FRACTIONAL INTEGRATION AND DUAL INTEGRAL EQUATIONS

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1. In the analysis of mixed boundary value problems by the use of Hankel transforms we often encounter pairs of dual integral equations which can be written in the symmetrical form

$$\int_0^\infty \xi^{-2\alpha}\Psi(\xi)J_\nu(\rho\xi)d\xi = F(\rho), \qquad 0 < \rho < 1 \tag{1.1}$$

$$\int_0^\infty \xi^{-2\beta}\Psi(\xi)J_\nu(\rho\xi)d\xi = G(\rho), \qquad \rho > 1.$$

Equations of this type seem to have been formulated first by Weber in his paper **(1)** in which he derives (by inspection) the solution for the case in which $\alpha - \beta = \frac{1}{2}$, $\nu = 0$, $F \equiv 1$, $G \equiv 0$.

The first direct solution of a pair of equations of this type was given by Beltrami **(2)** for the same values of $\alpha - \beta$ and ν with $G(\rho) \equiv 0$ but with $F(\rho)$ arbitrary.

The general case but with $G(\rho) \equiv 0$ has been considered by Titchmarsh **(3)**, Busbridge **(4)** and Gordon **(5)**. Recently, Copson **(6)** has given an elegant solution of this general case by using a method which is a generalization of an elementary method suggested by Sneddon **(7)** for the cases which are of physical interest.

The solution of the pair of equations (1.1) with general values of α, β satisfying $-1 < \alpha - \beta < 1$ was considered by Noble **(8)** who reduced the problem to that of solving an integral equation of Schlömilch type. Noble's analysis involves heavy manipulation and cannot be regarded as simple. Williams **(9)** has derived a solution, valid for the same range of parameters, by a formal application of the theory of Mellin transforms; the manipulation in this paper appears to be much simpler than that in Noble's paper, but this is because much of it has already been absorbed in the calculation of certain Mellin transforms. A straightforward extension of Copson's method **(6)** to the case in which $G(\rho)$ is not identically zero has recently been given by Lowengrub and Sneddon **(10)**. It should also be observed that a solution of the

Received October 16, 1961. This paper is the result of discussions between the authors when they were both Research Lecturers at the Seminar and Congress of the Canadian Mathematical Congress held in the University of Montreal in August and September 1961. The authors would like to take this opportunity to express their gratitude to the Officers and Programme Committee of the Congress for extending to them an invitation enabling them to share in the stimulating work of the Congress.

pair of equations (1.1) in the special case $\alpha - \beta = \pm 1$ was given by Tranter **(11)**, but it should be noted that his method is more cumbersome than Noble's.

All of these methods are complicated and require an elaborate apparatus of integral transforms. For that reason it is difficult to see the relations which must exist among the various methods or even to appreciate precisely why a certain approach would prove to be fruitful. The purpose of the present paper is to point out that a systematic use of certain operators introduced into the theory of fractional integration enables us to see more clearly the basic structure of any such method of solving dual integral equations and to appreciate more easily the connections which exist among the various methods. The key relations between the operators of fractional integration and the modified operator of Hankel transforms, which form the basis of the method presented here, were first derived by Erdélyi and Kober **(12)** and by Erdélyi **(13)**. They are summarized in §2 below.

To get the equations into a form to which this theory is immediately applicable we make the substitutions

$$\psi(y) = y^{-\frac{1}{2}}\Psi(2y^{\frac{1}{2}}), f(x) = 2^{2\alpha}x^{-\alpha}F(x^{\frac{1}{2}}), g(x) = 2^{2\beta}x^{-\beta}G(x^{\frac{1}{2}}) \tag{1.2}$$

by means of which we transform the equations (1.1) to

$$x^{-\alpha}\int_0^\infty y^{-\alpha}\psi(y)J_\nu(2\sqrt{(xy)})dy = f(x), \qquad 0 < x < 1, \tag{1.3}$$

$$x^{-\beta}\int_0^\infty y^{-\beta}\psi(y)J_\nu(2\sqrt{(xy)})dy = g(x), \qquad x > 1.$$

The results of §2 are applied in §3 to the solution of this pair of equations; the various types of solution, due to the authors cited above, are then clearly recognizable. In §5 one of the solutions obtained by this operational method is written out in explicit form and identified with Titchmarsh's solution **(3; 4)**.

Dual integral equations of a more complicated type arise in the discussion of mixed boundary value problems. They can be written in the form

$$\int_0^\infty \xi^{-2\alpha}[1 + K(\xi)]\Psi(\xi)J_\nu(\xi\rho)d\xi = F(\rho), \qquad 0 < \rho < 1 \tag{1.4}$$

$$\int_0^\infty \xi^{-2\beta}\Psi(\xi)J_\nu(\xi\rho)d\xi = G(\rho), \qquad \rho > 1,$$

where the functions $K(\xi)$, $F(\rho)$, and $G(\rho)$ are prescribed in the ranges stated.

A solution of these equations in the case $\nu = 0$ and $G(\rho) \equiv 0$ was given by Tranter **(14)** in 1950, the solution for general ν following four years later **(15)**. Tranter's method consisted in expressing $\Psi(\xi)$ as an infinite series of Bessel functions of the type

$$\Psi(\xi) = \xi^{1-k}\sum_{m=0}^\infty a_m J_{\nu+2m+k}(\xi)$$

and then deriving an infinite set of linear algebraic equations whose solution yielded the values of the coefficients a_m. Cooke **(16)** gave a solution which is the integral analogue of Tranter's method; in it the unknown function satisfies a linear integral equation of the second kind of Fredholm's type. A special case of physical interest was discussed by the same method derived independently by Lebedev and Ufliand **(17)**. A general solution, derived by a different method, was also given by Noble **(18)**. It should be noted that in all of those investigations it was assumed that $G(\rho) \equiv 0$.

In §4 of this paper we derive a solution of the pair of equations (1.4), with $G(\rho)$ not identically zero, in the sense that we reduce the problem to that of solving a linear integral equation of the second kind of Fredholm's type. To be able to apply the methods of §2 we first write the equations (1.4) in the form

$$(1.5)\qquad \begin{aligned} x^{-\alpha}\int_0^\infty y^{-\alpha}[1+k(y)]\psi(y)J_\nu(2\sqrt{(xy)})dy &= f(x), \qquad 0<x<1\\ x^{-\beta}\int_0^\infty y^{-\beta}\psi(y)J_\nu(2\sqrt{(xy)})dy &= g(x), \qquad x>1 \end{aligned}$$

by making the substitutions (1.2) and the additional substitution

$$(1.6)\qquad K(2\sqrt{y}) = k(y).$$

2. We shall give here a brief summary of the definitions and properties of the operators occurring in our work. A more detailed description will be found in **(12)** and **(13)**. For the sake of simplicity all the relevant parameters will be taken to be real and the definitions will be given in a form appropriate for quadratically integrable functions.

Let I_1 denote the interval $(0, 1)$, and I_2 the interval $(1, \infty)$, and let L_2 denote the space of functions which are quadratically integrable on $(0, \infty)$, two such functions being identified if they are equal almost everywhere. For a function f in L_2 we shall occasionally write $f_1 + f_2$, where

$$(2.1)\qquad \begin{aligned} f_1 &= f \text{ on } I_1, \; = 0 \text{ on } I_2\\ f_2 &= 0 \text{ on } I_1, \; = f \text{ on } I_2. \end{aligned}$$

The operators $I_{\eta,\alpha}$ and $K_{\eta,\alpha}$ are defined by the formulae

$$(2.2)\qquad \begin{aligned} I_{\eta,\alpha}f(x) &= \frac{x^{-\eta-\alpha}}{\Gamma(\alpha)}\int_0^x (x-y)^{\alpha-1}y^\eta f(y)dy\\ K_{\eta,\alpha}f(x) &= \frac{x^\eta}{\Gamma(\alpha)}\int_x^\infty (y-x)^{\alpha-1}y^{-\eta-\alpha}f(y)dy \qquad \alpha>0, \qquad \eta > -\tfrac{1}{2}. \end{aligned}$$

If f belongs to L_2 then $I_{\eta,\alpha}f$ and $K_{\eta,\alpha}f$ exist and belong to a subspace $L_2^{(-\alpha)}$ of L_2. For a precise description of $L_2^{(-\alpha)}$ see **(13**, p. 300**)**. Here it will be sufficient to recall that the elements of $L_2^{(-\alpha)}$ are $[\alpha]$ times continuously differentiable.*

*$[\alpha]$ is the largest integer $\leqslant \alpha$.

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For $\alpha = 0$,

$$I_{\eta,0}f = f, \qquad K_{\eta,0}f = f,$$

and for $\alpha < 0$ we define $g = I_{\eta,\alpha}f$ and $h = K_{\eta,\alpha}f$ as the solutions of the integral equations $f = I_{\eta+\alpha,-\alpha}g$ and $f = K_{\eta+\alpha,-\alpha}h$. These solutions exist if $\eta + \alpha > -\frac{1}{2}$ and f belongs to $L_2^{(\alpha)}$.

We set $L_2^{(\alpha)} = L_2$ if $\alpha \geqslant 0$, and see that for any real α and under the condition $\eta > -\frac{1}{2} + \max(0, -\alpha)$, $I_{\eta,\alpha}f$ and $K_{\eta,\alpha}f$ are defined and belong to $L_2^{(-\alpha)}$ provided that f is in $L_2^{(\alpha)}$. For the so extended operators

$$\begin{aligned} I_{\eta,\alpha}x^{\beta}f(x) &= x^{\beta}I_{\eta+\beta,\alpha}f(x) \\ I_{\eta,\alpha}I_{\eta+\alpha,\beta} &= I_{\eta,\alpha+\beta} \\ K_{\eta,\alpha}K_{\eta+\alpha,\beta} &= K_{\eta,\alpha+\beta} \\ I_{\eta,\alpha}^{-1} &= I_{\eta+\alpha,-\alpha}, \quad K_{\eta,\alpha}^{-1} = K_{\eta+\alpha,-\alpha} \end{aligned} \tag{2.3}$$

provided all operations make sense.

If n is a positive integer,

$$I_{\eta,-n}f(x) = x^{-\eta+n}\frac{d^n}{dx^n}\{x^{\eta}f(x)\}$$

$$K_{\eta,-n}f(x) = (-1)^n x^{\eta}\frac{d^n}{dx^n}\{x^{-\eta+n}f(x)\}$$

by explicit computation, and this result in combination with (2.3) leads to explicit expressions for $I_{\eta,\alpha}f$, $K_{\eta,\alpha}f$, $K_{\eta,\alpha}g$ when $\alpha + n \geqslant 0$, where n is a positive integer. $I_{\eta,\alpha} = I_{\eta+\alpha+n,-n}\, I_{\eta,\alpha+n}$ and similarly for K, so that

$$\begin{aligned} I_{\eta,\alpha}f(x) &= x^{-\eta-\alpha}\frac{d^n}{dx^n}\{x^{\eta+\alpha+n}I_{\eta,\alpha+n}f(x)\} \\ K_{n,\alpha}f(x) &= (-1)^n x^{\eta+\alpha+n}\frac{d^n}{dx^n}\{x^{-\eta-\alpha}K_{\eta,\alpha+n}f(x)\}. \end{aligned} \tag{2.4}$$

For $\alpha \geqslant 0$ we define the modified operator $S_{\eta,\alpha}$ of Hankel transforms by

$$S_{\eta,\alpha}f(x) = x^{-\frac{1}{2}\alpha}\int_0^{\infty} y^{-\frac{1}{2}\alpha}J_{2\eta+\alpha}(2\sqrt{(xy)})f(y)dy\ ; \tag{2.5}$$

and for $\alpha < 0$, we define $g = S_{\eta,\alpha}f$ as the solution of the integral equation $f = S_{\eta+\alpha,-\alpha}g$. If $\eta > -\frac{1}{2} + \max(0, -\alpha)$, and f belongs to $L_2^{(\alpha)}$, then $S_{\eta,\alpha}f$ exists and belongs to $L_2^{(-\alpha)}$. Also,

$$S_{\eta,\alpha}^{-1} = S_{\eta+\alpha,-\alpha}. \tag{2.6}$$

Between operators of fractional integration on the one hand and operators of Hankel transforms we have the following relations:

$$I_{\eta+\alpha,\beta}\, S_{\eta,\alpha} = S_{\eta,\alpha+\beta} \tag{2.7}$$

$$K_{\eta,\alpha}\, S_{\eta+\alpha,\beta} = S_{\eta,\alpha+\beta} \tag{2.8}$$

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(2.9) $$S_{\eta+\alpha,\beta}\, S_{\eta,\alpha} = I_{\eta,\alpha+\beta}$$

(2.10) $$S_{\eta,\alpha}\, S_{\eta+\alpha,\beta} = K_{\eta,\alpha+\beta}$$

provided the conditions for the existence of the various operations are satisfied.

3. Using the S-operator defined by equation (2.5) we write the dual integral equations (1.3) in the form

(3.1) $$S_{\frac{1}{2}\nu-\alpha,2\alpha}\psi = f, \qquad S_{\frac{1}{2}\nu-\beta,2\beta}\psi = g$$

where f is given on I_1, and g is given on I_2. Note that $\alpha - \beta$ is determined uniquely, but increasing α and β by the same amount means merely a change in the definition of f, g, ψ.

Titchmarsh's solution of (3.1) is obtained by observing that by virtue of (2.7) and (2.8),

$$I_{\frac{1}{2}\nu+\alpha,\beta-\alpha}f = I_{\frac{1}{2}\nu+\alpha,\beta-\alpha}S_{\frac{1}{2}\nu-\alpha,2\alpha}\psi = S_{\frac{1}{2}\nu-\alpha,\alpha+\beta}\psi$$
$$K_{\frac{1}{2}\nu-\alpha,\alpha-\beta}g = K_{\frac{1}{2}\nu-\alpha,\alpha-\beta}S_{\frac{1}{2}\nu-\beta,2\beta}\psi = S_{\frac{1}{2}\nu-\alpha,\alpha+\beta}\psi.$$

If we now define a function h by

(3.2) $$h = I_{\frac{1}{2}\nu+\alpha,\beta-\alpha}f \text{ on } I_1, \quad = K_{\frac{1}{2}\nu-\alpha,\alpha-\beta}g \text{ on } I_2,$$

then h can be calculated from the data. Also $S_{\frac{1}{2}\nu-\alpha,\alpha+\beta}\psi = h$, and so by Hankel's inversion theorem, or equation (2.6),

(3.3) $$\psi = S_{\frac{1}{2}\nu+\beta,-\alpha-\beta}h.$$

Noble's solution is based virtually on the same computation. We have established that

(3.4) $$I_{\frac{1}{2}\nu+\alpha,\beta-\alpha}f = K_{\frac{1}{2}\nu-\alpha,\alpha-\beta}g$$

is a consequence of (3.1). Setting $f = f_1 + f_2$, $g = g_1 + g_2$ as in (2.1), f_1 and g_2 are given so that (3.4) may be regarded as an equation for f_2 and g_1, viz.

(3.5) $$I_{\frac{1}{2}\nu+\alpha,\beta-\alpha}f_2 - K_{\frac{1}{2}\nu-\alpha,\alpha-\beta}g_1 = K_{\frac{1}{2}\nu-\alpha,\alpha-\beta}g_2 - I_{\frac{1}{2}\nu+\alpha,\beta-\alpha}f_1.$$

If this equation is evaluated on I_1, the first term on the left-hand side vanishes, and

(3.6) $$K_{\frac{1}{2}\nu-\alpha,\alpha-\beta}g_1 = I_{\frac{1}{2}\nu+\alpha,\beta-\alpha}f_1 - K_{\frac{1}{2}\nu-\alpha,\alpha-\beta}g_2 \text{ on } I_1$$

determines g_1. Hence $g_1 + g_2 = g$, from which, if desired, ψ may be obtained by Hankel's inversion formula as

$$\psi = S_{\frac{1}{2}\nu+\beta,-2\beta}g.$$

On the other hand, if we evaluate (3.5) on I_2, the second term on the left-hand side vanishes, and

(3.7) $$I_{\frac{1}{2}\nu+\alpha,\beta-\alpha}f_2 = K_{\frac{1}{2}\nu-\alpha,\alpha-\beta}g_2 - I_{\frac{1}{2}\nu+\alpha,\beta-\alpha}f_1 \text{ on } I_2$$

can be used for the computation of f_2 and hence of f and $\psi = S_{\frac{1}{2}\nu+\alpha,-2\alpha}f$.

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Copson's (or Gordon's) solution is obtained if we regard (3.3) as a trial solution of (3.1), with an unknown h. Substitution of (3.3) into the first equation of the pair (3.1) results in

$$f = S_{\frac{1}{2}\nu-\alpha,2\alpha}\, S_{\frac{1}{2}\nu+\beta,-\alpha-\beta}h = I_{\frac{1}{2}\nu+\beta,\alpha-\beta}h$$

by (2.9). This is a functional equation for h from which h_1 (that is, h on I_1) may be found. Similarly,

$$g = S_{\frac{1}{2}\nu-\beta,2\beta}\, S_{\frac{1}{2}\nu+\beta,-\alpha-\beta}h = K_{\frac{1}{2}\nu-\beta,\beta-\alpha}h$$

by (2.10), and this determines h_2 (that is, h on I_2). If $\alpha > \beta$, h_1 is determined by an integral equation while $h_2 = K_{\frac{1}{2}\nu-\alpha,\alpha-\beta}g_2$ on I_2, is given explicitly; if $\alpha < \beta$, h_1 is given explicitly while h_2 is the solution of an integral equation.

4. The same technique enables us to discuss the slightly more general dual integral equations (1.5) which may be written as

$$S_{\frac{1}{2}\nu-\alpha,2\alpha}(1 + k)\psi = f, \quad S_{\frac{1}{2}\nu-\beta,2\beta}\psi = g \tag{4.1}$$

where $k = k(x)$ is a given function on $x > 0$, f is given on I_1, and g is given on I_2. The solution we give here is essentially Cooke's **(16)** solution.

We again use (3.3), with unknown $h = h_1 + h_2$, as a trial solution. Substitution of (3.3) in (4.1) results in

$$\begin{aligned} f &= S_{\frac{1}{2}\nu-\alpha,2\alpha}S_{\frac{1}{2}\nu+\beta,-\alpha-\beta}h + S_{\frac{1}{2}\nu-\alpha,2\alpha}(kS_{\frac{1}{2}\nu+\beta,-\alpha-\beta}h)\\ g &= S_{\frac{1}{2}\nu-\beta,2\beta}S_{\frac{1}{2}\nu+\beta,-\alpha-\beta}h = K_{\frac{1}{2}\nu-\beta,\beta-\alpha}h. \end{aligned}$$

The second one of these equations determines h on I_2, so that

$$h_2 = K_{\frac{1}{2}\nu-\alpha,\alpha-\beta}g_2 \text{ on } I_2$$

may be regarded as known. The first equation, evaluated on I_1, then becomes

$$I_{\frac{1}{2}\nu+\beta,\alpha-\beta}h_1 + S_{\frac{1}{2}\nu-\alpha,2\alpha}(kS_{\frac{1}{2}\nu+\beta,-\alpha-\beta}h_1) = f - S_{\frac{1}{2}\nu-\alpha,2\alpha}(kS_{\frac{1}{2}\nu+\beta,-\alpha-\beta}h_2) \text{ on } I_1.$$

Using (2.7), we may re-write this in the form

$$\begin{aligned} h_1 &+ S_{\frac{1}{2}\nu-\alpha,\alpha+\beta}(kS_{\frac{1}{2}\nu+\beta,-\alpha-\beta}h_1)\\ &= I_{\frac{1}{2}\nu+\alpha,\beta-\alpha}f - S_{\frac{1}{2}\nu-\alpha,\alpha+\beta}(kS_{\frac{1}{2}\nu+\beta,-\alpha-\beta}h_2) = F, \text{ on } I_1, \end{aligned} \tag{4.2}$$

say. The second term on the left-hand side is a double integral. If the order of integrations in this double integral is interchanged, (4.2) turns out to be an integral equation of the second kind of Fredholm's type.

Indeed, from (2.5), we have on I_1

$$\begin{aligned} &S_{\frac{1}{2}\nu-\alpha,\alpha+\beta}(kS_{\frac{1}{2}\nu+\beta,-\alpha-\beta}h_1)(x)\\ &= x^{-\frac{1}{2}\alpha-\frac{1}{2}\beta}\int_0^\infty dz\, k(z)\, J_{\nu+\beta-\alpha}(2\surd(xz)) \int_0^1 dy\; y^{\frac{1}{2}\alpha+\frac{1}{2}\beta} J_{\nu+\beta-\alpha}(2\surd(yz))h_1(y)dy\\ &= \int_0^1 K(x, y)h_1(y)dy, \end{aligned}$$

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where

$$K(x, y) = \left(\frac{y}{x}\right)^{\frac{1}{2}\alpha+\frac{1}{2}\beta} \int_0^\infty J_{\nu+\beta-\alpha}(2\surd(xz))J_{\nu+\beta-\alpha}(2\surd(yz))k(z)\,dz. \tag{4.3}$$

Equation (4.2) now takes the form

$$h_1(x) + \int_0^1 K(x, y)h_1(y)dy = F(x) \qquad 0 < x < 1, \tag{4.4}$$

where $F(x)$ is known.

5. For the practical application of the results obtained in §**3** we should wish to express the various forms of the solution in explicit form. The advantage of our approach is the avoidance of all intermediate explicit computations, combined with the utilization of the known analytic expression of the operators appearing in the final result. We shall illustrate this by considering Titchmarsh's solution of (3.1).

The solution (3.3) appears in the form

$$\psi(x) = x^{\frac{1}{2}\alpha+\frac{1}{2}\beta} \int_0^\infty J_{\nu+\alpha+\beta}(2\surd(xy))y^{\frac{1}{2}\alpha+\frac{1}{2}\beta}h(y)dy. \tag{5.1}$$

In the computation of h two cases must be distinguished.

First let us assume $\beta > \alpha$, and let n be the smallest integer $\geqslant \beta - \alpha$. From (3.2)

$$h_1(x) = I_{\frac{1}{2}\nu+\alpha,\beta-\alpha}f(x) = \frac{x^{-\frac{1}{2}\nu-\beta}}{\Gamma(\beta-\alpha)} \int_0^x (x-y)^{\beta-\alpha-1}y^{\frac{1}{2}\nu+\alpha}f(y)dy \qquad 0 < x < 1 \tag{5.2}$$

$$\begin{aligned} h_2(x) &= K^{-1}_{\frac{1}{2}\nu-\beta,\beta-\alpha}g(x) = K_{\frac{1}{2}\nu-\alpha,\alpha-\beta}g(x) && 1 < x < \infty \\ &= (-1)^n x^{\frac{1}{2}\nu-\beta+n} \frac{d^n}{dx^n}\{x^{-\frac{1}{2}\nu+\beta}g(x)\} && \text{if } \alpha-\beta+n = 0 \\ &= (-1)^n x^{\frac{1}{2}\nu-\beta+n} \frac{d^n}{dx^n}\left\{\frac{x^{\beta-\alpha}}{\Gamma(\alpha-\beta+n)} \int_x^\infty (y-x)^{\alpha-\beta+n-1}y^{-\frac{1}{2}\nu+\beta-n}g(y)dy\right\} && \text{if } \alpha-\beta+n > 0. \end{aligned}$$

The conditions for the existence of the operators involved in this work are satisfied if f is quadratically integrable over $(0, 1)$, g is n times continuously differentiable and g and $g^{(n)}$ are quadratically integrable over $[1, \infty)$, $2|\alpha| - 1 < \nu$ and $\alpha + \beta + \nu > -1$.

Secondly, if $\alpha > \beta$, and n is the least positive integer $\geqslant \alpha - \beta$, we have from (3.2)

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$$(5.3)\qquad h_1(x) = I_{\frac{1}{2}\nu+\alpha,\beta-\alpha}f(x) \qquad 0 < x < 1$$

$$= x^{-\frac{1}{2}\nu-\beta}\frac{d^n}{dx^n}\{x^{\frac{1}{2}\nu+\beta+n}f(x)\} \qquad \text{if } n-\alpha+\beta = 0$$

$$= \frac{x^{-\frac{1}{2}\nu-\beta}}{\Gamma(\beta-\alpha+n)}\frac{d^n}{dx^n}\int_0^x (x-y)^{\beta-\alpha+n-1}y^{\frac{1}{2}\nu+\alpha}f(y)dy \qquad \text{if } n-\alpha+\beta > 0$$

$$h_2(x) = K_{\frac{1}{2}\nu-\alpha,\alpha-\beta}g(x) = \frac{x^{\frac{1}{2}\nu-\alpha}}{\Gamma(\alpha-\beta)}\int_x^\infty (y-x)^{\alpha-\beta-1}y^{-\frac{1}{2}\nu+\beta}g(y)dy \qquad 1 < x < \infty.$$

All conditions are satisfied if f is n times continuously differentiable, f and $f^{(n)}$ are quadratically integrable on $(0, 1]$, g is quadratically integrable on $(1, \infty)$, $-2|\alpha| - 1 < \nu$ and $\nu + \alpha + \beta > -1$.

Added in proof. (i) An integral equation equivalent to (4.4) was considered in a special case by E. R. Love (*The electrostatic field of two equal circular coaxial conducting disks*, Quart. J. Mech. and Appl. Math. 2 (1949), 428); and Cooke's solution of (4.1) is based on Love's work. After seeing the manuscript of the present paper, Professor Love (*Dual integral equations*, unpublished) proved that when $K(\xi) = \pm\exp(-\kappa\xi)$ in (1.4), the Fredholm integral equation (4.4) may be recast in a form which is a generalization of Love's integral equation, and proved that the kernel of the recast integral equation has norm less than unity so that the Neumann series converges in this case.

(ii) Dual integral equations differing from (1.1) in that the Bessel functions appearing in the two equations are of different orders have been considered by A. S. Peters (*Certain dual integral equations and Sonine's integral*, Technical Report IMM-NYU 285. Institute of Mathematical Sciences, New York University, August 1961). Such integral equations can also be solved by our method.

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