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NECESSARY AND SUFFICIENT CONDITIONS FOR OSCILLATIONS OF DELAY DIFFERENTIAL INEQUALITIES

JURANG YAN

A necessary and sufficient condition is obtained for a first order linear delay differential inequality to be oscillatory. Our main result improves and extends some known results.

1. INTRODUCTION

The oscillation theory of first order linear delay differential inequalities and equations has been extensively developed by many authors. We mention in particular the papers by Koplatadze and Chanturia [1], Ladas [2], and Ladas and Stavroulakis [3]. for a survey of results concerned with oscillations we refer to Zhang [5] and the references therein. Most of the known results, however, give sufficient conditions for oscillations.

The aim of this paper is to obtain necessary and sufficient conditions under which the first order delay differential inequality

$$x(t)[x'(t)+p(t)x(t-\tau(t))]\leqslant 0 \tag{1}$$

is oscillatory, where:

(a) $$p(t)\in C[[t_0,\infty),\mathbb{R}] \text{ and } p(t)\geqslant 0;$$

(b) $$\tau(t)\in C[[t_0,\infty),\mathbb{R}],\quad \tau(t)\geqslant 0 \text{ and } \lim_{t\to\infty}[t-\tau(t)]=\infty.$$

Conditions (a) and (b) are assumed to hold throughout this paper.

As is customary, a solution is said to be *oscillatory* if it has arbitrarily large zeros and *nonoscillatory* if it is eventually positive or eventually negative. Inequality (1) is said to be *oscillatory* if every solution of (1) is oscillatory.

Our main result is the following:

THEOREM. *A necessary and sufficient condition for inequality (1) to be oscillatory is that for every sufficiently large T there exists a $\bar{t}>T$ such that*

$$\int_{\bar{t}-\tau(\bar{t})}^{\bar{t}} p(s)\,ds>\frac{1}{e}. \tag{2}$$

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As an immediate consequence of the theorem we obtain the following corollary for the inequalities

$$x'(t) + p(t)x(t - \tau(t)) \leqslant 0, \tag{3}$$

$$x'(t) + p(t)x(t - \tau(t)) \geqslant 0 \tag{4}$$

and the equation

$$x'(t) + p(t)x(t - \tau(t)) = 0. \tag{5}$$

COROLLARY. *A necessary and sufficient condition for (3) to have no eventually positive solution, (4) to have no eventually negative solution and (5) to be oscillatory, is that condition (2) is satisfied.*

2. Proof of the theorem

Consider the integral inequality

$$u(t) \geqslant p(t) \exp\left(\int_{t-\tau(t)}^{t} u(s)\mathrm{d}s\right). \tag{6}$$

The function $u(t)$ said to be a nonnegative solution on $[t_1 \infty)$ ($t_1 \geqslant t_0$) if $u(t) \geqslant 0$ is continuous and bounded on $[t_1, t_2)$, continuous on $[t_2, \infty)$ and satisfies (6) for all $t \geqslant t_2$, where $t_2 = \inf\{t \mid t - \tau(t) \geqslant t_1\}$.

Using an argument similar to that given by the author ([4]), the following lemma is easily established:

LEMMA 1. *Inequality (1) has a nonoscillatory solution if and only if inequality (6) has a nonnegative solution on $[t_1, \infty)$ for some large t_1.*

For some $t_1 \geqslant t_0$, we define the following sequence of functions on $[t_1, \infty)$.

$$\begin{aligned} &u_0(t) = 0, \qquad t \geqslant t_2 \\ &u_k(t) = \begin{cases} p(t)\exp\left(\int_{t-\tau(t)}^{t} u_{k-1}(s)\mathrm{d}s\right), & t \geqslant t_2, \\ 0, \quad t_1 \leqslant t < t_2 \end{cases} \\ &\qquad k = 1, 2, \ldots. \end{aligned} \tag{7}$$

where $t_2 = \inf\{t \mid t - \tau(t) \geqslant t_1\}$. By a simple induction, we have that

$$0 \leqslant u_k(t) \leqslant u_{k+1}(t), \qquad t \geqslant t_1, \quad k = 1, 2, \ldots. \tag{8}$$

LEMMA 2. *Inequality (6) has a nonnegative solution on* $[t_1,\infty)$ *if and only if the sequence (7) is convergent for any* $t \geqslant t_1$.

PROOF: Suppose that $u(t)$ is a nonnegative solution on $[t_1,\infty)$. Thus $u_0(t) \leqslant u(t)$, $t \in [t_1,\infty)$. Then, it follows by induction that

$$u_0(t) \leqslant u_k(t) \leqslant u(t), \qquad t \geqslant t_1, \quad k = 1,2,\ldots.$$

Hence, from (8), we see that the sequence (7) converges to a finite limit $\tilde{u}(t)$, $t \geqslant t_1$.

Conversely, suppose that the sequence (7) converges to a finite limit $u(t)$ for any $t \geqslant t_1$. Hence $u_k(t) \leqslant u(t)$, $t \geqslant t_1$, and, by the monotone convergence theorem,

$$\lim_{k\to\infty}\int_{t-\tau(t)}^{t} u_k(s)\mathrm{d}s = \int_{t-\tau(t)}^{t} u(s)\mathrm{d}s, \qquad t \geqslant t_2.$$

From (7), we have that

$$u(t) = \begin{cases} p(t)\exp\left(\int_{t-\tau(t)}^{t} u(s)\mathrm{d}s\right), & t \geqslant t_2, \\ 0 \qquad t_1 \leqslant t < t_2, \end{cases}$$

which implies that (6) has a nonnegative solution on $[t_1,\infty)$. ∎

By combining Lemma 1 and Lemma 2 we have the following result:

LEMMA 3. *Inequality (1) is oscillatory if and only if, for every large* T, *there exists a* $\tilde{t} > T$ *such that*

$$\lim_{k\to\infty} u_k(\tilde{t}) = \infty.$$

PROOF OF THEOREM: Let

$$\int_{\tilde{t}-\tau(\tilde{t})}^{\tilde{t}} p(s)\mathrm{d}s = \tilde{\alpha}, \qquad \tilde{t} > T \geqslant t_3, \tag{9}$$

where $t_3 = \inf\{t \mid t - \tau(t) \geqslant t_2\}$. From (9), we are able to choose a constant α with $\dfrac{1}{e} < \alpha < \tilde{\alpha}$. Now, choose an interval $\Delta = (\tilde{t}-\delta, \tilde{t}+\delta)$ $(\tilde{t}-\delta \geqslant T)$ such that

$$\int_{t-\tau(t)}^{t} p(s)\mathrm{d}s \geqslant \alpha \qquad t \in \Delta. \tag{10}$$

Consider the sequence (7) on the interval Δ,

$$\begin{aligned} u_0(t) &= 0, \qquad t \in \Delta, \\ u_1(t) &= \xi_1 p(t) \qquad t \in \Delta, \text{ where } \xi_1 = 1, \\ u_2(t) &= p(t)\exp\left(\int_{t-\tau(t)}^{t} p(s)\mathrm{d}s\right) \geqslant p(t)e^{\xi_1\alpha} \\ &= \xi_2 p(t), \text{ where } \xi_2 = e^{\xi_1\alpha}. \end{aligned}$$

By a simple induction, it is easy to show that

$$u_{k+1}(t) \geqslant \xi_{k+1} p(t), \quad t \in \Delta \tag{11}$$

where

$$\xi_{k+1} = e^{\xi_k \alpha}. \tag{12}$$

Obviously, $\{\xi_n\}_{n=1}^{\infty}$ is increasing. Let $\lim\limits_{n\to\infty} \xi_n = \xi$, where ξ is a positive constant or infinite. We claim that ξ is infinite. Indeed, if this were false, the sequence would converge to some finite limit $\xi > 0$. Then, in view of (12), it follows that $\xi = e^{\alpha\xi}$ and hence we derive that $(\ln \xi)/\xi = \alpha$. But $\max\limits_{\xi>0}\{(\ln\xi)/\xi\} = \dfrac{1}{e}$. This implies $\alpha \leqslant \dfrac{1}{e}$, which is impossible. Therefore $\lim\limits_{n\to\infty} \xi_n = \infty$.

From (11), we find that

$$\lim_{\to\infty} u_k(t) = \infty, \qquad t \in \Delta.$$

Thus, by Lemma 3, inequality (1) is oscillatory.

Conversely, suppose that there exists a large $T \geqslant t_3$, where $t_3 = \inf\{t \mid t - \tau(t) \geqslant t_2\}$ such that

$$\int_{t-\tau(t)}^{t} p(s)\mathrm{d}s \leqslant \frac{1}{e}, \qquad t \geqslant T. \tag{13}$$

Consider the sequence (7) on $[T, \infty)$,

$$\begin{aligned} u_0(t) &= 0, \qquad t \geqslant T, \\ u_1(t) &= p(t) \leqslant ep(t), \qquad t \geqslant T. \end{aligned}$$

From (7) and (13), we have that

$$u_2(t) = p(t) \exp\left(\int_{t-\tau(t)}^{t} ep(s)\mathrm{d}s\right) \leqslant ep(t), \qquad t \geqslant T.$$

By induction, we have that

$$u_k(t) \leqslant e\, p(t), \qquad t \geqslant T, \quad k = 1, 2, \ldots.$$

Consequently, $u_n(t)$ converges pointwise to a finite limit for each $t \geqslant T$. By Lemma 3, (1) has a nonoscillatory solution. The proof of the Theorem is complete. ∎

3. Remarks

1. The Theorem in this paper improves and extends the previous results of Koplatdze and Chanturia [**1**] and Ladas [**2**].

2. One easily generalises the results in this paper to inequalities of the form

$$x(t)[x(t) + a(t)x(t) + p(t)x(t - \tau(t))] \leqslant 0,$$

where $p(t)$ and $\tau(t)$ satisfy conditions (a) and (b). The function $a(t) \in C\,[[t_0, \infty), \mathbb{R})$ and is allowed to change sign on $[t_0, \infty)$. Hence our results improve and extend the results of Ladas and Stravroulakis, [**3**].

3. Our techniques can be applied to advanced differential inequalities of the form

$$x(t)[x'(t) + a(t)x(t) - p(t)x(t + \tau(t))] \leqslant 0,$$

where $p(t)$, $\tau(t)$ and $a(t)$ satisfy the above conditions.

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Department of Mathematics,
Shanzi University,
Taiyuan, Shanxi,
People's Republic of China.