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GROUP RINGS WITH ONLY TRIVIAL UNITS OF FINITE ORDER

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1. Introduction. We denote by ZG the integral group ring of the finite group G. S.D. Berman [**1**] showed that every unit of finite order μ in G is trivial (i.e., $\mu = \pm g$ for some g in G) if and only if either G is abelian or G is a Hamiltonian 2-group. In this note, we give a new and shorter proof for the "only if" part. In fact, we prove the following

THEOREM. *Let G be a finite group. Suppose that for γ in $(ZG)^*$ (the group of units of ZG), $\gamma^{-1}g\gamma$ is in G for all g in G. Then G is either abelian or a Hamiltonian 2-group.*

We also characterize all the finite groups in the set $\mathscr{C}$, which Sehgal has defined as the set of groups G with the property that for any isomorphism $\theta : ZG \rightarrow ZH$, for each g in G, $\theta(g) = \pm h$, where h is in H [**3**, p. 1182]. In fact, we obtain the following

COROLLARY. *Let G be a finite group. Then the following are equivalent.*

(1) *G is in $\mathscr{C}$.*
(2) *Every inner automorphism of ZG is the extension of an automorphism of G.*
(3) *G is either abelian or a Hamiltonian 2-group.*
(4) *ZG contains only trivial units of finite order.*

For a group G as in the Corollary, we remark that by Sehgal [**2**, Theorem 2] we see that every normalized automorphism of ZG is the extension of an automorphism of G.

2. Proof of the theorem. We assume the hypothesis of the theorem. We claim that if $\mu \in ZG$, $\mu^2 = 0$, then $\mu = 0$. Since $(1 + \mu)(1 - \mu) = 1$, $1 + \mu$ is in ZG^*. By assumption the mapping ϕ defined by $\phi(g) = (1 + \mu)g(1 + \mu)^{-1}$ is an automorphism of G. As G is finite, ϕ has finite order k. So $\phi^k(g) = (1 + k\mu)g(1 - k\mu) = g$ for all g in G. Thus we have that $1 + k\mu$ is in the centre of ZG and so μ is in the centre of QG. But the centre of QG is a direct sum of fields; thus $\mu = 0$.

We now show that every cyclic subgroup is normal in G, and that will imply that G is Hamiltonian. For, given g in G as a generator of a cyclic group of order n, and for any h in G, let $\mu = (1 - g)h(1 + g + g^2 + \ldots + g^{n-1})$; then $\mu^2 = 0$ and consequently by the above $\mu = 0$. Hence, we must have $h = ghg^r$ for some positive integer r. It follows that the cyclic group generated by g is normal in G.

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If G is a Hamiltonian group, then G is the direct product of a quaterion group $H = \langle a, b; a^4 = 1, b^2 = a^2, b^{-1}ab = a^3\rangle$, an abelian group S of odd order and an abelian group of exponent 2. We now show that S is trivial under our assumption.

Suppose there exists an s in S of order p, an odd prime. Let $t = as$. Then T, the group generated by t, has order $4p$. Let ζ_d be a primitive dth root of unity. Consider the mapping θ from QT onto $K = \oplus_{d|4p}Q(\zeta_d)$ given by:

$$\theta(t) = \sum \zeta_d (d|4p)$$

and extended to the whole QT in the obvious way to make it a homomorphism. By the Chinese remainder theorem θ is an isomorphism.

Let $R = \oplus Z[\zeta_d](d|4p)$; then $\theta(ZT)$ and R are two orders in K with $\theta(ZT)$ contained in R. Clearly there exists an integer l such that lR is contained in $\theta(ZT)$. Let g, h be in R^* and $g + lR = h + lR$; then $g^{-1}h$ is in $\theta(ZT)^*$. Since the index of R over lR is finite, so also is the index of R^* over $\theta(ZT)^*$.

Let $\zeta_{4p} = \zeta$. Applying the Dirichlet-Minkowski unit theorem to both $Z[\zeta]$ and $Z[\zeta^2]$, we can choose a v in $Z[\zeta]^*$ such that for all i, v^i is not in $Z[\zeta^2]^*$. Let $\gamma = \theta^{-1}(1 + 1 + \ldots + 1 + v)$ be in $\theta^{-1}(R)$. Then we can find an integer k with γ^k in ZT^*. Again as in the previous proof, $\omega = (\gamma^k)^l$ is in the centre of ZG for some integer l. So $\psi\theta(\omega)$ is not in $Z[\zeta^2]^*$ where ψ is the projection mapping from R onto $Z[\zeta]$. It follows that $\psi\theta(\omega)$ is not even in $Z[\zeta^2]$. We now show that $\psi\theta(\omega)$ is in $Z[\zeta^2]$.

Now, ω is in ZT. Let W be the group generated by t^2. Then $\omega = \alpha + t\beta$ with α, β in ZW (which is contained in the centre of ZG). Thus $b(\alpha + t\beta) = (\alpha + t\beta)b$ which implies that $(1 - a^2)\beta = 0$, hence $\beta = (1 + a^2)\sigma$ for some σ in ZW. Now, $\omega = \alpha + a(1 + a^2)\sigma s = f(t^2) + (t^p + t^{-p})g(t^2)$ where f and g are polynomials over Z.

Thus

$$\begin{aligned} \psi\theta(\omega) &= f(\zeta^2) + (\zeta^p + \zeta^{-p})g(\zeta^2) \\ &= f(\zeta^2), \text{ since } \zeta^p + \zeta^{-p} = 0, \end{aligned}$$

i.e., $\psi\theta(\omega)$ is in $Z[\zeta^2]$. This is a contradiction.

Hence, we have shown that S is trivial and $|G| = 2^m$ for some m. The theorem is thus proved.

We now prove the Corollary by showing that (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) $\Rightarrow$ (1). The implication (2) $\Rightarrow$ (3) is our theorem and (3) $\Rightarrow$ (4) follows easily (see Berman [**1**]). The other implications are obvious.

References

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