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A CROSSING THEOREM FOR DISTRIBUTION FUNCTIONS AND THEIR MOMENTS

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It is proved here that for two distribution functions with equal moments up to order n, the number of crossings is at least n, and if exactly n, the remaining odd or even moments (for n even or odd respectively) do not cross again. This both generalises and extends a number of previous results.

1. Introduction

The full statement of the theorem proved in this paper is as follows.

THEOREM 1. *Suppose* $F(x)$ *and* $G(x)$ *are (arbitrary) distribution functions with their* rth *moments about some point given by* $\mu_{F,r}$ *and* $\mu_{G,r}$ *respectively, provided these exist. If* $\mu_{F,r} = \mu_{G,r}$ *for* $r = 1, \ldots, n$, *and* $F \not\equiv G$, *then*

(i) $F(x) - G(x)$ *changes sign at least* n *times for* $x \in (-\infty, \infty)$,

(ii) if $F(x) - G(x)$ *changes sign exactly* n *times with the* nth *sign change being from negative to positive (assuming* x *traverses* $(-\infty, \infty)$ *from left to right), then* $\mu_{F,n+2k-1} < \mu_{G,n+2k-1}$, $k = 1, \ldots$, *provided the moments exist.*

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Relationships between the crossings of moments and those of their densities or distribution functions have been previously investigated in the special cases of absolute moments of symmetric densities ([1], [6], [7]), and odd moments of absolutely continuous distributions with two crossings ([5], [7], [8]). Each of these cases involves either implicitly or explicitly the variation diminishing properties of the kernel x^r, which is strictly totally positive for $x \in [0, \infty)$ and all real r (see [4], p. 15). Although the restriction to $x \geq 0$ indicates why the above special cases have been examined, it does not in fact limit considerations to absolute moments, or odd and even moments separately, and the full variation diminishing theorem (see [3], [4], p. 233) is the main tool used in the proof of the above theorem.

For distribution functions F and G with means μ_F and μ_G, it is well known (see, for example, [9], p. 5) that $\mu_F = \mu_G$ implies F and G cross at least once unless $F \equiv G$. The corresponding result when F and G have the same first two moments has been proved in the special case of $G^{-1}(F(x))$ convex ([10], p. 10) but does not appear to have been considered further or stated. Neither does the general result for identical moments up to order n, even though the proof is quite straightforward.

2. Proof of Theorem 1

The variation diminishing theorem involves two ways of counting sign changes, and Karlin's ([4]) notation is defined here for ease of reference. If $h(x)$ is a real function defined on an interval I of the real line,

$$S^-(h) \equiv S^-(h(x)) \equiv \sup S^-[h(x_1), \ldots, h(x_m)] ,$$

where the supremum is taken over all sequences $x_1 < \ldots < x_m$ in I, m arbitrary but finite, and $S^-[y_1, \ldots, y_m]$ is the number of sign changes of the indicated sequence, with zero terms discarded. A stronger way of counting sign changes is given by

$$S^+(h) \equiv S^+(h(x)) \equiv \sup S^+[h(x_1), \ldots, h(x_m)] ,$$

where the supremum is taken over $x_1, \ldots, x_m$ as above, but $S^+[y_1, \ldots, y_m]$ is the maximum number of sign changes of the indicated sequence with zero terms taking arbitrary signs.

Proof of Theorem 1. The moments of F and G may be taken about any finite point c. Without loss of generality, c is here taken to be 0.

(i) If the rth moment $\mu_{F,r}$ of an arbitrary distribution function F exists, it may be expressed as

$$\mu_{F,r} = r\int_0^\infty x^{r-1}\big(1-F(x)\big)dx - r\int_{-\infty}^0 x^{r-1}F(x)dx .$$

Hence

$$(1)\qquad \mu_{F,r} - \mu_{G,r} = -r\int_{-\infty}^\infty x^{r-1}\big(F(x)-G(x)\big)dx .$$

Suppose $\mu_{F,r} = \mu_{G,r}$, $r = 1, \ldots, n$, and $F - G$ changes sign exactly m times at $a_1, \ldots, a_m$, with $F(x) - G(x) \geq 0$ for $x > a_m$.

Then, generalizing from Dyson [2],

$$(2)\qquad \phi_m(x) = \big(x-a_1\big) \ldots \big(x-a_m\big)[F(x)-G(x)] \geq 0 , \text{ all } x ,$$

for m even or odd, and the inequality in (2) is strict for at least some x. Therefore

$$\int_{-\infty}^\infty \phi_m(x)dx > 0 .$$

But $\int_{-\infty}^\infty \phi_m(x)dx = 0$ if $m = 1, \ldots, n-1$. Therefore

$$S^-(F-G) \geq n .$$

(ii) Suppose now $S^-(F-G) = n$, with the last change being to positive values, and $\mu_{F,r} = \mu_{G,r}$, $r = 1, \ldots, n$. Let X, Y be random variables with distribution functions F, G. Let $F_+(x)$ be the distribution function of $X_+ = \max(X, 0)$, and $F_-(x)$ that of $X_- = \max(-X, 0) = -\min(X, 0)$; let $\mu_{F_+,r} = EX_+^r$, $\mu_{F_-,r} = EX_-^r$. Similarly for G_+, G_-, $\mu_{G_+,r}$, $\mu_{G_-,r}$. Let $\lambda_r = \mu_{F_+,r} - \mu_{G_+,r}$, $\eta_r = \mu_{F_-,r} - \mu_{G_-,r}$. Then

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(3) $$\lambda_r = (-1)^{r-1}\eta_r\ , \quad r = 1, \ldots, n\ .$$

Suppose zero is not a change of sign of $F - G$ and that $S^-(F_+-G_+) = l$; then $S^-(F_--G_-) = n - l$.

The final sign of $F_+ - G_+$ is greater than or equal to 0 , and the initial sign (for x in a right hand neighbourhood of 0) is opposite to that of $F_- - G_-$.

From the variation diminishing theorem [4],

(4) $$S^-(\lambda_r) \le S^+(\lambda_r) \le l\ , \quad S^-(\eta_r) \le S^+(\eta_r) \le n - l\ ,$$

and if $S^-(\lambda_r) = l$, λ_r exhibits the same sequence of signs as $G_+ - F_+$; similarly for $S^-(\eta_r)$.

Therefore, since $\lambda_1 = \eta_1$ and $F_+ - G_+$ and $F_- - G_-$ are initially of opposite sign,

(5) $$S^-(\lambda_r) + S^-(\eta_r) \le n - 1\ .$$

A lemma is now proved for the sequences λ_r, η_r .

LEMMA. *Suppose the real sequences* λ_r, η_r *are related by*

$$\lambda_r = (-1)^{r-1}\eta_r\ , \quad r = 1, \ldots\ .$$

Then $S_n = S^+(\lambda_r,\ r = 1, \ldots, n) + S^+(\eta_r,\ r = 1, \ldots, n) \ge n - 1$, *and equality occurs when* λ_0 *and* λ_n *are both non-zero, and all zeros occur singly and between opposite signs, the same statement of course also holding for* η_r .

Proof. Consideration of $n = 2$ shows that $S_2 \ge 1$ with equality occurring for $\lambda_1, \lambda_2 \ne 0$ (and $\eta_1, \eta_2 \ne 0$).

For $n = 3$, $S_3 = 2$ when λ_1, λ_2 and λ_3 are all non-zero, or when $\lambda_2 = 0$ and λ_1 and λ_3 are of opposite sign. In other cases $S_3 > 2$.

Similarly for $n = 4$. Examples of sign sequences of $(\lambda_1, \ldots, \lambda_4)$

for which $S_4 > 3$ are $(+ve, 0, 0, -ve)$ and $(+ve, 0, +ve, +ve)$.

Suppose the result is true for S_{n-1} .

For $\lambda_{n-1} \neq 0$, $\lambda_n \neq 0$ increases S_{n-1} by 1 while $\lambda_n = 0$ increases S_{n-1} by 2 . In the latter case if λ_{n+1} is then of opposite sign to λ_{n-1} , $S_{n+1} = S_n = S_{n-1} + 2$.

For $\lambda_{n-1} = 0$, $S_{n-1} \geq n$. If λ_n is then of opposite sign to λ_{n-2} , S_{n-1} is not increased; any other λ_n increases S_{n-1} by 2 .

Hence the result holds for all n .

COROLLARY. *When* $S_n = n - 1$,

$$S^+(\lambda_r,\ r = 1,\ \ldots,\ n) = S^-(\lambda_r,\ r = 1,\ \ldots,\ n) ,$$

and $S^+(\eta_r,\ r = 1,\ \ldots,\ n) = S^-(\eta_r,\ r = 1,\ \ldots,\ n)$.

Returning to the proof of the theorem, from (4) and the above lemma,

$$n - 1 \leq S_n \leq n .$$

Suppose $S_n = n$. Then $S^+(\lambda_r,\ r = 1,\ \ldots,\ n) = l$ and $S^+(\eta_r,\ r = 1,\ \ldots,\ n) = n - l$. Therefore

$$S^+(\lambda_r,\ r = n+1,\ \ldots) = 0 = S^+(\eta_r,\ r = n+1,\ \ldots) .$$

(a) If $\lambda_n \neq 0$, $\operatorname{sign}(\lambda_{n+k}) = \operatorname{sign}(\lambda_n)$ and $\operatorname{sign}(\eta_{n+k}) = (-1)^{n-1}\operatorname{sign}(\lambda_n)$. From *(i)*, $\int_{-\infty}^{\infty} \phi_n(x)dx > 0$, that is

$$\mu_{F,n+1} < \mu_{G,n+1} .$$

Therefore

$$\mu_{F,n+2k-1} < \mu_{G,n+2k-1} ,\quad k = 1,\ \ldots .$$

(b) If $\lambda_n = 0$, the proof of the above lemma shows that

$$S^+(\lambda_r,\ r = 1,\ \ldots,\ n-1) = S^-(\lambda_r,\ r = 1,\ \ldots,\ n-1) = l - 1 ,$$

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and since $S^{+}(\lambda_r,\ r = 1,\ \ldots,\ n) = l$, then $S^{-}(\lambda_r,\ r = 1,\ \ldots,\ n+1) = l$.

Similarly, $S^{-}(\eta_r,\ r = 1,\ \ldots,\ n+1) = n - l$; but this is a contradiction of (5).

Suppose $S_n = n - 1$. Then

$$S^{-}(\lambda_r,\ r = 1,\ \ldots,\ n) + S^{-}(\eta_r,\ r = 1,\ \ldots,\ n) = n - 1 ,$$

and $\lambda_n \neq 0$. Hence there can be no further changes of sign or zeros in the sequences λ_r, η_r for $r > n$, and case (a) above applies.

Finally, if zero is a change of sign of $F - G$, $S_n \leq n - 1$, and the result follows from the above.

3. Comments

Theorem 1 gives in particular that if two standardised distribution functions cross twice, the differences between their odd (central) moments of order greater than or equal to 3 are all either positive or negative. If they cross three times and have equal third (central) moments, the differences between their even moments of order greater than or equal to 4 are of the same sign. It is interesting to note that there are no conditions on the number of crossings of the distributions on the left or right of 0 , contrary to what might be inferred from [7].

Part *(i)* of Theorem 1 may also be proved using the variation diminishing theorem (in a much longer proof), so that *(i)* could be generalised to apply when the co-incident moments are not necessarily of consecutive orders. However part *(ii)* depends on alternating odd and even co-incident moments, although other generalisations may be possible in particular cases.

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