NEKHOROSHEV–STABILITY OF L_4 and L_5 IN THE SPATIAL RESTRICTED PROBLEM

GIANCARLO BENETTIN, FRANCESCO FASSÒ and MASSIMILIANO GUZZO
Università di Padova, Dipartimento di Matematica Pura e Applicata, Via G. Belzoni 7, 35131 Padova, Italy

The Lagrangian equilateral points L_4 and L_5 of the restricted circular three–body problem are elliptic for all values of the reduced mass μ below Routh's critical mass $\mu_R \approx .0385$. In the spatial case, because of the possibility of Arnold diffusion, KAM theory does not provide Lyapunov–stability. Nevertheless, one can consider the so–called 'Nekhoroshev–stability': denoting by d a convenient distance from the equilibrium point, one asks whether

$$d(0) \leq \epsilon \quad \Longrightarrow \quad d(t) \leq \epsilon^a \quad \text{for} \quad |t| \leq \exp \epsilon^{-b}$$

for any small $\epsilon > 0$, with positive a and b. Until recently this problem, as more generally the problem of Nekhoroshev–stability of elliptic equilibria of Hamiltonian systems, was studied only under some arithmetic conditions on the frequencies, and thus on μ (see e.g .Giorgilli, 1989). Our aim was instead considering all values of μ up to μ_R. As a matter of fact, Nekhoroshev–stability of elliptic equilibria, without any arithmetic assumption on the frequencies, was proved recently under the hypothesis that the fourth order Birkhoff normal form of the Hamiltonian exists and satisfies a 'quasi–convexity' assumption (Fassò *et al*, 1998; Guzzo *et al*, 1998; Niedermann, 1998). However, in the case of L_4 and L_5 such assumption is not satisfied for any $\mu < \mu_R$. Therefore, our study rests in a crucial way on two extensions of the above result:

(i) The first extension replaces quasi–convexity by a weakened requirement, called *directional quasi–convexity* (DQC), which is specific for the case of an equilibrium point, and has no analogue in the general Nekhoroshev theorem. This property consists in testing quasi–convexity not in the whole plane of fast drift, but only in its intersection with the action space of the elliptic equilibrium, that is, the 'first octant' where all actions are nonnegative. DQC is a natural hypothesis for the case of elliptic equilibria, which appears to play the same role as quasi–convexity in Nekhoroshev theorem, and leads to good stability estimates (e.g. $b = 1/n$).

(ii) The second extension relaxes instead quasi–convexity to a simple 'steepness' condition on the 3–jet of the sixth order Birkhoff normal form, much in the line of Nekhoroshev's original theorem. For a system with three degrees of freedom it is necessary that the Birkhoff normal form can be constructed up to order eight at least, and we obtain $b = 1/20$; this result improves whenever it is possible to construct higher order normal forms.

Precise statements of these results are given in (Bennetin *et al.*, 1998), where one also finds the application to L_4 and L_5. The result of such an analysis, which

J. Henrard and S. Ferraz-Mello (eds.), Impact of Modern Dynamics in Astronomy, 445–446.
© 1999 *Kluwer Academic Publishers. Printed in the Netherlands.*

https://doi.org/10.1017/S0252921100073097 Published online by Cambridge University Press

was performed by numerically constructing the Birkhoff normal forms, is that the system is DQC for all μ outside a relatively small interval $[\mu_1, \mu_2]$, with $\mu_1 \approx .0109$ and $\mu_2 \approx .0164$, but at one point, namely $\mu_{(1,2,0)} \approx .0243$ where the Birkhoff normal form does not exist because of the resonance $(1,2,0)$. The hypotheses for the applicability of extension (ii) are satisfied everywhere in the interval $[\mu_1, \mu_2]$, but at three points $\mu_{(1,3,0)} \approx .0135$, $\mu_{(0,3,1)} \approx .0149$, and $\mu_{(3,3,-2)} \approx .0116$ where certain resonances prevent the existence of normal forms of order eight, and at a point $\mu_3 \approx .01478$ where the considered steepness property is violated. Summarizing, L_4 and L_5 are Nekhoroshev–stable for all values of $\mu < \mu_R$ but possibly five isolated values. However, the estimated stability times are not uniform in μ, depending on whether the system is DQC ($b = 1/3$) or not ($b = 1/20$, in the absence of special properties); it may be interesting to observe that the reduced mass of the Earth–Moon system falls inside the interval $[\mu_1, \mu_2]$ so that, on the basis of our analysis, rather poor stability properties might be found.

References

G. Benettin, F. Fassò and M. Guzzo: 1998, *Nekhoroshev–stability of L4 and L5 in the spatial restricted three–body problem,* to appear in Regular and Chaotic Dynamics. Postscript file available at `http://www.math.unipd.it:80/~fasso/#Publications`.

F. Fassò, M. Guzzo, G. Benettin: 1998, Nekhoroshev–stability of elliptic equilibria of Hamiltonian systems, *Comm. in Math. Phys.*, **197**, 347-360.

A. Giorgilli, A. Delshams, E. Fontich, L. Galgani, C. Simó: 1989, *J. Diff. Eq.*, **77**, 167-198

M. Guzzo, F. Fassò and G. Benettin: 1998, *Mathemathical Physics Electronic Journal*, **4**, Paper 1

L. Niedermann: 1998, Nonlinear stability around an elliptic equilibrium point in an Hamiltonian system, *Nonlinearity*, **11**, 1465-1479.

https://doi.org/10.1017/S0252921100073097 Published online by Cambridge University Press