THE COARSENESS OF THE COMPLETE GRAPH

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To P. Erdös on duplicating the cube for the last time, 26:3:67

1. The *coarseness*, $c(G)$, of a graph G is the maximum number of edge-disjoint, non-planar subgraphs of G. We consider only the complete graph, K_p, on p vertices here. For $p = 3r$, Erdös conjectured that the coarseness was $\binom{r}{2}$, but it has been shown (**1**) that

$$c(K_p) = \binom{r}{2} + [\tfrac{1}{5}r], \qquad p = 3r \geqq 30, \tag{1}$$

where square brackets denote integer part.

In § 2 we investigate the other cases where $p = 3r$, determining the coarseness exactly for $r \leqq 5$ and within one otherwise. In § 3 we consider the case $p = 3r + 1$, with exact results except when $r = 4$ or $r = 3n + 2 \geqq 5$, when the result is again within one. For $r = 6n$ see Rosa (**3**). Section 4 provides the complete solution for $p = 3r + 2$ and the results are summarized in the final section.

2. The "gas, water, electricity" graph $K_{3,3}$, formed by joining each of three vertices to each of three others, is non-planar. On joining each of r triads of points to every other, forming $\binom{r}{2}$ $K_{3,3}$ graphs, one sees immediately that

$$c(K_p) \geqq \binom{r}{2}, \qquad p = 3r. \tag{2}$$

We show that equality holds in (2) for $r \leqq 5$, considering first the case $r = 5$.

Kuratowski's theorem (**2**) states that a non-planar graph contains a subgraph homeomorphic to K_5 or $K_{3,3}$. Figures 1 and 2 are examples of graphs homeomorphic to K_5 and to $K_{3,3}$, respectively, the numbered vertices being "essential", and the lettered ones "false". On noting that each "false" vertex requires an additional edge (with two extremities), it will be seen that the arguments which follow, and which are applied to "essential" vertices, carry at least as much weight, if "false" vertices are also present.

Suppose that K_{15} contains eleven edge-disjoint non-planar subgraphs (i.e., that inequality holds in (2)), of which x are K_5 homeomorphs and y are $K_{3,3}$ homeomorphs. Since these have at least ten and nine edges, respectively,

$$(3) \qquad (?) \qquad x + y = 11,$$

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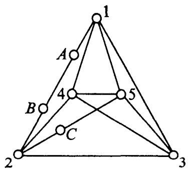

FIGURE 1

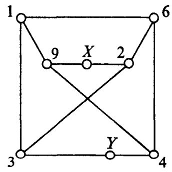

FIGURE 2

$$10x + 9y \leqq \binom{15}{2} = 105. \tag{4}$$

Together, these imply $x \leqq 6$. We consider three separate cases i ($i = 1, 2, 3$) and obtain a contradiction in each.

$$3i - 3 \leqq x \leqq 3i - 1, \qquad i = 1, 2, 3. \tag{4 + i}$$

In cases 1 and 2, the arguments involve only the numbers of edges at vertices in the non-planar graphs. Here and later, when we speak of a vertex "belonging to" a K_5 or $K_{3,3}$ graph, or homeomorph, we mean that the vertex is "essential" in the previously mentioned sense, i.e., that its valence is greater than two.

Case 1 ($x = 0, 1, 2$). There are at least $15 - 5x$ vertices which belong to no K_5 graph. Since $K_{3,3}$ is trivalent, each such vertex has at least two incident edges which are in none of the $K_{3,3}$ graphs to which it belongs. Therefore

$$y \leqq \{105 - 10x - (15 - 5x)\}/9,$$
$$x + y \leqq (90 + 4x)/9 \leqq 98/9 < 11,$$

by (5), contrary to (3).

Case 2 ($x = 3, 4, 5$). Let z be the number of vertices which belong to three K_5 graphs. Each such vertex then belongs to no other non-planar graph, while any other vertex belongs to at most four of them. Hence

$$5x + 6y \leqq 3z + 4(15 - z).$$

By (3), $6 \leqq x - z$, and since in this case $x < 6$ and z is non-negative, we again have a contradiction.

Case 3 ($x = 6$). In this case each of the graphs must be a "genuine" K_5 or $K_{3,3}$ and each vertex belongs to exactly two of the K_5 graphs. We label the six K_5 graphs as 0, 1, 2, 3, 4, 5 so that each vertex can be identified by an unordered pair of these digits. In a given $K_{3,3}$ no digit can appear three times, for suppose 05, 15, 25 do appear. These all belong to the same K_5, so must be in the same triad of the $K_{3,3}$. However, only the vertex 34 is adjacent to each of these, and not contained in a K_5 with any. Since a $K_{3,3}$ has twelve occurrences of digits, each must occur twice, both occurrences being in the same triad. Without loss we may assume that one $K_{3,3}$ has 01, 02, 12 and 34, 35, 45 as its triads. The vertex 01 must belong to a second

$K_{3,3}$, which necessarily contains at least one of 34, 35, 45 in the other triad, which is impossible. This completes the proof that K_{15} has coarseness 10.

Now assume

$$(?) \qquad c(K_{3r}) \geqq \binom{r}{2} + 1, \qquad r < 5.$$

To the r triads of points involved, adjoin $5 - r$ triads, each joined to each other and to the initial r to produce $K_{3,3}$ graphs, as in (2), and we have

$$c(K_{15}) \geqq \binom{r}{2} + 1 + \binom{5-r}{2} + r(5-r) = 11,$$

which we have just seen to be impossible. Note that (1) and (2) are equal for $r \leqq 4$.

For $6 \leqq r \leqq 9$ we are unable to decide between equality in (2) and (1).

3. For $p = 3r + 1$, we first give a maximal decomposition of K_{19} into nineteen $K_{3,3}$ graphs, due to D. Kleitman and others. Number the vertices $0, 1, 2, \ldots, 18$ and form one $K_{3,3}$ by joining 0, 1, 2 to 3, 6, 9. Permute the vertices cyclically, modulo 19, to obtain the other eighteen graphs.

To obtain a maximal decomposition of K_{28}, label the vertices with a digit from $\{0, 1, \ldots, 6\}$ and a letter from $\{A, B, C, D\}$. First form six $K_{3,3}$ graphs with the following pairs of triads

$0A\ 2B\ 1C$	$0A\ 4B\ 6C$	$0A\ 0B\ 6C$	$0A\ 6B\ 5D$	$0A\ 6B\ 6D$	$0A\ 3C\ 2D$
$1A\ 0B\ 0D$	$2A\ 3C\ 3D$	$4B\ 5C\ 4D$	$3A\ 2C\ 6D$	$5B\ 6C\ 1D$	$6B\ 1C\ 5D$

and then permute the digits cyclically modulo 7 to obtain 6×6 other graphs. These forty-two $K_{3,3}$ graphs are edge-disjoint.

We use these decompositions to prove

$$(8) \qquad c(K_p) \geqq \binom{r}{2} + 2[\tfrac{1}{3}r], \qquad p = 3r + 1 \geqq 19.$$

Let $p = 3r + 1$, $r = 3n + m$ with $0 \leqq m \leqq 2$, $n \geqq 2$. If n is even, take $\frac{1}{2}n$ copies of K_{19} with one vertex common to all, and m triads of vertices. Partition the eighteen other vertices of each K_{19} into $\frac{1}{2}n$ sets of six triads, and join each triad to every other not in the same set, giving

$$19(\tfrac{1}{2}n) + 6^2\binom{\frac{1}{2}n}{2} + 6m(\tfrac{1}{2}n) + \binom{m}{2} = \binom{r}{2} + 2n$$

$K_{3,3}$ graphs. If n is odd (and at least 3), take one K_{28} and $\frac{1}{2}(n-3)$ copies of K_{19}, with a vertex common to all, and m triads of vertices. Partition as before into $\frac{1}{2}(n-3)$ sets of six triads and a set of nine. Join the triads, giving

$$42 + 19\{\tfrac{1}{2}(n-3)\} + 54\{\tfrac{1}{2}(n-3)\} + 36\binom{\frac{1}{2}(n-3)}{2} + 3mn + \binom{m}{2} = \binom{r}{2} + 2n$$

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$K_{3,3}$ graphs, and completing the proof of (8).

Since a non-planar graph has at least nine edges,

$$(9) \qquad c(K_p) \leqq \binom{r}{2} + [\tfrac{2}{3}r], \qquad p = 3r + 1.$$

If $r \not\equiv 2 \pmod 3$ and $p \geqq 19$, the inequalities (8) and (9) imply equality in each. For $p = 9n + 7$ there is a gap of one. If equality is to be attained in (9), it can be shown that the subgraphs are all homeomorphic to $K_{3,3}$ graphs, and that just two or three are homeomorphs, the remainder being genuine $K_{3,3}$ graphs.

We now turn to the small values of r. It is readily seen that $c(K_4) = 0$, $c(K_7) = 1$. Since K_{10} has forty-five edges, if it were the union of five non-planar graphs, each would be a genuine $K_{3,3}$ graph. We show that it is not possible to obtain even four such graphs. If there were four, they would have $4 \times 6 = 24$ vertices, with three edges at each; so there are vertices which belong to three $K_{3,3}$ graphs. Label such a vertex 0 and let the others be $1, 2, \ldots, 9$. Without loss we may assume that three of the $K_{3,3}$ graphs contain vertices

$$\begin{matrix} 0 & . & . \\ 1 & 2 & 3 \end{matrix} \qquad \begin{matrix} 0 & . & . \\ 4 & 5 & 6 \end{matrix} \qquad \begin{matrix} 0 & . & . \\ 7 & 8 & 9 \end{matrix}$$

and that the first of these is either

$$\begin{matrix} 0 & 4 & 7 \\ 1 & 2 & 3 \end{matrix} \quad \text{or} \quad \begin{matrix} 0 & 4 & 5 \\ 1 & 2 & 3 \end{matrix}.$$

By considering vertex 4, we see that the second is to be completed by two of the three vertices 7, 8, 9. In the first case, $\begin{matrix} 0 & 4 & 7 \\ 1 & 2 & 3 \end{matrix}$, if 7 is included in

$$\begin{matrix} 0 & 7 & . \\ 4 & 5 & 6 \end{matrix}, \quad \text{then} \quad \begin{matrix} 0 & . & . \\ 7 & 8 & 9 \end{matrix}$$

cannot be completed, since 7 is already connected to 1, 2, 3, 4, 5, and 6. Hence the first two are

$$\begin{matrix} 0 & 4 & 7 \\ 1 & 2 & 3 \end{matrix} \quad \text{and} \quad \begin{matrix} 0 & 8 & 9 \\ 4 & 5 & 6 \end{matrix}$$

and again $\begin{matrix} 0 & . & . \\ 7 & 8 & 9 \end{matrix}$ cannot be completed. Without loss we may therefore assume that three of the $K_{3,3}$ graphs are

$$\begin{matrix} 0 & 4 & 5 \\ 1 & 2 & 3 \end{matrix} \qquad \begin{matrix} 0 & 7 & 8 \\ 4 & 5 & 6 \end{matrix} \qquad \begin{matrix} 0 & 1 & 2 \\ 7 & 8 & 9 \end{matrix}.$$

The remaining edges form the planar graph shown in Figure 3, so that four edge-disjoint $K_{3,3}$ graphs cannot be found. However, $c(K_{10}) = 4$, since we

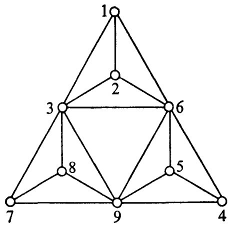

FIGURE 3

may decompose K_{10} into four $K_{3,3}$ homeomorphs, for example, the first four in the scheme

0	4	5	0	7	8	0	2	3	3	6	9
		6					1		5		8
1	2	3	4	5	6	7	8	9	4	1	2
10	14	15	8	9	10	10	12	13	6	7	13
		7					11		9		15
11	12	13	7	14	15	8	9	6	12	11	14
0	1	2	0	1	2	3	4	5	3	4	5
10	11	12	13	14	15	10	11	12	13	14	15

where the three vertices in the top row are each joined to the three in the last row, except that 5 is joined to 3 via 6 in the first, 2 is joined to 8 via 1 in the third, and the fourth is illustrated by Figure 2, with $X = 8$ and $Y = 5$. To see that $c(K_{13}) \geqq 7$, adjoin three new vertices, forming three more $K_{3,3}$ graphs on joining them to the triads 123, 456, and 789. The whole of the above scheme shows that $c(K_{16}) \geqq 12$. The second set of four $K_{3,3}$ homeomorphs is obtained from the first by adding 10 to the digits 0, 1, 2, 3, 4, 5 and permuting the digits 6, 7, 8, 9 cyclically. The $K_{6,6}$ graph formed from the sets of vertices 0, 1, 2, 3, 4, 5 and 10, 11, 12, 13, 14, 15 furnishes the final four $K_{3,3}$ graphs of the scheme. All edges are used except the four (1, 7), (5, 9), (6, 15), and (8, 11).

4. For the case $p = 3r + 2$, the coarseness is

(10) $$c(K_p) = \binom{r}{2} + [(14r + 1)/15], \qquad p = 3r + 2.$$

We first show that it is no greater than this, by obtaining a contradiction to the assumption that we can find $x\ K_5$ graphs and $y\ K_{3,3}$ graphs, as in § 2, such that

(11) (?) $$x + y > \binom{r}{2} + (14r + 1)/15.$$

As in (4),

(12) $$10x + 9y \leqq \binom{3r+2}{2},$$

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and (11) and (12) imply

$$x < \tfrac{1}{5}(3r + 2). \tag{13}$$

There are at least $3r + 2 - 5x$ vertices which belong to no K_5 graph. Since each vertex has $3r + 1$ incident edges, the trivalence of $K_{3,3}$ ensures that each such vertex has an unused incident edge.

Therefore,

$$10x + 9y \leqq \tfrac{1}{2}(9r^2 + 9r + 2) - \tfrac{1}{2}(3r + 2 - 5x),$$

$$x + y \leqq \binom{r}{2} + (15r + 3x)/18 < \binom{r}{2} + (14r + 1)/15$$

by (13), contradicting (11).

To complete the proof of (10) we next decompose K_{50} into 135 non-planar graphs. Join i, $i + 10$, $i + 20$, $i + 30$, $i + 40$ for $i = 0, 1, 2, \ldots, 9$ to form ten K_5 graphs. Form one hundred $K_{3,3}$ graphs

$$\begin{matrix} i & i+1 & i+2 \\ i+3 & i+6 & i+9 \end{matrix} \qquad \begin{matrix} i & i+1 & i+2 \\ i+13 & i+16 & i+19 \end{matrix}$$

by letting i run through a complete set of residues modulo 50. Finally, form twenty-five more $K_{3,3}$ graphs

$$\begin{matrix} i & i+1 & i+2 \\ i+23 & i+26 & i+29 \end{matrix} \qquad (i = 0, 1, 2, \ldots, 24),$$

where the additions are again modulo 50. This proves (10) for $r = 16$. For $r = 1$, choose K_5 and proceed inductively up to $r = 15$. First, if there is an edge AB of K_p which is not in a non-planar graph, we add three points C, D, E forming a K_5 graph with A, B and $\frac{1}{3}(p - 2)$ $K_{3,3}$ graphs with triads of other points of K_p. In this case

$$c(K_{p+3}) \geqq c(K_p) + \tfrac{1}{3}(p + 1).$$

If there is no such edge AB, we take just the $\frac{1}{3}(p - 2)$ $K_{3,3}$ graphs, and in this case

$$c(K_{p+3}) \geqq c(K_p) + \tfrac{1}{3}(p - 2).$$

This gives the following table, with $p = 3r + 2$,

r	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
$c(K_p)$	1	2	5	9	14	20	27	35	44	54	65	76	89	103	118

The only constructions of the second type are for $r = 2$ and $r = 12$.

To confirm (10) generally, let $p = 3r + 2$, $r = 15n + m + 1$, $0 \leqq m \leqq 14$, $n \geqq 0$. Take n copies of the K_{50} construction, together with one K_{3m+5} (as above), so that each of these $n + 1$ complete graphs have a K_5 graph in common. Partition the remaining vertices of the complete graphs into n sets

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of fifteen triads and one set of m triads, and join each triad to every other not in the same set, to produce a total of

$$134n + 15^2\binom{n}{2} + 15mn + \binom{m+1}{2} + (14(m+1)+1)/15 = \binom{r}{2} + (14r+1)/15$$

non-planar graphs.

5. We conclude with a summary of what is known about the coarseness of the complete graph.

For $p = 3r$, $\quad c(K_p) = \binom{r}{2} + [\frac{1}{5}r], \qquad r \geqq 10,$

$$c(K_p) = \binom{r}{2}, \qquad r \leqq 5.$$

For $p = 3r+1$, $\quad \binom{r}{2} + 2[\frac{1}{3}r] \leqq c(K_p) \leqq \binom{r}{2} + [\frac{2}{3}r], \qquad r \neq 3, 4,$

$$c(K_{10}) = 4, \qquad 7 \leqq c(K_{13}) \leqq 8.$$

For $p = 3r+2$, $\quad c(K_p) = \binom{r}{2} + (14r+1)/15.$

We conjecture the following results in the only cases which are still in doubt:

	p	$= 13$	18	21	24	27	$9n+7$
(?)	$c(K_p)$	$= 7$	15	21	28	36	$\frac{1}{2}(9n^2 + 13n + 2)$,

and note that if any of these are incorrect, then the true values are greater by one.

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