_N(|u|^2 u)\big\|_{X^{s',b'}} + \big\| P_N\big(|u|^2 u - |u^N|^2 u^N\big)\big\|_{X^{s',b'}}\Big) \\
&\quad + \|u_0 - u_{0,N}\|_{H^{s'}} \\
&\leq \mathrm{I} + \mathrm{II} + \mathrm{III}
\end{aligned}$$

for $b' < \frac{1}{2}$ such that $b + b' < 1$.

Term III is easier to estimate

$$\mathrm{III} = \Big\| \sum_{|n|>N} e^{inx}\widehat{(u_0)}_n \Big\|_{H^{s'}} \leq N^{s'-s}\|u_0\|_{H^s} \leq N^{s'-s}A.$$

For term I, we first observe that $P_N\big(|v|^2 v\big) = |v|^2 v$ for $v = P_{\frac{N}{3}}u$, from the convolution property of frequency restriction. Then we write

$$\begin{aligned}
\mathrm{I} &\leq \big\| |u|^2 u - P_N(|v|^2 v)\big\|_{X^{s',b'}} + \big\| P_N(|v|^2 v - |u|^2 u)\big\|_{X^{s',b'}} \\
&= \big\| |u|^2 u - |v|^2 v\big\|_{X^{s',b'}} + \big\| P_N(|v|^2 v - |u|^2 u)\big\|_{X^{s',b'}} \\
&= \mathrm{I}_1 + \mathrm{I}_2 \leq 2\mathrm{I}_1.
\end{aligned}$$

Estimating term I_1 using $X^{s,b}$ estimates and local well-posedness theory (see [12, Lemma 3 and Proposition 5]), we see that

$$\begin{aligned}
\mathrm{I}_1 &\lesssim (T_{LWP})^{1-b-b'}\big(\|u\|_{X^{s',b}} + \|v\|_{X^{s',b}}\big)^2 \|u - v\|_{X^{s',b}} \\
&\lesssim (T_{LWP})^{1-b-b'} A^2 \|u - P_{\frac{N}{3}}u\|_{X^{s',b}} \\
&\lesssim (T_{LWP})^{1-b-b'} A^2 \|u_0 - P_{\frac{N}{3}}u_0\|_{H^{s'}} \lesssim (T_{LWP})^{1-b-b'} A^3 N^{s'-s}.
\end{aligned}$$

Thus we get $\mathrm{I} \lesssim (T_{LWP})^{1-b-b'} A^3 N^{s'-s}$. Similarly, for the second term we have

$$\begin{aligned}\mathrm{II} &\lesssim (T_{LWP})^{1-b-b'} \big(\|u\|_{X^{s',b}} + \|u^N\|_{X^{s',b}} \big)^2 \|u - u^N\|_{X^{s',b}} \\ &\lesssim (T_{LWP})^{1-b-b'} A^2 \|u - u^N\|_{X^{s',b}},\end{aligned}$$

and collecting all the terms, we get

$$\begin{aligned}\|u - u_N\|_{X^{s',b}} &\le CN^{s'-s} A + C_2 (T_{LWP})^{1-b-b'} A^2 \|u - u^N\|_{X^{s',b}} \\ &\quad + C_1 (T_{LWP})^{1-b-b'} A^3 N^{s'-s} \\ &\le CAN^{s'-s} + \frac{1}{2}\|u - u^N\|_{X^{s',b}} \le 2CAN^{s'-s}\end{aligned}$$

for T_{LWP} small enough independent of N, s, and s'. Repeating this argument, since the implicit constant C can be taken independent of T_{LWP} and N, we see that at any T_{LWP} time, the norm at most doubles, and thus, at time T we get

$$\|u - u_N\|_{H^{s'}} \lesssim 2^{\frac{T}{T_{LWP}}} CAN^{s'-s} \approx e^{C'(1+A)^\delta T} AN^{s'-s},$$

which gives the result. ∎

Now, we define a probability measure on H^s using the Hamiltonian. For that we will mainly follow Zhidkov's arguments; see [23].

3.1 Construction of the Measure on H^s

First we fix the notation that we will use for the rest of the paper. Let $F = (-\Delta)^{\alpha-s}$ on H^s. We see that F has the orthonormal eigenfunctions $e_n = e^{inx}/\langle n\rangle^s$ in H^s with the eigenvalues $|n|^{2\alpha-2s}$. We also denote $u_n = (u, e_n)_{H^s}$.

Definition 3.3 A set $M \subset H^s$ is called *cylindrical* if there exists an integer $k \ge 1$ such that,

$$M = \{u \in H^s : [u_{-k}, \dots, u_{-2}, u_{-1}, u_1, u_2, \dots, u_k] \in D\},$$

for a Borel set $D \subset \mathbb{R}^{2k}$.

We denote by $\mathcal{A}$ the algebra containing all such cylindrical sets. Then we define the additive normalized measure w on the algebra $\mathcal{A}$ as follows. For $M \subset \mathcal{A}$, cylindrical,

$$w(M) = (2\pi)^{-k} \prod_{|n|=1}^{k} |n|^{\alpha-s} \int_D e^{-\frac{1}{2}\sum_{n=1}^{k} |n|^{2\alpha-2s}|u_n|^2} \prod_{|n|=1}^{k} du_n.$$

By the definition of the cylindrical sets, we see that the minimal σ-algebra $\overline{\mathcal{A}}$ containing $\mathcal{A}$ is the Borel σ-algebra; see [23]. Although the measure is additive by definition, it does not necessarily follow that it is countably additive.

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Theorem 3.4 *The Gaussian measure w is countably additive on $\mathcal{A}$ if and only if* $\sum_n |n|^{2s-2\alpha} < \infty$, i.e., $s < \alpha - \frac{1}{2}$.

Proof (*cf.* [23]) Let $\sum_n |n|^{2s-2\alpha} < \infty$. We first show that for any $\epsilon > 0$, there exists a compact set $K_\epsilon \subset H^s$ with $w(M) < \epsilon$ for any cylindrical set M such that $M \cap K_\epsilon = \varnothing$.

Let $b_n = |n|^{\tilde{\epsilon}}$ such that $a = \sum_n |n|^{2s-2\alpha+\tilde{\epsilon}} < \infty$. Then for an arbitrary $R > 0$, take the cylindrical sets of the form

$$M = \Big\{ u \in H^s : [u_{-k}, \dots, u_{-2}, u_{-1}, u_1, \dots, u_k] \in D, \text{ where } \sum_{|n|=1}^{k} |n|^{\tilde{\epsilon}} u_n^2 > R^2 \Big\}.$$

Then we see that

$$\begin{aligned} w(M) &= (2\pi)^{-k} \prod_{|n|=1}^{k} |n|^{\alpha-s} \int_{\sum_{n=1}^{k} |n|^{\tilde{\epsilon}} u_n^2 > R^2} e^{-\frac{1}{2}\sum_{n=1}^{k} |n|^{2\alpha-2s}|u_n|^2} \prod_{|n|=1}^{k} du_n \\ &\le (2\pi)^{-k} \prod_{|n|=1}^{k} |n|^{\alpha-s} \int_{\mathbb{R}^n} \sum_{n=1}^{k} \Big(\frac{|n|^{\tilde{\epsilon}}}{R^2} u_n^2 \Big) e^{-\frac{1}{2}\sum_{n=1}^{k} |n|^{2\alpha-2s}|u_n|^2} \prod_{|n|=1}^{k} du_n \\ &\le R^{-2} \sum_n |n|^{2s-2\alpha+\tilde{\epsilon}} = aR^{-2}. \end{aligned}$$

Here, to pass to the third line we used integration by parts with

$$f = \frac{-u_n}{|n|^{2\alpha-2s}} \quad \text{and} \quad dg = -|n|^{2\alpha-2s} u_n e^{-\frac{1}{2}|n|^{2\alpha-2s} u_n^2} du_n.$$

Then, for $R > \sqrt{\frac{a}{\epsilon}}$, we have $w(M) < \epsilon$.

Hence, if we take $K_\epsilon = \{u \in H^s : \sum_n |n|^{\tilde{\epsilon}} u_n^2 \le R^2\}$, we get the desired compact set.

Now let $A_1 \supset A_2 \supset \cdots \supset A_m \supset \cdots$ be a sequence of cylindrical sets in H^s such that $\bigcap_{m=1}^{\infty} A_m = \varnothing$. Then for any $\epsilon > 0$, there exists closed cylindrical sets $C_m \subset A_m$ for all m such that $w(A_m/C_m) < \epsilon 2^{-m-2}$. Let $D_m = \bigcap_{k=1}^{m} C_k$. Then $w(A_m/D_m) \le w(\bigcup_{k=1}^{m}(A_k/C_k)) < \epsilon/2$. Let $E_m = D_m \cap K_{\epsilon/2}$; then E_m's are compact with $E_m \subset A_m$ and $w(A_m/E_m) < \epsilon$. Since $\bigcap_m A_m = \varnothing$, $\bigcap_m E_m = \varnothing$, and since (E_m) is a nested sequence of compact sets, we see that $E_m = \varnothing$ for all $m > m_0$ for some $m_0 \in \mathbb{N}$.

Hence, $w(A_m) < w(E_m) + \epsilon < \epsilon$, for all $m > m_0$. Thus, $w(A_m) \to 0$, *i.e.*, w is countably additive.

For the converse, assume w is countably additive and also $\sum_n |n|^{2s-2\alpha} = \infty$, *i.e.*, $s \ge \alpha - \frac{1}{2}$. Then consider two cases.

Case 1: $(s \le \alpha)$**.** In this case we see that $|n|^{2s-2\alpha} \le 1$ for any n. Consider the cylindrical sets of the form,

$$M_k = \Big\{ u \in H^s : \Big| \sum_{|n|=1}^{k} (u_n^2) - \lambda_k \Big| < 2\sqrt{\lambda_k} \Big\},$$

where $\lambda_k = \sum_{|n|=1}^{k} |n|^{2s-2\alpha}$.

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Then we have

$$
\begin{aligned}
w(M_k^c) &= w\Big(\Big\{u \in H^s : \Big|\sum_{|n|=1}^{k}(u_n^2) - \lambda_k\Big| \geq 2\sqrt{\lambda_k}\Big\}\Big) \\
&\leq \int_{\mathbb{R}^{2n}} \frac{\big(\sum_{|n|=1}^{k}(u_n^2) - \lambda_k\big)^2}{4\lambda_k} e^{-\frac{1}{2}\sum_{|n|=1}^{k}|n|^{2\alpha-2s}|u_n|^2} \prod_{|n|=1}^{k} du_n \\
&= \frac{1}{4\lambda_k}\int_{\mathbb{R}^{2n}} \Big(\Big(\sum_{|n|=1}^{k} u_n^2\Big)^2 - 2\Big(\sum_{|n|=1}^{k} u_n^2\Big)\lambda_k + \lambda_k^2\Big) e^{-\frac{1}{2}\sum_{|n|=1}^{k}|n|^{2\alpha-2s}|u_n|^2} \prod_{|n|=1}^{k} du_n \\
&= \frac{1}{4\lambda_k}\Big(\Big(\lambda_k^2 + 2\sum_{|n|=1}^{k}|n|^{4s-4\alpha}\Big) - 2\lambda_k.\lambda_k + \lambda_k^2\Big) \\
&\leq \frac{1}{2}\frac{\sum_{|n|=1}^{k}|n|^{4s-4\alpha}}{\lambda_k} \leq \frac{1}{2},
\end{aligned}
$$

where to pass from the third line to the fourth line we used integration by parts again. Since $\lambda_k \to \infty$ as $k \to \infty$, there exist balls $B_{\lambda_k - 2\sqrt{\lambda_k}}(0)$ of arbitrarily large radii with $w(B_{\lambda_k - 2\sqrt{\lambda_k}}(0)) \leq w(M_k^c) \leq \frac{1}{2}$, which contradicts with the countably additivity of w.

Case 2: $(s > \alpha)$**.** In this case, for each $n \geq 1$, consider the cylindrical set

$$M_k = \{u \in H^s : |u_i| \leq k,\ |i| = 1, 2, \dots, a_k\},$$

where $a_k > 0$ is an integer. Then by a change of variables, we have

$$w(M_k) = (2\pi)^{-a_k}\prod_{|n|=1}^{a_k}\Big(\int_{-k|n|^{\alpha-s}}^{k|n|^{\alpha-s}} e^{-\frac{1}{2}|u_n|^2} du_n\Big) \leq \Big[(2\pi)^{-1}\int_{-k}^{k} e^{-\frac{1}{2}|x|^2} dx\Big]^{a_k},$$

since $s > \alpha$. By choosing a_k large enough, we can take $w(M_k) \leq 2^{-k-1}$ for each k and that $a_k \to \infty$ as $k \to \infty$. Then $\bigcup_{k=1}^{\infty} M_k = H^s$ and $w(H^s) = 1$, since H^s is a cylindrical set with full measure. But then $w(\bigcup_{k=1}^{\infty} M_k) \leq \sum_{k=1}^{\infty} w(M_k) \leq \frac{1}{2}$, which is a contradiction. Hence, the theorem follows. ∎

Now we define the sequence of finite dimensional measures (w_k) as follows: For any fixed $k \geq 1$, we take the σ-algebra, $\mathcal{A}_n$, of cylindrical sets in H^s of the form $M_k = \{u \in H^s : [u_{-k}, \dots, u_{-2}, u_{-1}, u_1, \dots, u_k] \in D\}$ for some Borel set $D \subset \mathbb{R}^{2k}$. Then

$$w_k(M_k) = (2\pi)^{-k}\prod_{|n|=1}^{k}|n|^{\alpha-s}\int_D e^{-\frac{1}{2}\sum_{|n|=1}^{k}|n|^{2\alpha-2s}|u_n|^2}\prod_{|n|=1}^{k} du_n.$$

Hence we get the sequence of finite-dimensional countably additive measures w_k on the σ-algebra $\mathcal{A}_k$. We can also extend these measures to the σ-algebra $\overline{\mathcal{A}}$ in H^s, by setting

$$w_k(A) = w_k(A \cap H_k^s), \quad \text{for } A \in \overline{\mathcal{A}},$$

where $H_k^s = \operatorname{span}(e_{-k}, \dots, e_{-1}, e_1, \dots, e_k)$, since $A \cap H_k^s$ is a Borel subset of H_k^s for $A \in \overline{\mathcal{A}}$; see [23].

The following proposition answers the immediate question as to whether or not the infinite dimensional Gaussian measure w and the finite measures w_k are related.

Proposition 3.5 *The sequence w_k converge weakly to the measure w on H^s for $s < \alpha - \frac{1}{2}$ as $k \to \infty$.*

Proof (*cf.* [23]) First, recall that a sequence of measures v_m is said to converge to a measure v weakly on H^s if and only if for any continuous bounded functional ϕ on H^s,

$$\int \phi(u) dv_m(u) \to \int \phi(u) dv(u).$$

Also recall that for any $\epsilon > 0$, if we take $K_\epsilon \subset H^s$ as in the Theorem 3.4, we see that $w(K_\epsilon) > 1 - \epsilon$, and moreover, $w_m(K_\epsilon) > 1 - \epsilon$ for all $n \geq 1$. Now let ϕ be an arbitrary continuous bounded functional on H^s with $B = \sup_{u \in H^s} \phi(u)$. Then for any $\epsilon > 0$ there exists $\delta = \delta(\epsilon) > 0$ such that

$$|\phi(u) - \phi(v)| < \epsilon \text{ for any } u \in K_\epsilon \text{ and } v \in H^s \text{ satisfying } \|u - v\|_{H^s} < \delta. \tag{3.1}$$

For any m, call $K_m = K_\epsilon \cap H^s_m$. Then by the definition of the measures w_m on $\overline{\mathcal{A}}$, we see that

$$\Big| \int_{H^s} \phi(u) dw_m(u) - \int_{K_m} \phi(u) dw_m(u) \Big| < \epsilon B, \tag{3.2}$$

for any $m \geq 1$. Define

$$K_{m,\epsilon} = \{ v \in H^s : v = v_1 + v_2,\ \ v_1 \in H^s_m,\ \ v_2^\perp \in H^s_m,\ \ \|v_2\|_{H^s} < \frac{\delta}{2},\ \ \text{dist}(v_1, K_m) < \frac{\delta}{2} \}.$$

Then $K_\epsilon \subset K_{m,\epsilon}$ for all sufficiently large m's. Thus, for m large enough

$$\Big| \int_{H^s} \phi(u) dw_m(u) - \int_{K_{m,\epsilon}} \phi(u) dw_m(u) \Big| < \epsilon B. \tag{3.3}$$

We now define the measure $w_m^\perp$ on $(H^s_m)^\perp$ as follows:

For a cylindrical set

$$M^\perp = \{ u \in (H^s_m)^\perp : [u_{-m-k}, \dots, u_{-m-2}, u_{-m-1}, u_{m+1}, u_{m+2}, \dots, u_{m+k}] \in F \},$$

where $F \subset \mathbb{R}^{2k}$ is a Borel set, and

$$w_m^\perp(M^\perp) = (2\pi)^{-k} \prod_{|n|=m+1}^{m+k} |n|^{\alpha - s} \int_F e^{-\frac{1}{2} \sum_{|n|=m+1}^{m+k} |n|^{2\alpha - 2s} |u_n|^2} \prod_{|n|=m+1}^{m+k} du_n.$$

Then we see that $w_m^\perp$ is a probability measure on $(H^s_m)^\perp$ and $w = w_m \otimes w_m^\perp$.

Thus, we get

$$\int_{K_{m,\epsilon}} \phi(u) dw(u) = \int_{u_m \in K_{m,\epsilon}} dw_m(u_m) \int_{u_m^\perp \in K^\perp_{m,\epsilon}(u_m)} \phi(u_m + u_m^\perp) dw_m^\perp(u_m^\perp),$$

where $K^\perp_{m,\epsilon}(u_m) = K_{m,\epsilon} \cap \{ u \in H^s : u = u_m + y, y \in (H^s_m)^\perp \}$. Then by (3.1),

$$\begin{aligned}
\int_{K_{m,\epsilon}} \phi(u) dw(u) &= \int_{u_m \in K_{m,\epsilon}} dw_m(u_m) \int_{u_m^\perp \in K^\perp_{m,\epsilon}(u_m)} \big(\phi(u_m + u_m^\perp) - \phi(u_m) \big) \\
&\quad + \phi(u_m) dw_m^\perp(u_m^\perp) \\
&\leq C\epsilon + \int_{u_m \in K_{m,\epsilon}} \phi(u_m) dw_m(u_m)
\end{aligned}$$

for C independent of m and ϵ.

Hence,

$$\int_{K_{m,\epsilon}} \phi(u)dw(u) - \int_{u_m\in K_{m,\epsilon}} \phi(u_m)dw_m(u_m) \le C\epsilon. \tag{3.4}$$

Therefore, combining (3.2), (3.3), and (3.4), we get the result. ∎

Now, we show that the measure μ is absolutely continuous with respect to the Gaussian measure w. Recall that

$$\begin{aligned} d\mu_N &= (2\pi)^{-N}\prod_{|n|=1}^{N}|n|^{\alpha-s}e^{-\frac12\sum_{|n|\le N}\left||n|^{\alpha-s}u_n(t)\right|^2-\frac{\gamma}{4}\int_{\mathbb{T}}|\sum_{|n|\le N}\frac{e^{inx}}{\langle n\rangle^s}u_n(t)|^4}du_0 \\ &\qquad \prod_{1\le|n|\le N}du_n \\ &= e^{-\frac{\gamma}{4}\int_{\mathbb{T}}|\sum_{|n|\le N}\frac{e^{inx}}{\langle n\rangle^s}u_n(t)|^4}(2\pi)^{-N}\prod_{|n|=1}^{N}|n|^{\alpha-s}e^{-\frac12\sum_{0<|n|\le N}|n|^{2\alpha-2s}|u_n(t)|^2}du_0 \\ &\qquad \prod_{1\le|n|\le N}du_n, \end{aligned}$$

and thus, μ_N is a weighted Gaussian measure.

For the defocusing NLS, since

$$|u_0|^2 \le \int_{\mathbb{T}}|u|^2 \lesssim \Big(\int_{\mathbb{T}}|u(t)|^4\Big)^{\frac12},$$

we have

$$\int_{u_0\in\mathbb{C}} e^{-\frac14\int_{\mathbb{T}}|\sum_n\frac{e^{inx}}{\langle n\rangle^s}u_n(t)|^4}du_0 \lesssim \int_{u_0\in\mathbb{C}} e^{-\frac14|u_0|^4}du_0 \lesssim C$$

uniformly in N. Thus, instead of working with the full measure μ_N it is enough to work with the measure w_N, which is also known as the Wiener measure.

For the focusing NLS, though, we do not have an a priori control over the weight $e^{\frac14\int_{\mathbb{T}}|\sum_{n\le N}e^{inx}\widehat{u_n(t)}|^4}$. We can overcome this problem by using a lemma of Lebovitz et al. (see[16]), which applies an L^2 cut-off to the set of initial data.

Lemma 3.6 $e^{\frac14\int|\sum_{1\le|n|\le N}e^{inx}\widehat{u_n(t)}|^4}\chi_{\{\|u\|_{L^2}\le B\}} \in L^1(dw_N)$ *uniformly in* N *for all* $B<\infty$.

Proof (*cf.* [21])

$$\begin{aligned} &\int e^{\frac14\int|\sum_{1\le|n|\le N}e^{inx}\widehat{u_n(t)}|^4}\chi_{\{\|u\|_{L^2}\le B\}}dw \\ &= \int_{(\int|\sum_{|n|=1}^{N}e^{inx}\widehat{u_n(t)}|^4\le K)} e^{\frac14\int|\sum_{1\le|n|\le N}e^{inx}\widehat{u_n(t)}|^4}\chi_{\{\|u\|_{L^2}\le B\}}dw \\ &\quad+\sum_{i=0}^{\infty}\int_{(\int|\sum_{|n|=1}^{N}e^{inx}\widehat{u_n(t)}|^4\in(2^iK,2^{i+1}K])} e^{\frac14\int|\sum_{1\le|n|\le N}e^{inx}\widehat{u_n(t)}|^4}\chi_{\{\|u\|_{L^2}\le B\}}dw \\ &\le e^{\frac14K^4}+\sum_{i=0}^{\infty}e^{\frac14(2^{i+1}K)^4}w\bigg(\Big\{\int\Big|\sum_{|n|=1}^{N}e^{inx}\widehat{u_n(t)}\Big|^4>2^iK, \|u\|_{L^2}<B\Big\}\bigg). \end{aligned}$$

Now to estimate the second term on the right hand side, choose N_0 dyadic, to be specified later. Now call $N_i = N_0.2^i$ and let a_i be such that $\sum_i a_i = \frac{1}{2}$. Then

$$w\Big(\big\{\|u\|_{L^4} > K, \|u\|_{L^2} < B\big\}\Big) \le \sum_{i=1}^{\infty} w\Big(\big\{\|P_{\{|n|\approx N_i\}}u\|_{L^4} > a_i K\big\}\Big),$$

and since we have

$$\|P_{\{|n|\approx N_i\}}u\|_{L^4} \lesssim N_i^{\frac{1}{4}} \|P_{\{|n|\approx N_i\}}u\|_{L^2},$$

by the Sobolev embessing, we see that

$$\begin{aligned} w\Big(\big\{(\|u\|_{L^4} > K, \|u\|_{L^2} < B\big\}\Big) &\le \sum_{i=1}^{\infty} w\Big(\big\{\|P_{\{|n|\approx N_i\}}u\|_{L^4} > a_i K\big\}\Big) \\ &\le \sum_{i=1}^{\infty} w\Big(\big\{\|P_{\{|n|\approx N_i\}}u\|_{L^2} \gtrsim a_i N_i^{-\frac{1}{4}} K\big\}\Big). \end{aligned}$$

Letting $a_i = C N_0^{\epsilon} N_i^{-\epsilon}$ and N_0 such that $K \approx N_0^{\frac{1}{4}} B$, *i.e.*, $N_0 \approx K^4 B^{-4}$, we get,

$$\begin{aligned} w\Big(\big\{(\|u\|_{L^4} > K, \|u\|_{L^2} < B\big\}\Big) &\le \sum_{i=1}^{\infty} w\Big(\Big\{\Big(\sum_{|n|\approx N_i} |\widehat{u_n}|^2\Big)^{\frac{1}{2}} \gtrsim a_i N_i^{-\frac{1}{4}} K\Big\}\Big) \\ &\approx \sum_{i=1}^{\infty} w\Big(\Big\{\Big(\sum_{|n|\approx N_i} |u_n|^2\Big)^{\frac{1}{2}} \gtrsim a_i N_i^{-\frac{1}{4}+s} K\Big\}\Big), \end{aligned}$$

and by the estimation of the tail of the Gaussian measure, (*cf.* (3.6)), we have

$$\begin{aligned} w\Big(\big\{(\|u\|_{L^4} > K, \|u\|_{L^2} < B\big\}\Big) &\lesssim \sum_{i=1}^{\infty} e^{-\frac{1}{4} a_i^2 N_i^{(2\alpha-2s)+2s-\frac{1}{2}} K^2} \le \sum_{i=1}^{\infty} e^{-\frac{1}{4} N_0^{2\epsilon} N_i^{2\alpha-\frac{1}{2}-2\epsilon} K^2} \\ &\le \sum_{i=1}^{\infty} e^{-\frac{1}{4} N_0^{2\alpha-\frac{1}{2}} 2^{(2\alpha-\frac{1}{2}-2\epsilon)i} K^2} \le e^{-\frac{1}{4} K^2 N_0^{2\alpha-\frac{1}{2}}} \\ &\approx e^{-\frac{1}{4} K^{2+4(2\alpha-\frac{1}{2})} B^{2-4s}}. \end{aligned}$$

Collecting terms, we obtain

$$\begin{aligned} \int e^{\frac{1}{4}\int |u|^4} \chi_{\{\|u\|_{L^2}\le B\}} dw &\le e^{\frac{1}{4}K^4} + \sum_{i=0}^{\infty} e^{\frac{1}{4}(2^{i+1}K)^4} w\Big(\big\{(\|u\|_{L^4} > 2^i K, \|u\|_{L^2} < B\big\}\Big) \\ &\le e^{\frac{1}{4}K^4} + \sum_{i=0}^{\infty} e^{\frac{1}{4}(2^{i+1}K)^4} e^{-\frac{1}{4}(2^i K)^{2+4(2\alpha-\frac{1}{2})} B^{2-8\alpha}} < \infty, \end{aligned}$$

since $\alpha > \frac{1}{2}$, which proves the lemma. ∎

Moreover, observe that for $\|u\|_{L^2} < B$, we get $|u_0|^2 \le \sum_n \frac{|u_n|^2}{\langle n\rangle^{2s}} \le B^2$. Hence, L^2 cut off also restricts u_0 to the ball $\{u_0 \in \mathbb{C} : |u_0| \le B\}$, uniformly in N. Therefore, combining these two results, we get that the measure μ_N is a weighted Gaussian measure with weight being uniformly in L^1 with respect to the Gaussian measure.

By the construction of the Gaussian measure, we see that for any compact set $E \subset H^s$, we have $w_N(E \cap H_N^s) \to w(E)$. Thus, using the result above we get $\mu_N(E \cap H_N^s) \to \mu(E)$.

Proof of Theorem 3.1 For the proof of the theorem and the invariance of the measure μ, we follow Bourgain's arguments in [3]. First, for any ϵ we will construct the sets $\Omega_N \subset H^s$ such that $\mu_N(\Omega_N^c) < \epsilon$ and,

$$\|u^N(t)\|_{H^s} \lesssim \Big(\log\Big(\frac{1+|t|}{\epsilon}\Big)\Big)^{\frac{1}{2}}. \tag{3.5}$$

For that, we fix a large time T and let $[-T_{LWP}, T_{LWP}]$ be the local well-posedness interval for equation (1.1). Then consider the set

$$\Omega^K = \{u \in H_N^s : \|u\|_{H^s} \leq K\},$$

where, again, $H_N^s = \operatorname{span}\{e_n : |n| \leq N\}$. We see that

$$\begin{aligned} w_N((\Omega^K)^c) &= (2\pi)^{-\frac{N}{2}} \prod_{|n|=1}^{N} |n|^{\alpha-s} \int_{\{u\in H_N^s:\|u\|_{H^s}>K\}} e^{-\frac{1}{2}\sum_{|n|=1}^N |n|^{2\alpha-2s}|u_n|^2} \prod_{|n|=1}^{N} du_n. \\ &= (2\pi)^{-\frac{N}{2}} \prod_{|n|=1}^{N} |n|^{\alpha-s} \int_{\{u\in H_N^s:\sum_{|n|\leq N}|u_n|^2>K^2\}} e^{-\frac{1}{2}\sum_{|n|=1}^N |n|^{2\alpha-2s}|u_n|^2} \prod_{|n|=1}^{N} du_n \\ &= (2\pi)^{-\frac{N}{2}} \int_{\{\sum_{|n|\leq N}\frac{|v_n|^2}{\langle n\rangle^{2\alpha-2s}}>K^2\}} e^{-\frac{1}{2}\sum_{|n|=1}^N |v_n|^2} \prod_{|n|=1}^{N} dv_n. \\ &\leq (2\pi)^{-\frac{N}{2}} \int_{\{\sum_{|n|\leq N}|v_n|^2>K^2\}} e^{-\frac{1}{2}\sum_{|n|=1}^N |v_n|^2} \prod_{|n|=1}^{N} dv_n \\ &= (2\pi)^{-\frac{N}{2}} \int_{S_{2N}} \int_K^\infty r^{2N-1} e^{-\frac{1}{2}r^2}\, dr\, dS_{2N} \\ &= (2\pi)^{-\frac{N}{2}} \int_{S_{2N}} \int_K^\infty r\, \underbrace{r^{2N-2} e^{-\epsilon(r-\epsilon)-\frac{1}{2}\epsilon^2}}_{\leq C} e^{-\frac{1}{2}(r-\epsilon)^2}\, dr\, dS_{2N} \\ &\lesssim (2\pi)^{-\frac{N}{2}} \int_{S_{2N}} \int_K^\infty r e^{-\frac{1}{2}(r-\epsilon)^2}\, dr\, dS_{2N} \\ &\lesssim (2\pi)^{-\frac{N}{2}} \int_{S_{2N}} \int_K^\infty (r-\epsilon) e^{-\frac{1}{2}(r-\epsilon)^2}\, dr\, dS_{2N} \leq e^{-\frac{1}{2}(K-\epsilon)^2} \lesssim e^{-\frac{1}{4}K^2} \end{aligned} \tag{3.6}$$

for ϵ small enough. Thus, $\mu_N((\Omega^K)^c) \lesssim e^{-\frac{1}{4}K^2}$.

Since μ_N is invariant under the solution operator, S_N of the truncated equation, if we define the set,

$$\Omega_N' = \Omega^K \cap S_N^{-1}(\Omega^K) \cap S_N^{-2}(\Omega^K) \cap \cdots \cap S_N^{-\frac{T}{T_{LWP}}}(\Omega^K),$$

Ω_N' satisfies the property $\mu_N((\Omega_N')^c) \leq \frac{T}{T_{LWP}} \mu_N((\Omega^K)^c) < TK^\theta e^{-\frac{1}{4}K^2}$, since the local well-posedness interval $[-T_{LWP}, T_{LWP}]$ depends uniformly on the H^s norm of the initial data. Thus, if we pick $K = ((4+2\theta)\log(\frac{T}{\epsilon}))^{\frac{1}{2}}$ for ϵ small, we get

$\mu_N((\Omega_N')^c) < \epsilon$, and by the construction of the set Ω_N' we have

$$\|u^N(t)\|_{H^s} \lesssim \Big(\log\Big(\frac{T}{\epsilon} \Big) \Big)^{\frac{1}{2}},$$

for all $|t| < T$. Moreover, if we take $T_j = 2^j$ and $\epsilon_j = \frac{\epsilon}{2^{j+1}}$, and construct $\Omega_{N,j}$'s, we see that $\Omega_N = \bigcap_{j=1}^{\infty} \Omega_{N,j}$ satisfies (3.5).

Also by Lemma 3.2, we see that for any $s' < s$, we have

$$\|u(t)\|_{H^{s'}} < 2A \le C_{s'}\Big(\log\Big(\frac{T}{\epsilon} \Big) \Big)^{\frac{1}{2}}.$$

Again by taking an increasing sequence of times, we get

$$\|u(t)\|_{H^{s'}} \le C_{s'}\Big(\log\Big(\frac{1+|t|}{\epsilon} \Big) \Big)^{\frac{1}{2}}.$$

Hence, if we intersect this result with an increasing sequence of $s < \alpha - \frac{1}{2}$, and taking $\Omega = \bigcap_N \Omega_N$ where (Ω_N)s are defined as above with $\mu_N(\Omega_N^c) < \frac{\epsilon}{2^N}$, we get that $\mu(\Omega) < \epsilon$ and that the solutions to equation (1.1) has the norm growth bound

$$\|u(t)\|_{H^s} \le C_s\Big(\log\Big(\frac{1+|t|}{\epsilon} \Big) \Big)^{\frac{1}{2}},$$

for initial data $u_0 \in \Omega$. Moreover, interpolating this bound with $\|u(t)\|_{L^2} = \|u_0\|_{L^2}$, we have

$$\|u(t)\|_{H^s} \le C\Big(\log\Big(\frac{1+|t|}{\epsilon} \Big) \Big)^{s+},$$

which proves Theorem 3.1. ∎

3.2 Invariance of μ Under the Solution Flow

Let K be a compact set and B_ϵ denote the ϵ ball in H^s. Let S be the flow map for equation (1.1) and let S_N be the flow map for equation (1.2). Then by the weak convergence of the measure

$$\mu(S(K) + B_\epsilon) = \lim_{N\to\infty} \mu_N\big((S(K) + B_\epsilon) \cap H_N^s \big).$$

Also, by the uniform convergence of the solutions of (1.2) to (1.1) in H^{s_1} for any $s_1 < s$, we get $S_N(P_N K) \subset S(K) + B_{\epsilon/2}$, for $N \ge N_0$ sufficiently large. Then for ϵ_1 small enough,

$$S_N\big((K + B_{\epsilon_1}) \cap H_N^s \big) \subset S_N(P_N K) + B_{\epsilon/2} \subset S(K) + B_\epsilon.$$

Hence,

$$\mu_N\Big(S_N\big((K + B_{\epsilon_1}) \cap H_N^s \big) \Big) \le \mu_N\big(S(K) + B_\epsilon \big),$$

and by the invariance of μ_N, we get

$$\mu_N\big((K + B_{\epsilon_1}) \cap H_N^s \big) \le \mu_N\big(S(K) + B_\epsilon \big),$$

and letting $N \to \infty$, by the convergence of the measures μ_N to μ,

$$\mu(K) \le \mu(K + B_{\epsilon_1}) \le \mu(S(K) + B_\epsilon),$$

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which say, by the arbitrariness of ϵ, that $\mu(K) \leq \mu(S(K))$. By the time reversibility, we also have the inverse inequality and, thus $\mu(K) = \mu(S(K))$, which gives the invariance of μ under the solution operator.

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Department of Mathematics, University of Illinois, Urbana, IL 61801, U.S.A.
e-mail: demirba2@illinois.edu