Canad. Math. Bull. Vol. **26** (2), 1983

A TYPICAL NOWHERE DIFFERENTIABLE FUNCTION

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ABSTRACT. We construct an explicit continuous function F such that for each point x, every extended real number is a derived number of F at x and F has an infinite left and an infinite right derived number at x.

Define the continuous real valued function

$$F(x) = \sum_{n=1}^{\infty} 2^{-n!} \cos(2^{(2n)!}x)$$

for all real x, and let D^+F, D_+F, D^-F, D_-F denote the usual four Dini derivates of F. In this note we prove the following properties of F.

(i) At each real x, either $D^+F(x) = -D_-F(x) = \infty$ or $D^-F(x) = -D_+F(x) = \infty$, and the set of all points x where either of these equations does not hold is a first category set of measure zero.

(ii) At each real x, $[D_-F(x), D^-F(x)] \cup [D_+F(x), D^+F(x)] = [-\infty, \infty]$.

(iii) Each of the four sets $\{x : F'_+(x) = \infty\}$, $\{x : F'_+(x) = -\infty\}$, $\{x : F'_-(x) = \infty\}$, $\{x : F'_-(x) = -\infty\}$ contains a perfect set in every interval, and hence has power c in every interval.

In [3] it was shown that in the sense of Baire category, most continuous functions satisfy (i), (ii) and (iii). However it is difficult to find concrete examples of such functions in the literature. F is easily defined and is much like Weierstrass' nowhere differentiable function W, although W does not satisfy (ii) everywhere. Indeed W has cusps at countably many points [1]. Moreover, every local extreme point of W is a cusp, and at least three of the four Dini derivates of W are infinite at each point [1]. On the other hand, any function satisfying (ii) must have two Dini derivates $= 0$ at local extreme points.

In [2] some additional properties of continuous functions satisfying (i), (ii) and (iii) are given, and some open questions about most such functions are discussed.

Received by the editors June 8, 1981.

Subject classification: 26A27, 26A15.

Key words and phrases: nowhere differentiable function, continuous function, Dini derivate, derived number.

Proof. (i) Let $F_n(x)=\sum_{j=1}^{n} 2^{-j!}\cos(2^{(2j)!}x)$ for $n\geq 1$, $F_0(x)=0$, $f_n(x)=F_n(x)-F_{n-1}(x)$ for $n\geq 1$, and for $x_1\neq x_2$, $I(g, x_1, x_2)=(g(x_2)-g(x_1))(x_2-x_1)^{-1}$.

Fix $n>1$. Let $-\frac{1}{2}\pi 2^{-(2n)!}\leq x\leq \frac{1}{2}\pi 2^{-(2n)!}$ and $x_1=\pi 2^{-(2n)!}$. Then

$$|(F-F_n)(x_1)-(F-F_n)(x)|\leq \sum_{j=n+1}^{\infty} 2^{-j!}\leq 2^{1-(n+1)!}$$

and

$$(F-F_{n-1})(x_1)-(F-F_n)(x)\leq f_n(x_1)-f_n(x)+2^{1-(n+1)!}\leq -2^{-n!}+2^{1-(n+1)!}\leq -\tfrac{1}{2}2^{-n!}.$$

Thus $I(F-F_{n-1}, x, x_1)\leq -\frac{1}{2}2^{-n!}/3\cdot\frac{1}{2}\pi 2^{-(2n)!}$ and it follows that

$$(1) \qquad I(F-F_{n-1}, x, x_1)\leq -2^{(2n)!-n!}/3\pi.$$

But $|F'_{n-1}|\leq \sum_{j=1}^{n-1} 2^{-j!}\cdot 2^{(2j)!}\leq 2\cdot 2^{(2n-2)!-(n-1)!}$, so

$$(2) \qquad I(F_{n-1}, x, x_1)\leq 2^{1+(2n-2)!-(n-1)!}.$$

It follows that

$$I(F, x, x_1)=I(F-F_{n-1}, x, x_1)+I(F_{n-1}, x, x_1)\leq -2^{(2n)!-n!}/3\pi+2^{1+(2n-2)!-(n-1)!}$$

and hence

$$(3) \qquad I(F, x, x_1)\leq -n.$$

In a similar manner, $I(F, x_2, x)>n$ where $x_2=-\pi 2^{-(2n)!}$. Now put

$$E_n=\bigcup_{k=-\infty}^{\infty}[(4k-1)\tfrac{1}{2}\pi 2^{-(2n)!}, (4k+1)\tfrac{1}{2}\pi 2^{-(2n)!}].$$

For any $x\in E_n$, there exist x_1 and x_2 with $x_2<x<x_1$, $x_1-x_2=2\pi 2^{-(2n)!}$, and $I(F, x, x_1)<-n$, $I(F, x_2, x)>n$. It follows that for any $x\in\bigcap_{m=1}^{\infty}\bigcup_{n=m}^{\infty}E_n=E$, we have $D^{\triangle}F(x)=\infty$ and $D_+F(x)=-\infty$. Likewise if

$$S_n=\bigcup_{k=-\infty}^{\infty}[(4k+1)\tfrac{1}{2}\pi 2^{-(2n)!}, (4k+3)\tfrac{1}{2}\pi 2^{-(2n)!}],$$

and if $x\in\bigcap_{m=1}^{\infty}\bigcup_{n=m}^{\infty}S_n=S$, we have $D^+F(x)=\infty$ and $D_-F(x)=-\infty$. But $E\cup S=R$ and $(R\setminus E)\cup(R\setminus S)$ is obviously a first category set of measure 0. This proves (i).

(ii) The period of $F-F_n$ divides $2^{1-(2n+2)!}\pi$, so for any x,

$$(4) \qquad (F-F_n)(x)=(F-F_n)(x+\pi 2^{1-(2n+2)!})=(F-F_n)(x-\pi 2^{1-(2n+2)!}).$$

Also $|F''_n|\leq\sum_{j=1}^{n} 2^{-j!}\cdot 2^{2(2j)!}\leq 2^{2(2n)!}$ and

$$|I(F_n, x, x+\pi 2^{1-(2n+2)!})-I(F_n, x, x-\pi 2^{1-(2n+2)!})|\leq 2^{2(2n)!}(\pi 2^{2-(2n+2)!})$$

and from this and (4) we obtain

$$(5) \qquad |I(F, x, x+\pi 2^{1-(2n+2)!})-I(F, x, x-\pi 2^{1-(2n+2)!})|<2^{-n}.$$

From (5) it follows that there is an extended real number that is both a left and right derived number of F at x, and

$$[D_-F(x), D^-F(x)] \cap [D_+F(x), D^+F(x)] \neq \emptyset.$$

Finally, (ii) follows from (i).

(iii) follows from (i) and [1, Prop. 3]. It would not be difficult to prove (iii) directly for F, but we will not do that here.

By studying the solution to this problem we see how other continuous functions of the form $\sum a_n \cos(b_n x)$ can be constructed to satisfy (i), (ii) and (iii). a_n must converge to 0 fast enough, b_n must diverge to ∞ fast enough, and a_n and b_n must relate to each other in the right way.

The author expresses his appreciation to Prof. Garg for his advice and encouragement.

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