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NONOSCILLATORY SOLUTIONS OF $x^{(m)}=(-1)^m Q(t)x$

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ABSTRACT. A continuous real vector function is said to be nonoscillatory on an interval if at least one of its components is of constant positive or negative sign there. In this note, various existence criteria for nonoscillatory solutions of the system $x^{(m)}=(-1)^m Q(t)x$ are established. In some cases, additional monotonicity properties for these solutions are also given.

A continuous real vector function is said to be nonoscillatory on an interval if at least one of its components is of constant positive or negative sign there. In analyzing linear differential equations or systems, nonoscillatory solutions are often desirable, especially when they satisfy certain monotonicity properties (for recent examples, see [2, 3, 5]). In this note, we shall be concerned with linear differential systems of the form

$$x^{(m)}=(-1)^m Q(t)x \qquad m\geq 1 \tag{1}$$

where $Q(t)=(q_{ij}(t))$ is an n-square continuous matrix function defined on $[a,\infty)$. Existence criteria for nonoscillatory solutions of (1) will be established under various assumptions. In some cases we shall also be able to deduce additional monotonicity properties for these solutions. Theorem 5 will be our main result which is basically a comparison theorem.

In the sequel, a real matrix $A=(a_{ij})$ is called positive (nonnegative), if $a_{ij}>0\ (a_{ij}\geq 0)$ for $i,j=1,2,\ldots,n$. We write $A>0\ (A\geq 0)$. If $x=\operatorname{col}(x_1,x_2,\ldots,x_n)$ and $y=\operatorname{col}(y_1,y_2,\ldots,y_n)$ are n-tuples, we will define the product xy to be the vector $xy=\operatorname{col}(x_1y_1,x_2y_2,\ldots,x_ny_n)$. For any n-square matrix $Q=(q_{ij})$, the matrix $\operatorname{diag} Q$ is defined to be the diagonal matrix $\operatorname{diag}[q_{11},q_{22},\ldots,q_{nn}]$.

The following lemma is fundamental and its proof is given in [1].

LEMMA 1. *If in* (1), $m=1$ *and* $Q(t)\geq 0$ *for* $t\geq a$, *then* (1) *has a nontrivial solution* $x(t)\geq 0$ *for* $t\geq a$.

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Clearly, the nontrivial solution $x(t)$ in Lemma 1 satisfies $x'(t) \le 0$ and hence at least one of its components is positive for $t \ge a$ (so that $x(t)$ is nonoscillatory).

LEMMA 2. *In the linear differential system*

$$p_0(t)z^{(m)} + \sum_{k=1}^{m-1} (-1)^{k+1} p_k(t) z^{(m-k)} = (-1)^m R(t) z, \tag{2}$$

let $R(t)$ be a nonnegative continuous n-square matrix function and let $p_0(t), \ldots, p_{m-1}(t)$ be continuous vector functions on $[a, \infty)$ satisfying $p_0(t) > 0$ and $p_k(t) \ge 0$ for $k = 2, \ldots, m-1$ and $a \le t \le \infty$ (while $p_1(t)$ is arbitrary). Then (2) has at least one nontrivial solution $z(t)$ satisfying $(-1)^j z^{(j)}(t) \ge 0$ for $j = 0, 1, \ldots, m-1$.

The proof of the above lemma is omitted since it follows from Lemma 1.1 in a way similar to an argument given in [1, p. 737]. As a corollary, we see that if $Q \ge 0$ for $t \ge a$, then (1) has a nontrivial solution $x(t)$ satisfying $(-1)^k x^{(k)}(t) \ge 0$ for $k = 0, 1, \ldots, m$.

Before stating the next theorem, we first recall [4, p. 122] that a nonnegative n-square matrix $A = (a_{ij})$ is said to be decomposable if the indices $1, 2, \ldots, n$ can be divided into two disjoint nonempty sets $\{i(1), i(2), \ldots, i(h)\}$ and $\{j(1), j(2), \ldots, j(k)\}$ $(h + k = n)$, such that $a_{i(\alpha)j(\beta)} = 0$ $(\alpha = 1, 2, \ldots, h;$ $\beta = 1, 2, \ldots, k)$. Otherwise A is said to be indecomposable.

THEOREM 3. *If $Q(t)$ is indecomposable on any subinterval $[b, \infty)$ of $[a, \infty)$, then (1) has at least one positive solution $x(t)$ satisfying $(-1)^k x^{(k)}(t) \ge 0$ for $k = 1, 2, \ldots, m$.*

Proof. Let $x(t)$ be the nontrivial solution of (1) satisfying $(-1)^k x^{(k)}(t) \ge 0$ for $k = 0, 1, \ldots, m$. Assume to the contrary that there is a nonempty subset I of $\{1, 2, \ldots, n\}$ such that for each i in I, the corresponding component $x_i(t)$ of $x(t)$ vanishes at some point $t_i \ge a$. Then $x_i(t) \equiv 0$ for $t \ge t_i$ since $x_i(t) \ge 0$ and $x_i'(t) \le 0$ for $t \ge a$. Hence for each i in I, $x_i(t) \equiv 0$ for $t \ge t^*$, where $t^* = \max\{t_i \mid i \text{ in } I\}$. Let $J = \{1, 2, \ldots, n\} - I$. Note that J is nonempty since otherwise $x(t) \equiv 0$ for $t \ge t^*$ would imply that $x(t)$ is trivial. Note further that for each j in J, $x_j(t) > 0$ for $t \ge a$. Consequently for each i in I and $t \ge t^*$,

$$0 = x_i^{(m)}(t) = (-1)^m \sum_{k=1}^{n} q_{ik}(t) x_k(t) = (-1)^m \sum_{j \in J} q_{ij}(t) x_j(t).$$

However, since $x_j(t) > 0$ and $q_{ij}(t) \ge 0$, we must have $q_{ij}(t) = 0$ for i in I and j in J, where $t \ge t^*$. This contradicts our assumption and shows that $x(t) > 0$ for $t \ge a$. Q.E.D.

So far we have assumed that $Q \ge 0$. We will now relax this assumption. Let T be a diagonal matrix $\operatorname{diag}[t_1, t_2, \ldots, t_n]$ where $t_i = \pm 1$.

THEOREM 4. *Suppose* $Q(t)=TH(t)T$, *where* $H(t)$ *is continuous and nonnegative for* $a\le t<\infty$. *Then* (1) *has a nontrivial solution* $x(t)$ *such that*

$$(-1)^k Tx^{(k)}(t)\ge 0$$

for $k=0,1,\ldots,m$; *and for* $t\ge a$.

The proof is elementary. We first note that $T=T^{-1}$. By setting $y=Tx$, (1) can be transformed into the system $y^{(m)}=(-1)^m H(t)y$, which has a nontrivial solution $y(t)$ satisfying $(-1)^k y^{(k)}(t)\ge 0$ for $k=0,1,\ldots,m$. But then $x=Ty$ is a solution of (1) satisfying the conclusion of the theorem.

Before proving the main theorem of this note, we remark that matrices of the form $Q=THT$ where $H\ge 0$ satisfies diag $Q\ge 0$. Furthermore, if the matrix $(\operatorname{sgn} h_{ij})$ is symmetric, so is $(\operatorname{sgn} q_{ij})$.

THEOREM 5. *Suppose* $Q(t)=(q_{ij}(t))$ *and* $P(t)=(p_{ij}(t))$ *in* (1) *and*

$$y^{(m)}=(-1)^m P(t)y \tag{3}$$

respectively, are continuous n*-square matrix functions satisfying*

$$\begin{cases} p_{ij}(t) \le q_{ij}(t) & i=j \\ p_{ij}(t)\le 0\le q_{ij}(t) & i\ne j \end{cases}$$

for $t\ge a$. *Suppose further that* (3) *has a solution* $y(t)>0$ *satisfying* $(-1)^{k+1}y^{(k)}(t)\ge 0$ *for* $k=2,\ldots,m-1$ *and for* $t\ge a$ (*while* $y'(t)$ *is arbitrary*). *Then* (1) *has a nonoscillatory solution* $x(t)$ *and a positive constant* K *such that* $0\le x(t)\le Ky(t)$ *for sufficiently large* t. *If, furthermore, that* $y'(t)\le 0$, *then* $x'(t)\le 0$ *for* $t\ge a$.

Proof. We shall seek a solution of (1) in the form $x(t)=y(t)z(t)$ where $z(t)$ is to be determined. Substitution of $x(t)$ into (1) leads to the following differential system for $z(t)$:

$$\sum_{k=0}^{m}\binom{m}{k}y^{(k)}z^{(m-k)}=(-1)^m Q(t)(yz).$$

Subtracting $y^{(m)}z=(-1)^m[P(t)y]z$ from both sides, we obtain

$$\sum_{k=0}^{m-1}\binom{m}{k}y^{(k)}z^{(m-k)}=(-1)^m Q(t)(yz)-(-1)^m[P(t)y]z. \tag{4}$$

Let $r_{ij}=q_{ij}y_j$ if $i\ne j$ and $r_{ij}=q_{ij}y_j-\sum_{k=1}^{n}P_{ik}y_k$ if $i=j$, and let $R(t)=(r_{ij}(t))$. Then (4) can be written in the form (2) where

$$(-1)^{k+1}p_k(t)=\binom{m}{k}y^{(k)}$$

for $k=2,3,\ldots,m-1$. Furthermore, our assumptions imply $R\ge 0$. It follows from Lemma 2 that (4) has a nontrivial solution $z(t)$ satisfying $(-1)^j z^{(j)}(t)\ge 0$

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for $j=0,1,\ldots,m-1$. The conclusion of the theorem now follows readily from $x(t)=y(t)z(t)$. Q.E.D.

The above theorem has many consequences. We list some of them here. The first of which is an improvement of Lemma 1.

COROLLARY 6. *If in* (1), $m=1$ *and* $q_{ij}(t)\geq 0$ *for* $i\neq j$ *and for* $t\geq a$, *then* (1) *has a nontrivial solution* $x(t)\geq 0$ *for* $t\geq a$.

COROLLARY 7. *Suppose in* (1), $m=2$ *and* $q_{ij}(t)\geq 0$ *for* $i\neq j$ *and for* $t\geq a$. *Suppose further that* $u_i''=q_{ii}(t)u_i$ $(1\leq i\leq n)$ *is nonoscillatory. Then* (1) *has a nontrivial solution which is nonnegative and nonoscillatory for large t.*

COROLLARY 8. *Suppose* $G(t)=(g_{ij}(t))$ *is an n-square continuous matrix function such that* $g_{ij}(t)\leq 0$ *in* $[a,\infty)$ *for* $i\neq j$, *and* $\int_a^\infty g_{ii}(s)\,ds<\infty$ *for each i,* $1\leq i\leq n$. *If the system*

$$x''+4F^2(t)x=0 \qquad F(t)=\int_t^\infty [\operatorname{diag} G(s)]\,ds$$

has a positive solution on $[a,\infty)$, *then the system*

$$y''+G(t)y=0$$

has a solution $y(t)$ *such that* $y(t)$ *is nonoscillatory and*

$$0\leq y(t)\leq\left[\exp\int_a^\infty F(s)\,ds\right]x^{1/2}(t), \qquad x^{1/2}=\operatorname{col}(x_1^{1/2},\ldots,x_n^{1/2})$$

for sufficiently large t.

Proof. For $t\geq a$, the vector $u(t)=[\exp\int_a^t F(s)\,ds]x^{1/2}(t)>0$. Furthermore, an easy calculation shows that $u(t)$ satisfies the system

$$u''+[\operatorname{diag} G(t)+(H(t)-F(t))^2]u=0$$

where $H=\operatorname{diag}[x_1'/2x_1,\ldots,x_n'/2x_n]$. Since for $t\geq a$, $G(t)\leq\operatorname{diag} G(t)+(H(t)-F(t))^2$, an application of Theorem 5 then leads to the desired conclusion.

Suppose $p(t)$ is a continuous function satisfying $(-1)^m p(t)\geq 0$ and $p\not\equiv 0$ on every subinterval $[b,\infty)$ of $[a,\infty)$. According to a result of Kim [3, Th. 1], if y is a nontrivial solution of

$$y^{(m)}+p(t)y=0 \tag{5}$$

such that $y\geq 0$ and $y(t)/t^2\to 0$ as $t\to\infty$, then $y(t)>0$ and $(-1)^{k+1}y^{(k)}(t)>0$ for $k=1,2,\ldots,m-1$ as $t\to\infty$. In this connection, we have the following corollary of Theorem 5.

THEOREM 9. *Let* $q(t)$ *be a continuous function satisfying* $(-1)^m q(t)\leq(-1)^m p(t)$

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for $t \geq a$. If $y(t)$ is a nontrivial solution of (5) such that $y \geq 0$ and $y(t)/t^2 \to 0$ as $t \to \infty$, then

$$x^{(m)} + q(t)x = 0 \tag{6}$$

has a nontrivial solution $x(t)$ satisfying $x \geq 0$ and $x(t)/t^2 \to 0$ as $t \to \infty$.

The above theorem can be used to derive existence criteria for $L_2(0, \infty)$ solutions of (6). Such criteria will supplement those of Read [5] and will be dealt with elsewhere.

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