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A characterization of pre-near-standardness in locally convex linear topological spaces

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Let X be a locally convex linear topological space. A point z in an ultralimit enlargement of X is pre-near-standard if and only it is finite and for every equicontinuous subset S' of the dual space X', a point z' belongs to ${}^*S' \cap \mu_{\sigma(X',X)}(0)$ only if $z'(z)$ is infinitesimal.

1. Introduction

In a recent paper Luxemburg [2] obtains a characterization of pre-near-standardness for normed spaces leading to several interesting applications in the standard theory. Our object here will be to derive a characterization for locally convex spaces generalizing ([2], 3.17.2). An introduction to the theory of non-standard analysis can be found in [3] and [4]. In addition we make implicit use of ultralimit (or suitably saturated) enlargements. All the necessary metamathematical background will be found in [2]. For the topological vector space concepts see for example Köthe [1].

Some definitions and notations will now be given. Let Ω be a family of subsets of a given set. We denote the (*intersection*) *monad* of Ω by $\mu(\Omega)$ (see [2]). By definition, $\mu(\Omega) = \bigcap({}^*E : E \in \Omega)$.

When Ω is the filter F of neighbourhoods of a point x in a topological space (X, τ), we often write $\mu_\tau(x)$ for $\mu(F)$. The definition of $\mu_\tau(x)$ accords with that of Robinson [4].

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If (X, τ) is a linear topological space, and $\{N_\delta : \delta \in \Delta\}$ the set of all neighbourhoods of the origin, then the set of vicinities $V_\delta = \{(x, y) : x-y \in N_\delta\}$ is a uniformity u on X which induces the topology τ. For any point $z \in {}^*X$ we define

$$\mu_u(z) = \bigcap\left({}^*V(z) : V \in u\right) ,$$

where ${}^*V(z) = ({}^*V)(z) = \{x : x-z \in {}^*V\}$.

We recall that a proper filter F of subsets of X is a *Cauchy filter* if for each $V \in u$ there exists $F \in F$ such that $F \times F \subset V$, or equivalently, if and only if $\mu(F) \times \mu(F) \subset \mu(u)$. A Cauchy filter is called *minimal* if it does not properly contain a Cauchy filter.

The point $z \in {}^*X$ is called *pre-near-standard* if there exists a filter F of subsets of X (necessarily a Cauchy filter) such that $\mu(F) \subset \mu_u(z)$. Clearly every near-standard point is pre-near-standard. The concept of pre-near-standardness was first introduced by Luxemburg [2]. A necessary and sufficient condition that a uniform space A be precompact is that every point of ${}^*A$ be pre-near-standard [2].

We shall make use of the following result:

THEOREM 1.1 ([2], 3.12.1). *A point $z \in {}^*X$ is pre-near-standard if and only if there exists a minimal Cauchy filter F such that* $\mu(F) = \mu_u(z)$.

Throughout the remainder of this paper the following notation will be used. Except where stated otherwise, X will denote a locally convex linear topological space whose generating family of seminorms is $(p_\lambda : \lambda \in \Lambda)$. X_λ will denote the seminormed space (X, p_λ), and S'_λ the unit ball of the topological dual X'_λ of X_λ. For an arbitrary topological space (X, τ), $\mu_\tau(x)$ denotes the τ-monad of $x \in X$. In particular, $\mu_d(x)$ is the monad of x in the discrete topology, and $\mu_{\sigma(X',X)}(0)$ is the monad of the origin of X' in its weak star topology. For any pair of real numbers a, b the relation $a =_1 b$ means that $a - b$ is infinitesimal.

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A point $z \in {}^*X$ will be called *finite* if $p_\lambda(z)$ is finite for every standard λ in Λ.

2. The main theorem

Our principal result will be the following:

THEOREM 2.1. *Let* z *be a point of* ${}^*X$. *Then* z *is pre-near-standard if and only if it is finite and for every equicontinuous set* S' *in* X', $z' \in {}^*S' \cap \mu_{\sigma(X',X)}(0)$ *implies* $z'(z) =_1 0$.

The theorem will be proved by means of some auxiliary theorems and lemmas.

Let $z \in {}^*X$ and let $\mathcal{B}(z)$ denote the collection of all finite intersections of sets $\{x : p_\lambda(z-x) < \varepsilon\}$ where λ and $\varepsilon > 0$ are both standard.

THEOREM 2.2. *The point* z *is pre-near-standard if and only if each* B *in* $\mathcal{B}(z)$ *contains a standard point.*

Proof. Suppose z is pre-near-standard. It is clear that $\mu_u(z) = \{x : p_\lambda(x-z) =_1 0$ for all standard $\lambda\}$. By Theorem 1.1, $\mu_u(z)$ is a filter monad and therefore $\mu_d(z) \subset \mu_u(z)$. Let $B \in \mathcal{B}(z)$. Then B is by definition internal and $\mu_u(z) \subset B$. Hence, since $\mu_d(z) \subset B$, B contains a standard point by ([2], 2.8.1).

Conversely, suppose that each member of $\mathcal{B}(z)$ contains a standard point. For each $B \in \mathcal{B}(z)$ select a standard $x_B \in B$. Now $\mathcal{B}(z)$ becomes a directed set if we define $B_1 \le B_2 \iff B_2 \subset B_1$ and then $(x_B : B \in \mathcal{B}(z))$ becomes a net. Let F be the associated filter of subsets $\{x_{B'} : B' \ge B\}$.

We will show that F is a Cauchy filter. We have

$$\mu(F) = \{x_B : B \in {}^*\mathcal{B}(z),\ B \ge B' \text{ for all standard } B'\} .$$

Therefore

$$x \in \mu(F) \Rightarrow x \in B \text{ for every standard } B \text{ in } {}^*\mathcal{B}(z)$$
$$\Rightarrow p_\lambda(x-z) < \varepsilon \text{ for all standard } \lambda \text{ and } \varepsilon > 0$$
$$\Rightarrow p_\lambda(x-z) =_1 0 \text{ for all standard } \lambda .$$

Hence if $x, y \in \mu(F)$ we have

$$p_\lambda(x-y) \le p_\lambda(x-z) + p_\lambda(z-y) =_1 0 ,$$

whence $(x, y) \in \mu(u)$. Consequently $\mu(F) \times \mu(F) \subset \mu(u)$, that is, F is Cauchy.

Now $x \in \mu(F) \Rightarrow p_\lambda(x-z) =_1 0$ for all standard $\lambda \Rightarrow x \in \mu_u(z)$. Thus $\mu(F) \subset \mu_u(z)$ showing that z is pre-near-standard.

We remark that Theorem 2.2 implies ([2], 3.17.1).

COROLLARY 2.3. *Every pre-near-standard point is finite.*

PROPOSITION 2.4. *If* $z \in {}^*X$ *is finite, then* $f(z)$ *is also finite for every standard* $f \in X'$.

Proof. Let $z \in {}^*X$ be finite, and let $f \in X'$. If $f(z)$ is infinite, then for all standard λ, $p_\lambda\big(z/f(z)\big) =_1 0$ since $p_\lambda(z)$ is finite. Thus $z/f(z) \in \mu_\tau(0)$. But $f\big(z/f(z)\big) = 1$, contradicting the continuity of f, ([3], 5.4.1).

DEFINITION 2.5. Let z be a finite point of ${}^*X$. The functional $\mathrm{st}_\omega(z)$, (*cf.* [2], p. 83), is defined by

$$\mathrm{st}_\omega(z)(f) = \mathrm{st}f(z) \quad \text{for } f \in X' ,$$

where $\mathrm{st}f(z)$ denotes the standard part of $f(z)$, ([4], p. 57). By Proposition 2.4, $\mathrm{st}_\omega(z)$ is a linear functional on X'.

We make use of the following elementary result:

LEMMA 2.6. *Let* S' *be a uniform space, and* $\big(g_\delta : \delta \in \Delta\big)$ *a net of continuous mappings of* S' *into the space of real numbers such that* $g_\delta \to g$ *uniformly on* S'. *Then* g *is continuous.*

THEOREM 2.7. *If* $z \in {}^*X$ *is pre-near-standard, then* $\mathrm{st}_\omega(z)$ *is*

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continuous in the $\sigma(X', X)$ *topology on every equicontinuous subset of* X' .

Proof. Let $z \in {}^*X$ be pre-near-standard and let S' be an equicontinuous subset of X' . By Theorem 2.2, for each B in $\mathcal{B}(z)$ we can select a standard x_B in B . Then $\hat{x}_B$ is $\sigma(X', X)$ continuous on S' . By Lemma 2.6 it is sufficient to show that $\hat{x}_B$ converges uniformly to $\mathrm{st}_w(z)$ on S' . This will follow if we can show that for given $\varepsilon > 0$, there exists $B \in \mathcal{B}(z)$ such that

$$B' \subset B \Rightarrow |\hat{x}_{B'}(f) - \mathrm{st}_w(z)(f)| < \varepsilon \quad \text{for all} \quad f \in S' .$$

Let N be a 0-neighbourhood such that

$$x - y \in N \Rightarrow |f(x) - f(y)| < \varepsilon/2 \quad \text{for all} \quad f \in S'$$

and such that N contains an element B_0 of $\mathcal{B}(0)$. Set $z + B_0 = B \in \mathcal{B}(z)$. Then for all $B' \subset B$, $x_{B'} - z \in B_0 \subset {}^*N$, whence $|f(x_{B'}) - f(z)| < \varepsilon/2$ $(f \in S')$. Consequently

$$|\hat{x}_B(f) - \mathrm{st}_w(z)(f)| = |f(x_B) - \mathrm{st}f(z)| \le$$

$$|f(x_{B'}) - f(z)| + |f(z) - \mathrm{st}f(z)| \le_1 \varepsilon/2 < \varepsilon ,$$

as required.

THEOREM 2.8. *Let* $z \in {}^*X$ *be pre-near-standard, and let* S' *be an equicontinuous subset of* X' . *Then* $\mathrm{st}_w(z)(z') =_1 z'(z)$ *for every* z' *in* ${}^*S'$.

Proof. Let z and S' be as stated, and let $\varepsilon > 0$ be given. Since S' is equicontinuous, there is a basic 0-neighbourhood N in X such that

$$(1) \qquad y \in N \Rightarrow |x'(y)| < \varepsilon/2 \quad (x' \in S') .$$

Then $|z'(y)| < \varepsilon/2$ for all z' in ${}^*S'$ and y in ${}^*N$. Next we have for $z' \in {}^*S'$

$$\mathrm{st}_w z(z') - z'(z) = \mathrm{st}_w(z-x)(z') + \mathrm{st}_w x(z') - z'(z-x) - z'(x)$$

whenever x is standard.

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Put $B = \{z\} - {}^*N$. Since z is pre-near-standard we can select $x \in X$ with $x \in B$ (Theorem 2.2). Putting $y = z - x$, y belongs to ${}^*N$ and so by (1), $|x'(y)| < \varepsilon/2$ $(x' \in S')$. Hence for (standard) $x' \in S'$,

$$|\mathrm{st}_w(z-x)(x')| = |\mathrm{st}x'(z-x)| =_1 |x'(y)| < \varepsilon/2 .$$

Transferring to ${}^*X'$ we obtain

(2) $$|\mathrm{st}_w(z-x)(z')| < \varepsilon/2 \quad (z' \in {}^*S') .$$

Next for all x' in X' we have $\mathrm{st}_w x(x') = x'(x)$. Hence in ${}^*X'$,

(3) $$\mathrm{st}_w x(z') = z'(x) .$$

Since $z-x \in {}^*N$, we have from (1)

(4) $$|z'(z-x)| < \varepsilon/2 \quad (z' \in {}^*S') .$$

Finally,

$$|\mathrm{st}_w z(z')-z'(z)| \le |\mathrm{st}_w(z-x)(z')| + |\mathrm{st}_w x(z')-z'(x)| + |z'(z-x)| < \varepsilon$$

by (2), (3) and (4) .

It follows that $\mathrm{st}_w z(z') =_1 z'(z)$ for all $z' \in {}^*S'$.

PROPOSITION 2.9. *Suppose* S' *is an equicontinuous subset of* X' *and* z *any finite point of* ${}^*X$ *such that*

(i) $\mathrm{st}_w z(z') =_1 z'(z)$ $(z' \in {}^*S')$;

(ii) $\mathrm{st}_w z$ *is* $\sigma(X', X)$ *continuous on* S' .

Then $z'(z) =_1 0$ *for all* $z' \in {}^*S' \cap \mu_{\sigma(X',X)}(0)$.

Proof. We have only to note that since $\mathrm{st}_w(z)$ is $\sigma(X', X)$ continuous on S' ,

$$\mathrm{st}_w z\left(\mu_{\sigma(X',X)}(0) \cap {}^*S'\right) \subset \mu\mathrm{st}_w z(0) = \mu(0)$$

by Robinson's continuity criterion ([4], p. 98). Consequently $z' \in \mu_{\sigma(X',X)}(0) \cap {}^*S' \Rightarrow \mathrm{st}_w z(z') =_1 0$ *and the result follows on applying* condition *(i)*.

Let us observe that X'_λ is embeddable as a linear subspace of X' .

The following lemma is easily verified:

LEMMA 2.10. *S'_λ is an equicontinuous subset of X'.*

THEOREM 2.11. *Let $z \in {}^*X$. Then z is pre-near-standard if and only if z is pre-near-standard in each ${}^*X_\lambda$.*

Proof. Let $z \in {}^*X$ be pre-near-standard. By Theorem 2.2, the set $\{x : p_\lambda(x-z) < \varepsilon\}$ contains a standard point, and hence z is pre-near-standard in ${}^*X_\lambda$.

Conversely, suppose that z is pre-near-standard in each ${}^*X_\lambda$. By Theorem 1.1, there exists a minimal Cauchy filter F_λ such that $\mu_{u_\lambda}(z) = \mu(\mathsf{F}_\lambda)$ where u_λ is the uniformity of (X, p_λ). Since $z \in \mu(\mathsf{F}_\lambda)$ for each λ, it follows that $\cap\mu(\mathsf{F}_\lambda) \neq \emptyset$, and so the union filter $\mathsf{F} = \bigvee \mathsf{F}_\lambda$ exists as the filter of all finite intersections $F_{\lambda_1} \cap \ldots \cap F_{\lambda_n}$, where $F_{\lambda_i} \in \mathsf{F}_{\lambda_i}$. Also

$$\mu(\mathsf{F}) = \cap\{\mu(\mathsf{F}_\lambda) : \lambda \in \Lambda\} . \tag{5}$$

(a) $\mu(u) = \cap\mu(u_\lambda)$.

It suffices to show that $u = \bigvee u_\lambda$. Indeed the uniformity u is generated by sets of the form

$$B = \left\{(x, y) : x-y \in \bigcap_{i=1}^{n} \{p_{\lambda_i} < \varepsilon_i\}\right\} =$$

$$\bigcap_{i=1}^{n} \left\{(x, y) : x-y \in \{p_{\lambda_i} < \varepsilon_i\}\right\} = \bigcap_{i=1}^{n} B_{\lambda_i} ,$$

say. But the B_{λ_i} form a base for u_{λ_i}, so (a) follows from the definition of $\bigvee u_{\lambda_i}$.

(b) $\mu_u(z) = \cap\mu_{u_\lambda}(z)$.

We have

$$\mu_{u}(z) = \mu(u)(\{z\}) = \{y : (y, z) \in \mu(u)\}$$
$$= \{y : (y, z) \in \cap\mu(u_\lambda)\} \text{ , using (a),}$$
$$= \cap\{y : (y, z) \in \mu(u_\lambda)\} = \cap\mu_{u_\lambda}(z) \text{ .}$$

(c) Combining (b) and (5),

$$(\forall\lambda)\mu_{u_\lambda}(z) = \mu(F_\lambda) \Rightarrow \mu_{u}(z) = \mu(F) \text{ .}$$

It only remains to show that F is a Cauchy filter in X. Let $U \in u$ and select a basic vicinity

$$V = \left\{(x, y) : x-y \in \bigcap_{i=1}^{n} \{p_{\lambda_i} < \varepsilon_i\}\right\}$$

such that $V \subset U$. Since $V_{\lambda_i} = \left\{(x, y) : x-y \in \{p_{\lambda_i} < \varepsilon_i\}\right\}$ is in u_{λ_i} and F_{λ_i} is Cauchy in X_{λ_i}, we can find F_i in F_{λ_i} such that $F_i \times F_i \subset V_{\lambda_i}$. Put $F = \bigcap_{i=1}^{n} F_i$. Then $F \in F$, and if $x, y \in F$ then $x, y \in F_i$ for all i and so $(x, y) \in V_{\lambda_i}$. Thus $(x, y) \in \cap V_{\lambda_i} = V$ and $F \times F \subset V$. It follows that F is Cauchy, and the theorem is proved.

Proof of Theorem 2.1. Assume z is a finite point of ${}^*X$ such that for every equicontinuous subset S' of X', $z' \in {}^*S' \cap \mu_{\sigma(X',X)}(0)$ implies $z'(z) =_1 0$. Fix λ and take $S' = S'_\lambda$. By Lemma 2.10, S' is an equicontinuous set. Since $S'_\lambda \subset X'_\lambda$, the topologies $\sigma(X', X)$ and $\sigma(X'_\lambda, X)$ coincide on S'_λ. Hence

$$z' \in {}^*S'_\lambda \cap \mu_{\sigma(X'_\lambda,X)}(0) \Rightarrow z' \in {}^*S' \cap \mu_{\sigma(X',X)}(0) \Rightarrow z'(z) =_1 0$$

by hypothesis. By ([2], 3.17.2, (c) ⇒ (a)), z is pre-near-standard in ${}^*X_\lambda$. By Theorem 2.11, this proves that z is pre-near-standard in ${}^*X$.

Conversely, let z be a pre-near-standard point of ${}^*X$. Then z is finite, by Corollary 2.3. Let S' be an equicontinuous subset of X' and

$z' \in {}^*S' \cap \mu_{\sigma(X',X)}(0)$. By Theorem 2.7, $\mathrm{st}_\omega(z)$ is $\sigma(X', X)$ continuous on S' , and $\mathrm{st}_\omega z(z') =_1 z'(z)$ for $z' \in {}^*S'$ by Theorem 2.8. Hence applying Proposition 2.9, $z'(z) =_1 0 \quad (z' \in {}^*S')$.

The proof of Theorem 2.1 is now complete.

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