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A HARDY-DAVIES-PETERSEN INEQUALITY FOR A CLASS OF MATRICES

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1. Let ω be the set of all real sequences $a \equiv \{a_n\}_{n \geqq 0}$. Unless otherwise indicated operations on sequences will be coordinatewise. If any component of a has the entry ∞ the corresponding component of a^{-1} has entry zero. The *convolution* of two sequences s and q is given by $s * q = \{\sum_{k=0}^{n} s_k\, q_{n-k}\}$. The *Toeplitz martix* associated with sequence s is the lower triangular matrix $T_s = \{t_{nk}\}_{n,k \geqq 0}$ defined by $t_{nk} = s_{n-k}$ $(n \geqq k)$, $t_{nk} = 0$ $(n < k)$. It can be seen that $T_s(q) = s * q$ for each sequence q and that T_s is invertible if and only if $s_0 \neq 0$. We shall denote a diagonal matrix with diagonal sequence s by D_s.

Let l_p and c_0 be sequence spaces with their usual significance. $\|\ \|_p$ would denote the l_p norm $1 \leqq p \leqq \infty$. If a sequence $u \notin l_p$ we shall take $\|u\|_p = \infty$. If some of the entries of u are $\pm\infty$ we shall also take $\|u\|_p = \infty$. Throughout K will be a positive constant.

For any matrix $B = (b_{nk})$ $(n, k = 0, 1, \ldots)$ and two sequences u and v of real numbers we shall call an inequality of the form

$$(1) \qquad \|Bu\|_p \leqq K\|vu\|_p$$

and *HPD inequality* (see [**5**, § 0] for nomenclature).

A non-negative sequence s which satisfies $s_n \leqq K\, s_m$ $(m \leqq n)$ will be called an *essentially non-increasing sequence* with K (e.n.i. with K for brevity). It should be noted that a non-negative sequence is e.n.i. if and only if it is equivalent to a non-increasing, non-negative sequence.

Unless otherwise mentioned $A = (a_{nk})$ will be a lower triangular matrix with non-negative entries and positive entries on the main diagonal. If for some K all the columns of A starting from the main diagonal and scanning downward are e.n.i. with K then A will be called *admissible with* K. As in [**5**], if f is a positive sequence we shall call the pair (A, f) an *MDP matrix with* K if and only if $D_f A$ is an admissible matrix with constant K.

2. Introduction. Davies and Petersen [**2**, Theorem 2], generalized Hardy's inequality [**3**, pp. 239-242] which can be put in the following form (see [**5**, Theorem 5.5]):

THEOREM A ([**5**, Theorem 5.5; **2**, Theorem 2]). *Suppose* $1 < p < \infty$ *and* (A, f) *is an MDP matrix with* K. *Then for any non-negative sequence* u

$$(2) \qquad \|A(f^{-p}b^{-1}d^{-1}u)\|_p \leqq p\, K^p\|u\|_p$$

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where d is the diagonal sequence of A and $b_n = \sum_{k=n}^{\infty} f_k^{-p}$. (If $f^{-1} \notin l_p$, then b^{-1} is the zero sequence).

As a supplement to Theorem A the following theorem was established:

THEOREM B [**5**, Theorem 5.8]. *Suppose A is admissible with K, b is the sequence of l_1 norms of the columns of A and $1 < p < \infty$. Then for any non-negative sequence u*

$$(3) \qquad \|A(b^{-1}u)\|_p \leqq p\,K^{p-1}\|u\|_p.$$

The following result is central to the concerns of this paper.

THEOREM 1. *Suppose s is positive e.n.i. with K and the infinite matrix A is given by $A = D_a T_s$ where a is a positive sequence, D is a diagonal matrix and T is a Toeplitz matrix. Then for $1 < p < \infty$, and any non-negative sequence u,*

$$(4) \qquad \|A(c^{-1}a^{p-1}u)\|_p \leqq p\,K^{p-1}\|u\|_p,$$

where c is the sequence defined by

$$(5) \qquad c_k = \sum_{n=k}^{\infty} s_{n-k}a_n^{\,p} = \sum_{n=0}^{\infty} s_n a_{n+k}^{\,p}.$$

Remarks. The entries of c are allowed to be ∞. Also c depends on p as well as on a and s.

In § 3 we shall prove Theorem 1. In § 4 and § 5 we shall compare Theorem 1 with similar *HPD* inequalities which are obtained by setting $A = D_a T_s$ in Theorem A and Theorem B respectively. In § 6 we shall apply Theorem 1 to generalized Nörlund matrix introduced by Borwein (**1**). In § 7 we construct two sequence spaces and with the help of Theorem 1 we shall show that both the sequence spaces contain l_p as a linear subspace.

3. We shall need the following lemma for the proof of Theorem 1.

LEMMA 1 [**2**, Lemma 2]. *If $1 < p < \infty$ and $z_0, z_1, \ldots, z_n$ are non-negative real numbers, then*

$$(6) \qquad \left(\sum_{k=0}^{n} z_k\right)^p \leqq p \sum_{k=0}^{n} z_k \left(\sum_{m=0}^{k} z_m\right)^{p-1}.$$

Since the proof of Theorem 1 uses ideas similar to that used by Davies and Petersen [**2**, Theorem 2], we briefly sketch the proof for the sake of completeness.

Proof of Theorem 1. By Lemma 1 and the definition of the matrix $A = D_a T_s$

we have for any natural number N,

$$\sum_{n=0}^{N} a_n^{\,p}\left(\sum_{k=0}^{n} s_{n-k}c_k^{-1}a_k^{p-1}u_k\right)^p$$
$$\leqq p\sum_{n=0}^{N} a_n^{\,p}\sum_{k=0}^{n} s_{n-k}c_k^{-1}a_k^{p-1}u_k\left(\sum_{m=0}^{k} s_{n-m}c_m^{-1}a_m^{p-1}u_m\right)^{p-1}$$
$$\leqq pK^{p-1}\sum_{n=0}^{N} a_n^{\,p}\sum_{k=0}^{n} s_{n-k}c_k^{-1}a_k^{p-1}u_k\left(\sum_{m=0}^{k} s_{k-m}c_m^{-1}a_m^{p-1}u_m\right)^{p-1}$$
$$= pK^{p-1}\sum_{k=0}^{N}\left(\sum_{m=0}^{k} s_{k-m}c_m^{-1}a_m^{p-1}u_m\right)^{p-1} c_k^{-1}a_k^{p-1}u_k\sum_{n=k}^{N} s_{n-k}a_n^{\,p}$$
$$\leqq pK^{p-1}\sum_{n=0}^{N}\left(a_n\sum_{k=0}^{n} s_{n-k}c_k^{-1}a_k^{p-1}u_k\right)^{p-1} u_n.$$

Now taking p' to be such that $p^{-1} + p'^{-1} = 1$ and using Hölder's inequality and dividing both sides of the resulting inequality by

$$\left(\sum_{n=0}^{N}\left(a_n\sum_{k=0}^{n} s_{n-k}c_k^{-1}a_k^{p-1}u_k\right)^p\right)^{1/p'}. \tag{7}$$

we obtain

$$\left(\sum_{n=0}^{N} a_n^{\,p}\left(\sum_{k=0}^{n} s_{n-k}c_k^{-1}a_k^{p-1}u_k\right)^p\right)^{1/p} \leqq pK^{p-1}\left(\sum_{n=0}^{N} u_k^{\,p}\right)^{1/p}. \tag{8}$$

It must be pointed out that the division by (7) in above requires that N be sufficiently large so that $c_k^{-1}u_k > 0$ for some $k \leqq n \leqq N$. If $c_k^{-1}u_k = 0$ for all k, the inequality to be proven is trivial.

Now on letting $N \to \infty$ in (8) we obtain the required result.

COROLLARY 2.3. *If $0 < c_k < \infty$ for all k, then*

$$\|Au\|_p \leqq p\, K^{p-1}\|ca^{1-p}u\|_p \quad (1 < p < \infty) \tag{9}$$

for any non-negative sequence u.

Proof. Replace u in the theorem by $ca^{1-p}u$.

4. Setting $A = D_aT_s$ in Theorem A we obtain the following corollary.

COROLLARY 1. *If s is e.n.i. with K, a is positive, $A = D_aT_s$, and $1 < p < \infty$, then for any non-negative sequence u,*

$$\|A\,(a^{p-1}b^{-1}u)\|_p \leqq p\, K^p s_0\|u\|_p, \tag{10}$$

with

$$b_n = \sum_{k=n}^{\infty} a_k^{\,p}.$$

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We observe that if s is e.n.i. with K then

$$c_k = \sum_{n=0}^{\infty} s_n a_{n+k}^p \leqq K s_0 \sum_{n=k}^{\infty} a_n^p = K b_k s_0$$

and consequently $\sup_n (c_n/b_n) < \infty$. Thus Theorem 1 implies Corollary 1. To show that Theorem 1 is in fact an improvement over Corollary 1 we furnish the following example where $\sup_n (c_n/b_n) < \infty$ but $\sup_n (b_n/c_n)$ is not finite.

Example 1. Let a be any positive sequence such that $a \in l_\infty \backslash \bigcup_{p>0} l_p$ and s be any decreasing positive sequence in l_1. Then b^{-1} is the zero sequence. Hence $\sup_n (c_n/b_n) < \infty$ although $\sup_n (b_n/c_n)$ is not finite.

5. Setting the matrix $A = D_a T_s$ in Theorem B we have the following corollary.

COROLLARY 2. *If a, s are positive sequences such that*

$$a_n s_{n-k} \leqq K a_m s_{m-k} \quad (k \leqq m \leqq n),$$

$1 < p < \infty$, *and the matrix $A = D_a T_s$, then for non-negative u,*

$$\text{(11)} \quad ||A(b^{-1}u)||_p \leqq p K^{p-1} ||u||_p,$$

where

$$\text{(12)} \quad b_k = \sum_{n=k}^{\infty} a_n s_{n-k}.$$

Even though the hypothesis of Corollary 2 is different from that of Theorem 1 it is interesting to know how the inequality (4) compares with the inequality (11) for sequences a and s which satisfy the hypotheses of both Corollary 2 and Theorem 1.

The following example illustrates that for some choices of a and s, inequality (4) is better than that of (11).

Example 2. Les $s_n = 1$ for all n and $a_n = (n+1)^{-1}$. Then

$$b^{-1} = \left\{ \left(\sum_{n=k}^{\infty} s_{n-k} a_n \right)^{-1} \right\}_k = \{0\},$$

while

$$a^{p-1} c^{-1} = a^{p-1} \left\{ \left(\sum_{n=k}^{\infty} s_{n-k} a_n^p \right)^{-1} \right\}_k$$

is not so for $1 < p < \infty$.

Remark 1. It is possible to construct sequences a and s such that b^{-1} is not the zero sequence but still $\sup_n (b_n^{-1}/c_n^{-1} a_n^{p-1}) < \infty$ although

$$\sup_n (c_n^{-1} a_n^{p-1}/b_n^{-1})$$

is not finite.

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Remark 2. When $s = e$, the sequence whose every entry is 1, the inequality (4) and inequality (10) are identical. Hence Example 2 shows that inequality (10) can be an improvement over (11) for matrices $A = D_a T_s$ satisfying the hypotheses required for the validity of (10) and (11).

Our next example would show that for some choices of a and s satisfying the hypotheses of Theorem 1 and Corollary 2 the inequality (11) can be better than (4).

Example 3. Let $s_n = r^n$ $(0 < r < 1)$ and $a_n = r^{-n}(n + 1)^{-\alpha}$ $(\alpha > 1)$. Then $\{r^n a_n\}$ is decreasing and $\sum r^n a_n < \infty$ which implies that the sequence b as defined in (12) is such that b^{-1} has no zero entries while $c^{-1} = \{0\}$ for each $p > 1$. Thus $\sup_n(c_n^{-1}a_n^{p-1}/b_n^{-1}) < \infty$ while $\sup_n(b_n^{-1}/c_n^{-1}a_n^{p-1})$ is not finite.

Remark 3. In view of Examples 2 and 3 we observe that there exist sequences a and s satisfying the requirements of Theorem 1 and Corollary 2 for which one of Theorem 1 or Corollary 2 gives an inequality better than the other. This leads us to the following.

CONJECTURE. *There exist sequences a and s satisfying the requirements of Theorem* 1 *and Corollary* 2 *for which the inequalities* (4) *and* (11) *are incomparable in the sense that neither* $\sup_n(a_n^{p-1}c_n^{-1}/b_n^{-1}) < \infty$ *nor*

$$\sup_n(b_n^{-1}/a_n^{p-1}c_n^{-1}) < \infty$$

for all $p > 1$.

6. For sequences s and q, Borwein [**1**] defined the *generalized Nörlund method* (N, s, q) with the help of the matrix $N = (\eta_{nk})$ given by $\eta_{nk} = s_{n-k}q_k/r_n$ $(k \leqq n)$ and $\eta_{nk} = 0$ $(k > n)$ where $r_n = (s * q)_n$. Clearly the matrix $N = D_{r^{-1}} T_s D_q$. In what follows we shall assume s and q to be positive sequences.

If in Theorem 1 and Corollary 1 we take the matrix N in place of A we obtain the following corollaries.

COROLLARY 3. *If s is e.n.i. with constant K, $1 < p < \infty$, then*

$$(13) \quad ||N(c^{-1}r^{1-p}q^{-1}u)||_p \leqq p\, K^{p-1}||u||_p$$

for non-negative u, with

$$c_k = \sum_{n=k}^{\infty} s_{n-k}r_n^{-p}.$$

COROLLARY 4. *If s is e.n.i. with constant K, $1 < p < \infty$, then*

$$(14) \quad ||N(b^{-1}r^{1-p}q^{-1}u)||_p \leqq p\, K^p s_0||u||_p,$$

with

$$b_n = \sum_{k=n}^{\infty} r_k^{-p}.$$

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Since the inequalities (13) and (14) hold with r, s, q being any positive sequences not necessarily related by $r = s * q$, we observe that, in view of Example 1 with the sequence a replaced by r^{-1}, inequality (13) is an improvement over (14) when r, s, q are not related. A natural question which arises at this stage is the following:

Is (13) an improvement over (14) even for the more restricted class of (N, s, q) matrices?

To show that the answer is in the affirmative we are required to provide examples of sequences b and c such that $\sup_n (c_n/b_n) < \infty$ while $\sup_n (b_n/c_n)$ is not finite under the requirement that $q = T_s^{-1}(r)$ be a positive sequence. We shall need the following result.

LEMMA 2. *If r, s are positive sequences, r is non-decreasing, and s is non-increasing, then $T_s^{-1}(r)$ is non-negative. If either r or s is strictly monotone, then $T_s^{-1}(r)$ is positive.*

Proof. Let $q = T_s^{-1}(r)$. Clearly $q_0 = r_0 s_0^{-1} > 0$. Assuming that $q_0, \ldots, q_{n-1} \geqq 0$, we have

$$\begin{aligned} q_n &= s_0^{-1}(r_n - (q_{n-1}s_1 + \ldots + q_0 s_n)) \\ &\geqq s_0^{-1}(r_n - (q_{n-1}s_0 + \ldots + q_0 s_{n-1})) \\ &= s_0^{-1}(r_n - r_{n-1}) \geqq 0. \end{aligned}$$

If either r or s is strictly monotone, we assume inductively that $q_0, \ldots, q_{n-1} > 0$, and then either the first or the second inequality in the preceding argument is strict.

This completes the proof.

The following example answers the question raised.

Example 4. Set $s_n = r_n^{-1} = (n + 1)^{-1}$ for all $n \geqq 0$. Clearly s is decreasing. For $1 < p < \infty$, $b_n = \sum_{k=n}^{\infty} r_k^{-p} = O((n + 1)^{1-p})$. By Lemma 2, $q = T_s^{-1}(r)$ is a positive sequence. Choose any $m > 1$. Then for $1/m + 1/m' = 1$ we have

$$(15)\quad c_n/b_n = O\left((n+1)^{p-1} \sum_{k=0}^{\infty} s_k r_{n+k}^{-p}\right) = O\left((n+1)^{p-1}\left(\sum_{k=1}^{\infty} k^{-m'}\right)^{1/m'} \times \left(\sum_{k=n+1}^{\infty} k^{-mp}\right)^{1/m}\right) = O((n+1)^{1/m-1}).$$

Thus $b^{-1}/c^{-1} \in c_0$ while $\sup_n (c_n/b_n)$ is not finite.

7. We shall apply Theorem 1 to two sequence spaces which we construct in this section.

7.1. Let us write

$$N(u) = \frac{1}{\log (n+2)} \sum_{k=0}^{n} \frac{u_k}{n - k + 1}.$$

Let $H_p = \{u \in \omega | N(|u|) \in l_p\}$. Since in [**4**] it is shown that for any lower triangular matrix A, finite sequence u, $A(|u|) \in l_p$ if and only if the columns of the matrix lie in l_p, we observe that the sequence space H_p is non-trivial for $1 < p < \infty$. One can norm the sequence space H_p by specifying

$$(16)\quad ||u||_{H_p} = \left(\sum_{n=0}^{\infty}\left(\frac{1}{\log(n+2)}\sum_{k=0}^{n}\frac{|u_k|}{n-k+1}\right)^p\right)^{1/p} \quad (1 < p < \infty).$$

Since

$$\sum_{n=k}^{\infty}\frac{1}{(n-k+1)(\log(n+2))^p} = \left(\sum_{n=k}^{2k}+\sum_{2k+1}^{\infty}\right) \times \frac{1}{(n-k+1)(\log(n+2))^p} = O((\log(k+2))^{-p+1}),$$

sequence c of Corollary 3 is of order $O\{(\log(k+2))^{-p+1}\}$, $(1 < p < \infty)$. On taking $q_k = 1$ for all k in Corollary 3, we obtain the inequality

$$(17)\quad \sum_{n=0}^{\infty}\left(\frac{1}{\log(n+2)}\sum_{k=0}^{n}\frac{|u_k|}{(n-k+1)}\right)^p \leqq C(p)\sum_{n=0}^{\infty}|u_k|^p \quad (1 < p < \infty),$$

where $C(p)$ is a constant depending only on p.

In view of the inequality (17), and (16), we can conclude that H_p contains l_p $(1 < p < \infty)$ as a linear subspace and the injection is continuous. Questions with regard to continuity of coordinate linear functionals on H_p, its completeness and separability etc. can be investigated along the lines of [**4**] with appropriate modifications.

7.2. Let

$$\sigma_n^{\alpha}(u) = \frac{1}{A_n^{\alpha}}\sum_{k=0}^{n} A_{n-k}^{\alpha-1}u_k, \quad (\alpha > -1)$$

where A_n^{α} is defined by the identity,

$$\sum_{n=0}^{\infty} A_n^{\alpha}x^n = (1-x)^{-\alpha-1} \quad (|x| < 1).$$

Let $\mathrm{ces}_{\alpha,p} = \{u \in \omega | \sigma(|u|) \in l_p\}$. It is easy to show for $1 < p < \infty$, $\mathrm{ces}_{\alpha,p}$ is non-trivial and can be normed by prescribing for each $u \in \mathrm{ces}_{\alpha,p}$,

$$||u||_{\mathrm{ces}_{\alpha,p}} = \left(\sum_{n=0}^{\infty}\left(\frac{1}{A_n^{\alpha}}\sum_{k=0}^{n} A_{n-k}^{\alpha-1}u_k\right)^p\right)^{1/p} \quad (1 < p < \infty).$$

By taking $q_n = 1$, $s_n = A_n^{\alpha-1}$ in Corollary 3 we conclude that for $0 < \alpha \leqq 1$ and a constant $B(p, \alpha)$ depending upon p and α,

$$(18)\quad \sum_{n=0}^{\infty}\left(\frac{1}{A_n^{\alpha}}\sum_{k=0}^{n} A_{n-k}^{\alpha-1}|u_k|\right)^p \leqq B(p,\alpha)\sum_{n=0}^{\infty}|u_n|^p,$$

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since

$$
\begin{aligned}
c_k &= \sum_{n=k}^{\infty} \frac{A_{n-k}^{\alpha-1}}{(A_n^{\alpha})^p} = \sum_{n=k}^{\infty} A_{n-k}^{\alpha-1} n^p \left(\int_0^1 x^{n-1}(1-x)^{\alpha} dx \right)^p \\
&\leqq \sum_{n=k}^{\infty} A_{n-k}^{\alpha-1} n \int_0^1 (1-x)^{\alpha p} x^{n-1} dx \\
&= \alpha p \sum_{n=0}^{\infty} A_n^{\alpha-1} \int_0^1 x^{n+k}(1-x)^{\alpha p-1} dx \\
&= \alpha p / (\alpha(p-1) A_k^{\alpha(p-1)}).
\end{aligned}
$$

In view of (18), $\text{ces}_{\alpha,p}$ $(0 < \alpha \leqq 1,\ 1 < p < \infty)$ contains l_p as a linear subspace. Apart from the constant, the case $\alpha = 1$ of our inequality is essentially Hardy's inequality and the sequence space $\text{ces}_{1,p}$ is the well-known ces_p (see [**6**]). Properties of the sequence space $\text{ces}_{\alpha,p}$ will be investigated elsewhere.

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Added in proof. M. Izumi, S. Izumi and G. M. Petersen (Tohoku Math. J., *21* (1969), p. 611) have obtained the following inequality:

$$
(19) \quad \sum_{n=1}^{\infty} \left(\frac{1}{\log (n+1)} \sum_{k=1}^{n} (n-k+1)^{-1} |u_k| \right)^p \leqq A(p) \sum_{n=1}^{\infty} (a_n^p n^{p-1} / \log (n+1)) \quad (p > 1)
$$

It can be seen that our inequality (17) is sharper than (19). Also we would like to remark that our inequality (18) is contained in G. H. Hardy, *An inequality for Hausdorff means*, J. London Math. Soc. (1943), p. 49 where the best possible constant is also given.

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