1

Graded rings and graded modules

Graded rings appear in many circumstances, both in elementary and advanced areas. Here are two examples.

1 In elementary school when we distribute 10 apples giving 2 apples to each person, we have

$$10 \text{ Apples} : 2 \text{ Apples} = 5 \text{ People}.$$

The psychological problem caused to many kids as to exactly how the word "People" appears in the equation can be overcome by correcting it to

$$10 \text{ Apples} : 2 \text{ Apples / People} = 5 \text{ People}.$$

This shows that already at the level of elementary school arithmetic, children work in a much more sophisticated structure, *i.e.,* the graded ring

$$\mathbb{Z}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$$

of Laurent polynomial rings! (see the interesting book of Borovik [23, §4.7] on this).

2 If A is a commutative ring generated by a finite number of elements of degree 1, then by the celebrated work of Serre [85], the category of quasicoherent sheaves on the scheme is equivalent to QGr-$A \cong$ Gr-A/ Fdim-A, where Gr-A is the category of graded modules over A and Fdim-A is the Serre subcategory of (direct limits of) finite dimensional submodules. In particular when $A = K[x_0, x_1, \dots, x_n]$, where K is a field, then QCoh-$\mathbb{P}^n$ is equivalent to QGr-$A[x_0, x_1, \dots, x_n]$ (see [85, 9, 79] for more precise statements and relations with noncommutative algebraic geometry).

This book treats graded rings and the category of graded modules over a

https://doi.org/10.1017/CBO9781316717134.002 Published online by Cambridge University Press

graded ring. This category is an abelian category (in fact a Grothendieck category). Many of the classical invariants constructed for the category of modules can be constructed, *mutatis mutandis*, starting from the category of graded modules. The general viewpoint of this book is that, once a ring has a natural graded structure, graded invariants capture more information than the non-graded counterparts.

In this chapter we give a concise introduction to the theory of graded rings. We introduce grading on matrices, study graded division rings and introduce gradings on graph algebras that will be the source of many interesting examples.

1.1 Graded rings

1.1.1 Basic definitions and examples

A ring A is called a Γ-*graded ring*, or simply a *graded ring*, if $A = \bigoplus_{\gamma\in\Gamma} A_\gamma$, where Γ is an (abelian) group, each A_γ is an additive subgroup of A and $A_\gamma A_\delta \subseteq A_{\gamma+\delta}$ for all $\gamma, \delta \in \Gamma$.

If A is an algebra over a field K, then A is called a *graded algebra* if A is a graded ring and for any $\gamma \in \Gamma$, A_γ is a K-vector subspace.

The set $A^h = \bigcup_{\gamma\in\Gamma} A_\gamma$ is called the set of *homogeneous elements* of A. The additive group A_γ is called the γ-*component* of A and the nonzero elements of A_γ are called *homogeneous of degree* γ. We write $\deg(a) = \gamma$ if $a \in A_\gamma \backslash \{0\}$. We call the set

$$\boxed{\Gamma_A = \{\, \gamma \in \Gamma \mid A_\gamma \neq 0 \,\}}$$

the *support* of A. We say the Γ-graded ring A has a *trivial grading*, or A is *concentrated in degree zero*, if the support of A is the trivial group, *i.e.*, $A_0 = A$ and $A_\gamma = 0$ for $\gamma \in \Gamma \backslash \{0\}$.

For Γ-graded rings A and B, a Γ-*graded ring homomorphism* $f : A \to B$ is a ring homomorphism such that $f(A_\gamma) \subseteq B_\gamma$ for all $\gamma \in \Gamma$. A graded homomorphism f is called a *graded isomorphism* if f is bijective and, when such a graded isomorphism exists, we write $A \cong_{\mathrm{gr}} B$. Notice that if f is a graded ring homomorphism which is bijective, then its inverse f^{-1} is also a graded ring homomorphism.

If A is a graded ring and R is a commutative graded ring, then A is called a *graded R-algebra* if A is an R-algebra and the associated algebra homomorphism $\phi : R \to A$ is a graded homomorphism. When R is a field concentrated in degree zero, we retrieve the definition of a graded algebra above.

Proposition 1.1.1 *Let $A = \bigoplus_{\gamma \in \Gamma} A_\gamma$ be a Γ-graded ring. Then*

(1) *1_A is homogeneous of degree 0;*
(2) *A_0 is a subring of A;*
(3) *each A_γ is an A_0-bimodule;*
(4) *for an invertible element $a \in A_\gamma$, its inverse a^{-1} is homogeneous of degree $-\gamma$, i.e., $a^{-1} \in A_{-\gamma}$.*

Proof (1) Suppose $1_A = \sum_{\gamma \in \Gamma} a_\gamma$ for $a_\gamma \in A_\gamma$. Let $b \in A_\delta$, $\delta \in \Gamma$, be an arbitrary nonzero homogeneous element. Then $b = b1_A = \sum_{\gamma \in \Gamma} ba_\gamma$, where $ba_\gamma \in A_{\delta+\gamma}$ for all $\gamma \in \Gamma$. Since the decomposition is unique, $ba_\gamma = 0$ for all $\gamma \in \Gamma$ with $\gamma \neq 0$. But as b was an arbitrary homogeneous element, it follows that $ba_\gamma = 0$ for all $b \in A$ (not necessarily homogeneous), and in particular $1_A a_\gamma = a_\gamma = 0$ if $\gamma \neq 0$. Thus $1_A = a_0 \in A_0$.

(2) This follows since A_0 is an additive subgroup of A with $A_0 A_0 \subseteq A_0$ and $1 \in A_0$.

(3) This is immediate.

(4) Let $b = \sum_{\delta \in \Gamma} b_\delta$, with $\deg(b_\delta) = \delta$, be the inverse of $a \in A_\gamma$, so that $1 = ab = \sum_{\delta \in \Gamma} ab_\delta$, where $ab_\delta \in A_{\gamma+\delta}$. By (1), since 1 is homogeneous of degree 0 and the decomposition is unique, it follows that $ab_\delta = 0$ for all $\delta \neq -\gamma$. Since a is invertible, $b_{-\gamma} \neq 0$, so $b = b_{-\gamma} \in A_{-\gamma}$ as required. □

The ring A_0 is called the *0-component ring* of A and plays a crucial role in the theory of graded rings. The proof of Proposition 1.1.1(4), in fact, shows that if $a \in A_\gamma$ has a left (or right) inverse then that inverse is in $A_{-\gamma}$. In Theorem 1.6.9, we characterise $\mathbb{Z}$-graded rings such that A_1 has a left (or right) invertible element.

Example 1.1.2 GROUP RINGS

For a group Γ, the group ring $\mathbb{Z}[\Gamma]$ has a natural Γ-grading

$$\mathbb{Z}[\Gamma] = \bigoplus_{\gamma \in \Gamma} \mathbb{Z}[\Gamma]_\gamma, \text{ where } \mathbb{Z}[\Gamma]_\gamma = \mathbb{Z}\gamma.$$

In §1.1.4, we construct crossed products which are graded rings and are generalisations of group rings and skew groups rings. A group ring has a natural involution which makes it an involutary graded ring (see §1.9).

In several applications (such as K-theory of rings, Chapter 6) we deal with $\mathbb{Z}$-graded rings with support in $\mathbb{N}$, the so called *positively graded rings*.

Example 1.1.3 TENSOR ALGEBRAS AS POSITIVELY GRADED RINGS

Let A be a commutative ring and M be an A-module. Denote by $T_n(M)$,

$n \geq 1$, the tensor product of n copies of M over A. Set $T_0(M) = A$. Then the natural A-module isomorphism $T_n(M) \otimes_A T_m(M) \to T_{n+m}(M)$, induces a ring structure on

$$T(M) := \bigoplus_{n \in \mathbb{N}} T_n(M).$$

The A-algebra $T(M)$ is called the *tensor algebra* of M. Setting

$$T(M)_n := T_n(M)$$

makes $T(M)$ a $\mathbb{Z}$-graded ring with support $\mathbb{N}$. From the definition, we have $T(M)_0 = A$.

If M is a free A-module, then $T(M)$ is a free algebra over A, generated by a basis of M. Thus free rings are $\mathbb{Z}$-graded rings with the generators being homogeneous elements of degree 1. We will systematically study the grading of free rings in §1.6.1.

Example 1.1.4 Formal matrix rings as graded rings

Let R and S be rings, M a R–S-bimodule and N a S–R-bimodule. Consider the set

$$T := \left\{ \begin{pmatrix} r & m \\ n & s \end{pmatrix} \middle| \, r \in R, s \in S, m \in M, n \in N \right\}.$$

Suppose that there are bimodule homomorphisms $\phi : M \otimes_S N \to R$ and $\psi : N \otimes_R M \to S$ such that $(mn)m' = m(nm')$, where we denote $\phi(m,n) = mn$ and $\psi(n,m) = nm$. One can then check that T with matrix addition and multiplication forms a ring with an identity. The ring T is called the *formal matrix ring* and denoted also by

$$T = \begin{pmatrix} R & M \\ N & S \end{pmatrix}.$$

For example, the Morita ring of a module is a formal matrix ring (see §2.3 and (2.6)).

Considering

$$T_0 = \begin{pmatrix} R & 0 \\ 0 & S \end{pmatrix}, \quad T_1 = \begin{pmatrix} 0 & M \\ N & 0 \end{pmatrix},$$

it is easy to check that T becomes a $\mathbb{Z}_2$-graded ring. In the cases that the images of ϕ and ψ are zero, these rings have been extensively studied (see [57] and references therein).

When $N = 0$, the ring T is called a *formal triangular matrix ring*. In this case there is no need to consider the homomorphisms ϕ and ψ. Setting further $T_i = 0$ for $i \neq 0, 1$ makes T also a $\mathbb{Z}$-graded ring.

One specific example of such a grading on (subrings of) formal triangular matrix rings is used in representation theory. Recall that for a field K, a finite dimensional K-algebra R is called *Frobenius algebra* if $R \cong R^*$ as right R-modules, where $R^* := \operatorname{Hom}_K(R, K)$. Note that R^* has a natural R-bimodule structure.

Starting from a finite dimensional K-algebra R, one constructs the *trivial extension* of R which is a Frobenius algebra and has a natural $\mathbb{Z}$-graded structure as follows. Consider $A := R \bigoplus R^*$, with addition defined component-wise and multiplication defined as

$$(r_1, q_1)(r_2, q_2) = (r_1 r_2, r_1 q_1 + q_2 r_2),$$

where $r_1, r_2 \in R$ and $q_1, q_2 \in R^*$. Clearly A is a Frobenius algebra with identity $(1, 0)$. Moreover, setting

$$\begin{aligned} A_0 &= R \oplus 0, \\ A_1 &= 0 \oplus R^*, \\ A_i &= 0, \text{ otherwise,} \end{aligned}$$

makes A into a $\mathbb{Z}$-graded ring with support $\{0, 1\}$. In fact this ring is a subring of the formal triangular matrix ring

$$T_0 = \begin{pmatrix} R & R^* \\ 0 & R \end{pmatrix},$$

consisting of elements $\begin{pmatrix} a & q \\ 0 & a \end{pmatrix}$.

These rings appear in representation theory (see [46, §2.2]). The graded version of this contraction is carried out in Example 1.2.9.

Example 1.1.5 THE GRADED RING A AS A_0-MODULE

Let A be a Γ-graded ring. Then A can be considered as an A_0-bimodule. In many cases A is a projective A_0-module, for example in the case of group rings (Example 1.1.2) or when A is a strongly graded ring (see §1.1.3 and Theorem 1.5.12). Here is an example that this is not the case in general. Consider the formal matrix ring T

$$T = \begin{pmatrix} R & M \\ 0 & 0 \end{pmatrix},$$

where M is a left R-module which is not a projective R-module. Then by Example 1.1.4, T is a $\mathbb{Z}$-graded ring with $T_0 = R$ and $T_1 = M$. Now T as a T_0-module is $R \oplus M$ as an R-module. Since M is not projective, $R \oplus M$ is not a projective R-module. We also get that T_1 is not a projective T_0-module.

https://doi.org/10.1017/CBO9781316717134.002 Published online by Cambridge University Press

1.1.2 Partitioning graded rings

Let A be a Γ-graded ring and $f : \Gamma \to \Delta$ be a group homomorphism. Then one can assign a natural Δ-graded structure to A as follows: $A = \bigoplus_{\delta\in\Delta} A_\delta$, where

$$A_\delta = \begin{cases} \bigoplus_{\gamma\in f^{-1}(\delta)} A_\gamma & \text{if } f^{-1}(\delta) \neq \varnothing; \\ 0 & \text{otherwise.} \end{cases}$$

In particular, for a subgroup Ω of Γ we have the following constructions.

Subgroup grading The ring $A_\Omega := \bigoplus_{\omega\in\Omega} A_\omega$ forms a Ω-graded ring. In particular, A_0 corresponds to the trivial subgroup of Γ.

Quotient grading Considering

$$A = \bigoplus_{\Omega+\alpha\in\Gamma/\Omega} A_{\Omega+\alpha},$$

where

$$A_{\Omega+\alpha} := \bigoplus_{\omega\in\Omega} A_{\omega+\alpha},$$

makes A a Γ/Ω-graded ring. (Note that if Γ is not abelian, then for this construction, Ω needs to be a normal subgroup.) Notice that with this grading, $A_0 = A_\Omega$. If $\Gamma_A \subseteq \Omega$, then A, considered as a Γ/Ω-graded ring, is concentrated in degree zero.

This construction induces a *forgetful* functor (or with other interpretations, a *block*, or a *coarsening* functor) from the category of Γ-graded rings $\mathcal{R}^\Gamma$ to the category of Γ/Ω-graded rings $\mathcal{R}^{\Gamma/\Omega}$, *i.e.,*

$$U : \mathcal{R}^\Gamma \to \mathcal{R}^{\Gamma/\Omega}.$$

If $\Omega = \Gamma$, this gives the obvious forgetful functor from the category of Γ-graded rings to the category of rings. We give a specific example of this construction in Example 1.1.8 and others in Examples 1.1.20 and 1.6.1.

Example 1.1.6 TENSOR PRODUCT OF GRADED RINGS

Let A be a Γ-graded and B a Ω-graded ring. Then $A \otimes_{\mathbb{Z}} B$ has a natural $\Gamma\times\Omega$-graded ring structure as follows. Since A_γ and B_ω, $\gamma \in \Gamma$, $\omega \in \Omega$, are $\mathbb{Z}$-modules then $A \otimes_{\mathbb{Z}} B$ can be decomposed as a direct sum

$$A \otimes_{\mathbb{Z}} B = \bigoplus_{(\gamma,\omega)\in\Gamma\times\Omega} A_\gamma \otimes B_\omega$$

(to be precise, $A_\gamma \otimes B_\omega$ is the image of $A_\gamma \otimes_{\mathbb{Z}} B_\omega$ in $A \otimes_{\mathbb{Z}} B$).

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Now, if $\Omega = \Gamma$ and

$$f : \Gamma \times \Gamma \longrightarrow \Gamma,$$
$$(\gamma_1, \gamma_2) \longmapsto \gamma_1 + \gamma_2,$$

then we get a natural Γ-graded structure on $A \otimes_{\mathbb{Z}} B$. Namely,

$$A \otimes_{\mathbb{Z}} B = \bigoplus_{\gamma \in \Gamma} (A \otimes B)_\gamma,$$

where

$$(A \otimes B)_\gamma = \Big\{ \sum_i a_i \otimes b_i \mid a_i \in A^h, b_i \in B^h, \deg(a_i) + \deg(b_i) = \gamma \Big\}.$$

We give specific examples of this construction in Example 1.1.7. One can replace $\mathbb{Z}$ by a field K, if A and B are K-algebras and A_γ, B_γ are K-modules.

Example 1.1.7 Let A be a ring with identity and Γ be a group. We consider A as a Γ-graded ring concentrated in degree zero. Then, by Example 1.1.6,

$$A[\Gamma] \cong A \otimes_{\mathbb{Z}} \mathbb{Z}[\Gamma]$$

has a Γ-graded structure, *i.e.,* $A[\Gamma] = \bigoplus_{\gamma \in \Gamma} A\gamma$. If A itself is a (nontrivial) Γ-graded ring $A = \bigoplus_{\gamma \in \Gamma} A_\gamma$, then by Example 1.1.6, $A[\Gamma]$ has also a Γ-grading

$$A[\Gamma] = \bigoplus\nolimits_{\gamma \in \Gamma} A^\gamma, \text{ where } A^\gamma = \bigoplus\nolimits_{\gamma = \zeta + \zeta'} A_\zeta \zeta'. \tag{1.1}$$

A specific example is when A is a positively graded $\mathbb{Z}$-graded ring. Then $A[x] \cong A \otimes \mathbb{Z}[x]$ is a $\mathbb{Z}$-graded ring with support $\mathbb{N}$, where

$$A[x]_n = \bigoplus_{i+j=n} A_i x^j.$$

This graded ring will be used in §6.2.4 when we prove the fundamental theorem of K-theory. Such constructions are systematically studied in [72] (see also [75, §6]).

Example 1.1.8 Let A be a $\Gamma \times \Gamma$-graded ring. Define a Γ-grading on A as follows. For $\gamma \in \Gamma$, set

$$A'_\gamma = \sum_{\alpha \in \Gamma} A_{\gamma - \alpha, \alpha}.$$

It is easy to see that $A = \bigoplus_{\gamma \in \Gamma} A'_\gamma$ is a Γ-graded ring. When A is $\mathbb{Z} \times \mathbb{Z}$-graded, then the $\mathbb{Z}$-grading on A is obtained from considering all the homogeneous components on a diagonal together, as Figure 1.1 shows.

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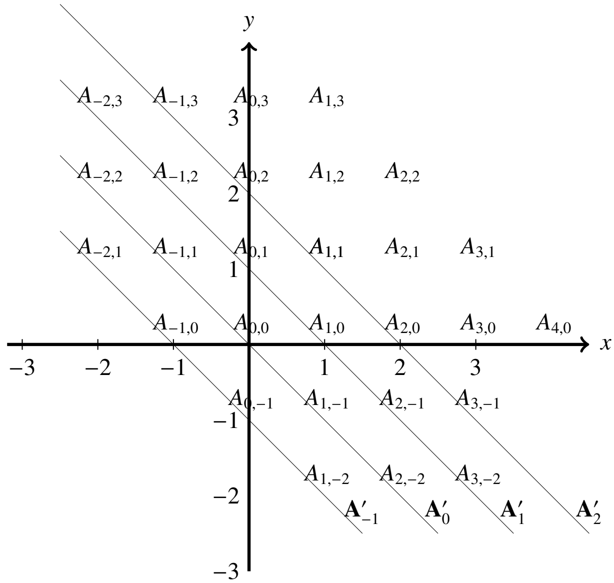

Figure 1.1

In fact this example follows from the general construction given in §1.1.2. Consider the homomorphism $\Gamma \times \Gamma \to \Gamma, (\alpha, \beta) \mapsto \alpha + \beta$. Let Ω be the kernel of this map. Clearly $(\Gamma \times \Gamma)/\Omega \cong \Gamma$. One can check that the $(\Gamma \times \Gamma)/\Omega$-graded ring A gives the graded ring constructed in this example (see also Remark 1.1.26).

Example 1.1.9 <u>The direct limit of graded rings</u>

Let A_i, $i \in I$, be a direct system of Γ-graded rings, *i.e.,* I is a directed partially ordered set and for $i \leq j$ there is a graded homomorphism $\phi_{ij} : A_i \to A_j$ which is compatible with the ordering. Then $A := \varinjlim A_i$ is a Γ-graded ring with homogeneous components $A_\alpha = \varinjlim A_{i\alpha}$. For a detailed construction of such direct limits see [24, II, §11.3, Remark 3].

As an example, the ring $A = \mathbb{Z}[x_i \mid i \in \mathbb{N}]$, where $A = \varinjlim_{i \in \mathbb{N}} \mathbb{Z}[x_1, \dots, x_i]$, with $\deg(x_i) = 1$ is a $\mathbb{Z}$-graded ring with support $\mathbb{N}$. We give another specific example of this construction in Example 1.1.10.

We will study in detail one type of these graded rings, *i.e.,* graded ultramatricial algebras (Chapter 5, Definition 5.2.1) and will show that the graded Grothendieck group (Chapter 3) classifies these graded rings completely.

Example 1.1.10 Let $A = \bigoplus_{\gamma \in \Gamma} A_\gamma$ and $B = \bigoplus_{\gamma \in \Gamma} B_\gamma$ be Γ-graded rings. Then $A \times B$ has a natural grading given by $A \times B = \bigoplus_{\gamma \in \Gamma} (A \times B)_\gamma$ where $(A \times B)_\gamma = A_\gamma \times B_\gamma$.

Example 1.1.11 LOCALISATION OF GRADED RINGS

Let S be a central multiplicative closed subset of the Γ-graded ring A, consisting of homogeneous elements. Then $S^{-1}A$ has a natural Γ-graded structure. Namely, for $a \in A^h$, define $\deg(a/s) = \deg(a) - \deg(s)$ and for $\gamma \in \Gamma$,

$$(S^{-1}A)_\gamma = \{\, a/s \mid a \in A^h, \deg(a/s) = \gamma \,\}.$$

It is easy to see that this is well-defined and makes $S^{-1}A$ a Γ-graded ring.

Many rings have a "canonical" graded structure, among them are crossed products (group rings, skew group rings, twisted group rings), edge algebras, path algebras, incidence rings, etc. (see [53] for a review of these ring constructions). We will study some of these rings in this book.

Remark 1.1.12 RINGS GRADED BY A CATEGORY

The use of groupoids as a suitable language for structures whose operations are partially defined has now been firmly recognised. There is a generalised notion of groupoid graded rings as follows. Recall that a *groupoid* is a small category with the property that all morphisms are isomorphisms. As an example, let G be a group and I a nonempty set. The set $I \times G \times I$, considered as morphisms, forms a groupoid where the composition is defined by

$$(i, g, j)(j, h, k) = (i, gh, k).$$

One can show that this forms a connected groupoid and any connected groupoid is of this form ([62, Ch. 3.3, Prop. 6]). If $I = \{1, \dots, n\}$, we denote $I \times G \times I$ by $n \times G \times n$.

Let Γ be a groupoid and A be a ring. A is called a Γ-*groupoid graded* ring if $A = \bigoplus_{\gamma \in \Gamma} A_\gamma$, where γ is a morphism of Γ, each A_γ is an additive subgroup of A and $A_\gamma A_\delta \subseteq A_{\gamma\delta}$ if the morphism $\gamma\delta$ is defined and $A_\gamma A_\delta = 0$ otherwise. For a group Γ, considering it as a category with one element and Γ as the set of morphisms, we recover the Γ-group graded ring A (see Example 2.3.1 for an example of a groupoid graded ring).

One can develop the theory of groupoid graded rings in parallel and similarly to the group graded rings. See [65, 66] for this approach. Since adjoining a zero to a groupoid gives a semigroup, a groupoid graded ring is a special case of rings graded by semigroups (see Remark 1.1.13). For a general notion of a ring graded by a category see [1, §2], where it is shown that the category of graded modules (graded by a category) is a Grothendieck category.

Remark 1.1.13 Rings graded by a semigroup

In the definition of a graded ring (§1.1.1), one can replace the group grading with a semigroup. With this setting, the tensor algebras of Example 1.1.3 are $\mathbb{N}$-graded rings. A number of results on group graded rings can also be established in the more general setting of rings graded by cancellative monoids or semigroups (see for example [24, II, §11]). However, in this book we only consider group graded rings.

Remark 1.1.14 Graded rings without identity

For a ring without identity, one defines the concept of the graded ring exactly as when the ring has an identity. The concept of the strongly graded ring is defined similarly. In several occasions in this book we construct graded rings without an identity. For example, Leavitt path algebras arising from infinite graphs are graded rings without an identity, §1.6.4. See also §1.6.1, the graded free rings. The unitisation of a (nonunital) graded ring has a canonical grading. This is studied in relation with graded K_0 of nonunital rings in §3.5 (see (3.25)).

1.1.3 Strongly graded rings

Let A be a Γ-graded ring. By Proposition 1.1.1, $1 \in A_0$. This implies $A_0 A_\gamma = A_\gamma$ and $A_\gamma A_0 = A_\gamma$ for any $\gamma \in \Gamma$. If these equalities hold for any two arbitrary elements of Γ, we call the ring a strongly graded ring. Namely, a Γ-graded ring $A = \bigoplus_{\gamma \in \Gamma} A_\gamma$ is called a *strongly graded ring* if $A_\gamma A_\delta = A_{\gamma+\delta}$ for all $\gamma, \delta \in \Gamma$. A graded ring A is called *crossed product* if there is an invertible element in every homogeneous component A_γ of A; that is, $A^* \cap A_\gamma \neq \varnothing$ for all $\gamma \in \Gamma$, where A^* is the group of all invertible elements of A. We define the support of invertible homogeneous elements of A as

$$\boxed{\Gamma_A^* = \{ \gamma \in \Gamma \mid A_\gamma^* \neq \varnothing \},} \tag{1.2}$$

where $A_\gamma^* := A^* \cap A_\gamma$. It is easy to see that Γ_A^* is a group and $\Gamma_A^* \subseteq \Gamma_A$ (see Proposition 1.1.1(4)). Clearly A is a crossed product if and only if $\Gamma_A^* = \Gamma$.

Proposition 1.1.15 *Let $A = \bigoplus_{\gamma \in \Gamma} A_\gamma$ be a Γ-graded ring. Then*

(1) *A is strongly graded if and only if $1 \in A_\gamma A_{-\gamma}$ for any $\gamma \in \Gamma$;*
(2) *if A is strongly graded then the support of A is Γ;*
(3) *any crossed product ring is strongly graded;*
(4) *if $f : A \to B$ is a graded homomorphism of graded rings, then B is strongly graded (resp. crossed product) if A is so.*

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Proof (1) If A is strongly graded, then $1 \in A_0 = A_\gamma A_{-\gamma}$ for any $\gamma \in \Gamma$. For the converse, the assumption $1 \in A_\gamma A_{-\gamma}$ implies that $A_0 = A_\gamma A_{-\gamma}$ for any $\gamma \in \Gamma$. Then for $\sigma, \delta \in \Gamma$,

$$A_{\sigma+\delta} = A_0 A_{\sigma+\delta} = (A_\sigma A_{-\sigma})A_{\sigma+\delta} = A_\sigma(A_{-\sigma}A_{\sigma+\delta}) \subseteq A_\sigma A_\delta \subseteq A_{\sigma+\delta},$$

proving $A_{\sigma\delta} = A_\sigma A_\delta$, so A is strongly graded.

(2) By (1), $1 \in A_\gamma A_{-\gamma}$ for any $\gamma \in \Gamma$. This implies $A_\gamma \neq 0$ for any γ, *i.e.*, $\Gamma_A = \Gamma$.

(3) Let A be a crossed product ring. By definition, for $\gamma \in \Gamma$, there exists $a \in A^* \cap A_\gamma$. So $a^{-1} \in A_{-\gamma}$ by Proposition 1.1.1(4) and $1 = aa^{-1} \in A_\gamma A_{-\gamma}$. Thus A is strongly graded by (1).

(4) Suppose A is strongly graded. By (1), $1 \in A_\gamma A_{-\gamma}$ for any $\gamma \in \Gamma$. Thus

$$1 \in f(A_\gamma) f(A_{-\gamma}) \subseteq B_\gamma B_{-\gamma}.$$

Again (1) implies B is strongly graded. The case of the crossed product follows easily from the definition. □

The converse of (3) in Proposition 1.1.15 does not hold. One can prove that if A is strongly graded and A_0 is a local ring, then A is a crossed product algebra (see [75, Theorem 3.3.1]). In §1.6 we give examples of a strongly graded algebra A such that A is crossed product but A_0 is not a local ring. We also give an example of a strongly $\mathbb{Z}$-graded ring A such that A_0 is not local and A is not crossed product (Example 1.6.22). Using graph algebras we will produce large classes of strongly graded rings which are not crossed product (see Theorems 1.6.15 and 1.6.16).

If Γ is a finitely generated group, generated by the set $\{\gamma_1, \dots, \gamma_n\}$, then (1) in Proposition 1.1.15 can be simplified to the following: A is strongly graded if and only if $1 \in A_{\gamma_i} A_{-\gamma_i}$ and $1 \in A_{-\gamma_i} A_{\gamma_i}$, where $1 \leq i \leq n$. Thus if $\Gamma = \mathbb{Z}$, in order for A to be strongly graded, we only need to have $1 \in A_1 A_{-1}$ and $1 \in A_{-1} A_1$. This will be used, for example, in Proposition 1.6.6 to show that certain corner skew Laurent polynomial rings (§1.6.2) are strongly graded.

Example 1.1.16 Constructing strongly graded rings via tensor products

Let A and B be Γ-graded rings. Then by Example 1.1.6, $A \otimes_{\mathbb{Z}} B$ is a Γ-graded ring. If one of the rings is strongly graded (resp. crossed product) then $A \otimes_{\mathbb{Z}} B$ is so. Indeed, suppose A is strongly graded (resp. crossed product). Then the claim follows from Proposition 1.1.15(4) and the graded homomorphism $A \to A \otimes_{\mathbb{Z}} B, a \mapsto a \otimes 1$.

As a specific case, suppose A is a $\mathbb{Z}$-graded ring. Then

$$A[x, x^{-1}] = A \otimes \mathbb{Z}[x, x^{-1}]$$

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is a strongly graded ring. Notice that with this grading, $A[x, x^{-1}]_0 \cong A$.

Example 1.1.17 Strongly graded as a Γ/Ω-graded ring

Let A be a Γ-graded ring. Using Proposition 1.1.15, it is easy to see that if A is a strongly Γ-graded ring, then it is also a strongly Γ/Ω-graded ring, where Ω is a subgroup of Γ. However the strongly gradedness is not a "closed" property, i.e, if A is a strongly Γ/Ω-graded ring and A_Ω is a strongly Ω-graded ring, it does not follow that A is strongly Γ-graded.

1.1.4 Crossed products

Natural examples of strongly graded rings are crossed product algebras (see Proposition 1.1.15(3)). They cover, as special cases, the skew group rings and twisted groups rings. We briefly describe the construction here.

Let A be a ring, Γ a group (as usual we use the additive notation), and let $\phi : \Gamma \to \operatorname{Aut}(A)$ and $\psi : \Gamma \times \Gamma \to A^*$ be maps such that for any $\alpha, \beta, \gamma \in \Gamma$ and $a \in A$,

(i) ${}^{\alpha}({}^{\beta}a) = \psi(\alpha,\beta)\, {}^{\alpha+\beta}a\, \psi(\alpha,\beta)^{-1}$,
(ii) $\psi(\alpha,\beta)\psi(\alpha+\beta,\gamma) = {}^{\alpha}\psi(\beta,\gamma)\, \psi(\alpha,\beta+\gamma)$,
(iii) $\psi(\alpha, 0) = \psi(0,\alpha) = 1$

Here for $\alpha \in \Gamma$ and $a \in A$, $\phi(\alpha)(a)$ is denoted by ${}^{\alpha}a$. The map ψ is called a *2-cocycle map*. Denote by $A^{\phi}_{\psi}[\Gamma]$ the free left A-module with the basis Γ, and define the multiplication by

$$(a\alpha)(b\beta) = a\, {}^{\alpha}b\, \psi(\alpha,\beta)(\alpha+\beta). \tag{1.3}$$

One can show that with this multiplication, $A^{\phi}_{\psi}[\Gamma]$ is a Γ-graded ring with homogeneous components $A\gamma$, $\gamma \in \Gamma$. In fact $\gamma \in A\gamma$ is invertible, so $A^{\phi}_{\psi}[\Gamma]$ is a crossed product algebra [75, Proposition 1.4.1].

On the other hand, any crossed product algebra is of this form (see [75, §1.4]): for any $\gamma \in \Gamma$ choose $u_\gamma \in A^* \cap A_\gamma$ and define $\phi : \Gamma \to \operatorname{Aut}(A_0)$ by $\phi(\gamma)(a) = u_\gamma a u_\gamma^{-1}$ for $\gamma \in \Gamma$ and $a \in A_0$. Moreover, define the cocycle map

$$\begin{aligned} \psi : \Gamma \times \Gamma &\longrightarrow A_0^*, \\ (\zeta, \eta) &\longmapsto u_\zeta u_\eta u_{\zeta+\eta}^{-1}. \end{aligned}$$

Then

$$A = A_0{}^{\phi}_{\psi}[\Gamma] = \textstyle\bigoplus_{\gamma\in\Gamma} A_0\gamma,$$

with multiplication

$$(a\zeta)(b\eta) = a\,{}^{\zeta}b\,\psi(\zeta,\eta)(\zeta+\eta),$$

where ${}^{\zeta}b$ is defined as $\phi(\zeta)(b)$.

Note that when Γ is cyclic, one can choose $u_i = u_1^i$ for $u_1 \in A^* \cap A_1$ and thus the cocycle map ψ is trivial, ϕ is a homomorphism and the crossed product is a skew group ring. In fact, if $\Gamma = \mathbb{Z}$, then the skew group ring becomes the so-called *skew Laurent polynomial ring*, denoted by $A_0[x, x^{-1}, \phi]$. Moreover, if u_1 is in the centre of A, then ϕ is the identity map and the crossed product ring reduces to the group ring $A_0[\Gamma]$. A variant of this construction, namely corner skew polynomial rings, is studied in §1.6.2.

Skew group rings If $\psi : \Gamma \times \Gamma \to A^*$ is a trivial map, *i.e.,* $\psi(\alpha,\beta) = 1$ for all $\alpha, \beta \in \Gamma$, then Conditions (ii) and (iii) trivially hold, and Condition (i) reduces to ${}^{\alpha}({}^{\beta}a) = {}^{\alpha+\beta}a$ which means that $\phi : \Gamma \to \mathrm{Aut}(A)$ becomes a group homomorphism. In this case $A^{\phi}_{\psi}[\Gamma]$, denoted by $A \star_{\phi} \Gamma$, is a *skew group ring* with multiplication

$$(a\alpha)(b\beta) = a\,{}^{\alpha}b\,(\alpha + \beta). \tag{1.4}$$

Twisted group ring If $\phi : \Gamma \to \mathrm{Aut}(A)$ is trivial, *i.e.,* $\phi(\alpha) = 1_A$ for all $\alpha \in \Gamma$, then Condition (i) implies that $\psi(\alpha,\beta) \in C(A) \cap A^*$ for any $\alpha, \beta \in \Gamma$. Here $C(A)$ stands for the centre of the ring A. In this case $A^{\phi}_{\psi}[\Gamma]$, denoted by $A_{\psi}[\Gamma]$, is a *twisted group ring* with multiplication

$$(a\alpha)(b\beta) = ab\psi(\alpha,\beta)(\alpha + \beta). \tag{1.5}$$

A well-known theorem in the theory of central simple algebras states that if D is a central simple F-algebra with a maximal subfield L such that L/F is a Galois extension and $[A : F] = [L : F]^2$, then D is a crossed product, with $\Gamma = \mathrm{Gal}(L/F)$ and $A = L$ (see [35, §12, Theorem 1]).

Some of the graded rings we treat in this book are of the form $K[x, x^{-1}]$, where K is a field. This is an example of a graded field.

A Γ-graded ring $A = \bigoplus_{\gamma\in\Gamma} A_\gamma$ is called a *graded division ring* if every nonzero homogeneous element has a multiplicative inverse. If A is also a commutative ring, then A is called a *graded field*.

Let A be a Γ-graded division ring. It follows from Proposition 1.1.1(4) that Γ_A is a group, so we can write $A = \bigoplus_{\gamma\in\Gamma_A} A_\gamma$. Then, as a Γ_A-graded ring, A is a crossed product and it follows from Proposition 1.1.15(3) that A is strongly Γ_A-graded. Note that if $\Gamma_A \neq \Gamma$, then A is not strongly Γ-graded. Also note that if A is a graded division ring, then A_0 is a division ring.

Remark 1.1.18 GRADED DIVISION RINGS AND DIVISION RINGS WHICH ARE GRADED

Note that a graded division ring and a division ring which is graded are different. By definition, A is a graded division ring if and only if $A^h \backslash \{0\}$ is a group. A simple example is the Laurent polynomial ring $D[x, x^{-1}]$, where

D is a division ring (Example 1.1.19). Other examples show that a graded division ring does not need to be a domain (Example 1.1.21). However, if the grade group is totally ordered, then a domain which is also graded has to be concentrated in degree zero. Thus a division ring which is graded by a totally ordered grade group Γ is of the form $A = \bigoplus_{\gamma\in\Gamma} A_\gamma$, where A_0 is a division ring and $A_\gamma = 0$ for $\gamma \neq 0$. This will not be the case if Γ is not totally ordered (see Example 1.1.20).

In the following we give some concrete examples of graded division rings.

Example 1.1.19 <u>THE VERONESE SUBRING</u>

Let $A = \bigoplus_{\gamma\in\Gamma} A_\gamma$ be a Γ-graded ring, where Γ is a torsion-free group. For $n \in \mathbb{Z}\backslash\{0\}$, the nth-*Veronese subring* of A is defined as $A^{(n)} = \bigoplus_{\gamma\in\Gamma} A_{n\gamma}$. This is a Γ-graded ring with $A^{(n)}_\gamma = A_{n\gamma}$. It is easy to see that the support of $A^{(n)}$ is Γ if the support of A is Γ. Note also that if A is strongly graded, so is $A^{(n)}$. Clearly $A^{(1)} = A$ and $A^{(-1)}$ is the graded ring with the components "flipped", *i.e.*, $A^{(-1)}_\gamma = A_{-\gamma}$. For the case of $A^{(-1)}$ we don't need to require the grade group to be torsion-free.

Let D be a division ring and let $A = D[x, x^{-1}]$ be the Laurent polynomial ring. The elements of A consist of finite sums $\sum_{i\in\mathbb{Z}} a_i x^i$, where $a_i \in D$. Then A is a $\mathbb{Z}$-graded division ring with $A = \bigoplus_{i\in\mathbb{Z}} A_i$, where $A_i = \{ax^i \mid a \in D\}$. Consider the nth-Veronese subring $A^{(n)}$ which is the ring $D[x^n, x^{-n}]$. The elements of $A^{(n)}$ consist of finite sums $\sum_{i\in\mathbb{Z}} a_i x^{in}$, where $a_i \in D$. Then $A^{(n)}$ is a $\mathbb{Z}$-graded division ring, with $A^{(n)} = \bigoplus_{i\in\mathbb{Z}} A_{in}$. Here both A and $A^{(n)}$ are strongly graded rings.

There is also another way to consider the $\mathbb{Z}$-graded ring $B = D[x^n, x^{-n}]$ such that it becomes a graded subring of $A = D[x, x^{-1}]$. Namely, we define $B = \bigoplus_{i\in\mathbb{Z}} B_i$, where $B_i = Dx^i$ if $i \in n\mathbb{Z}$ and $B_i = 0$ otherwise. This way B is a graded division ring and a graded subring of A. The support of B is clearly the subgroup $n\mathbb{Z}$ of $\mathbb{Z}$. With this definition, B is not strongly graded.

Example 1.1.20 <u>DIFFERENT GRADINGS ON A GRADED DIVISION RING</u>

Let $\mathbb{H} = \mathbb{R}\oplus\mathbb{R}i\oplus\mathbb{R}j\oplus\mathbb{R}k$ be the real *quaternion algebra*, with multiplication defined by $i^2 = -1$, $j^2 = -1$ and $ij = -ji = k$. It is known that $\mathbb{H}$ is a noncommutative division ring with centre $\mathbb{R}$. We give $\mathbb{H}$ two different graded division ring structures, with grade groups $\mathbb{Z}_2 \times \mathbb{Z}_2$ and $\mathbb{Z}_2$ respectively as follows.

$\mathbb{Z}_2 \times \mathbb{Z}_2$-grading Let $\mathbb{H} = R_{(0,0)} \oplus R_{(1,0)} \oplus R_{(0,1)} \oplus R_{(1,1)}$, where

$$R_{(0,0)} = \mathbb{R},\quad R_{(1,0)} = \mathbb{R}i,\quad R_{(0,1)} = \mathbb{R}j,\quad R_{(1,1)} = \mathbb{R}k.$$

It is routine to check that $\mathbb{H}$ forms a strongly $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded division ring.

$\mathbb{Z}_2$-grading Let $\mathbb{H} = \mathbb{C}_0 \oplus \mathbb{C}_1$, where $\mathbb{C}_0 = \mathbb{R} \oplus \mathbb{R}i$ and $\mathbb{C}_1 = \mathbb{C}j = \mathbb{R}j \oplus \mathbb{R}k$. One can check that $\mathbb{C}_0\mathbb{C}_0 = \mathbb{C}_0$, $\mathbb{C}_0\mathbb{C}_1 = \mathbb{C}_1\mathbb{C}_0 = \mathbb{C}_1$ and $\mathbb{C}_1\mathbb{C}_1 = \mathbb{C}_0$. This makes $\mathbb{H}$ a strongly $\mathbb{Z}_2$-graded division ring. Note that this grading on $\mathbb{H}$ can be obtained from the first part by considering the quotient grade group $\mathbb{Z}_2 \times \mathbb{Z}_2/0 \times \mathbb{Z}_2$ (§1.1.2). Quaternion algebras are examples of Clifford algebras (see Example 1.1.24).

The following generalises the above example of quaternions as a $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded ring.

Example 1.1.21 SYMBOL ALGEBRAS

Let F be a field, ξ be a primitive nth root of unity and let $a, b \in F^*$. Let

$$A = \bigoplus_{i=0}^{n-1} \bigoplus_{j=0}^{n-1} F x^i y^j$$

be the F-algebra generated by the elements x and y, which are subject to the relations $x^n = a$, $y^n = b$ and $xy = \xi yx$. By [35, Theorem 11.1], A is an n^2-dimensional central simple algebra over F. We will show that A forms a graded division ring. Clearly A can be written as a direct sum

$$A = \bigoplus_{(i,j)\in\mathbb{Z}_n\oplus\mathbb{Z}_n} A_{(i,j)}, \quad \text{where } A_{(i,j)} = F x^i y^j$$

and each $A_{(i,j)}$ is an additive subgroup of A. Using the fact that $\xi^{-kj} x^k y^j = y^j x^k$ for each j, k, with $0 \le j, k \le n-1$, we can show that

$$A_{(i,j)} A_{(k,l)} \subseteq A_{([i+k],[j+l])},$$

for $i, j, k, l \in \mathbb{Z}_n$. A nonzero homogeneous element $f x^i y^j \in A_{(i,j)}$ has an inverse

$$f^{-1} a^{-1} b^{-1} \xi^{-ij} x^{n-i} y^{n-j},$$

proving A is a graded division ring. Clearly the support of A is $\mathbb{Z}_n \times \mathbb{Z}_n$, so A is strongly $\mathbb{Z}_n \times \mathbb{Z}_n$-graded.

These examples can also be obtained from graded free rings (see Example 1.6.3).

Example 1.1.22 A GOOD COUNTER-EXAMPLE

In the theory of graded rings, in many instances it has been established that if the grade group Γ is finite (or in some cases, finitely generated), then a graded property implies the corresponding nongraded property of the ring (*i.e.,* the property is preserved under the forgetful functor). For example, one can prove that if a $\mathbb{Z}$-graded ring is graded Artinian (Noetherian), then the ring is Artinian (Noetherian). One good example which provides counter-examples to such phenomena is the following graded field.

Let K be a field and $A = K[x_1^{\pm 1}, x_2^{\pm 1}, x_3^{\pm 1}, \dots]$ a Laurent polynomial ring in countably many variables. This ring is a graded field with its "canonical" $\bigoplus_\infty \mathbb{Z}$-grading and thus it is graded Artinian and Noetherian. However, A is not Noetherian.

1.1.5 Graded ideals

Let A be a Γ-graded ring. A two-sided ideal I of A is called a *graded ideal* (or *homogeneous ideal*) if

$$I = \bigoplus_{\gamma \in \Gamma} (I \cap A_\gamma). \tag{1.6}$$

Thus I is a graded ideal if and only if for any $x \in I$, $x = \sum x_i$, where $x_i \in A^h$, implies that $x_i \in I$.

The notions of a *graded subring*, a *graded left* and a *graded right ideal* are defined similarly.

Let I be a graded ideal of A. Then the quotient ring A/I forms a graded ring, with

$$\boxed{A/I = \bigoplus_{\gamma \in \Gamma} (A/I)_\gamma, \quad \text{where} \quad (A/I)_\gamma = (A_\gamma + I)/I.} \tag{1.7}$$

With this grading $(A/I)_0 \cong A_0/I_0$, where $I_0 = A_0 \cap I$. From (1.6) it follows that an ideal I of A is a graded ideal if and only if I is generated as a two-sided ideal of A by homogeneous elements. Also, for a two-sided ideal I of A, if (1.7) induces a grading on A/I, then I has to be a graded ideal. By Proposition 1.1.15(4), if A is strongly graded or a crossed product, so is the graded quotient ring A/I.

Example 1.1.23 SYMMETRIC AND EXTERIOR ALGEBRAS AS $\mathbb{Z}$-GRADED RINGS

Recall from Example 1.1.3 that for a commutative ring A and an A-module M, the tensor algebra $T(M)$ is a $\mathbb{Z}$-graded ring with support $\mathbb{N}$. The *symmetric algebra* $S(M)$ is defined as the quotient of $T(M)$ by the ideal generated by elements $x \otimes y - y \otimes x$, $x, y \in M$. Since these elements are homogeneous of degree two, $S(M)$ is a $\mathbb{Z}$-graded commutative ring.

Similarly, the *exterior algebra* of M, denoted by $\bigwedge M$, is defined as the quotient of $T(M)$ by the ideal generated by homogeneous elements $x \otimes x$, $x \in M$. So $\bigwedge M$ is a $\mathbb{Z}$-graded ring.

https://doi.org/10.1017/CBO9781316717134.002 Published online by Cambridge University Press

Let I be a two-sided ideal of a Γ-graded ring A generated by a subset $\{a_i\}$ of not necessarily homogeneous elements of A. If Ω is a subgroup of Γ such that $\{a_i\}$ are homogeneous elements in Γ/Ω-graded ring A (see §1.1.2), then clearly I is a Γ/Ω-graded ideal and consequently A/I is a Γ/Ω-graded ring.

Example 1.1.24 CLIFFORD ALGEBRAS AS $\mathbb{Z}_2$-GRADED RINGS

Let V be a F-vector space and $q : V \to F$ be a quadratic form with its associated nondegenerate symmetric bilinear form $B : V \times V \to F$.

The *Clifford algebra* associated with (V, q) is defined as

$$\mathrm{Cl}(V, q) := T(V)/\langle v \otimes v + q(v)\rangle.$$

Considering $T(V)$ as a $\mathbb{Z}/2\mathbb{Z}$-graded ring (see §1.1.2), the elements $v \otimes v - q(v)$ are homogeneous of degree zero. This induces a $\mathbb{Z}_2$-graded structure on $\mathrm{Cl}(V, q)$. Identifying V with its image in the Clifford algebra $\mathrm{Cl}(V, q)$, V lies in the odd part of the Clifford algebra, *i.e.,* $V \subset \mathrm{Cl}(V, q)_1$.

If $\mathrm{char}(F) \neq 2$, as B is nondegenerate, there exist $x, y \in V$ such that $B(x, y) = 1/2$, and thus

$$xy + yx = 2B(x, y) = 1 \in \mathrm{Cl}(V, q)_1 \, \mathrm{Cl}(V, q)_1.$$

Similarly, if $\mathrm{char}(F) = 2$, there exist $x, y \in V$ such that $B(x, y) = 1$, so

$$xy + yx = B(x, y) = 1 \in \mathrm{Cl}(V, q)_1 \, \mathrm{Cl}(V, q)_1.$$

It follows from Proposition 1.1.15 that Clifford algebras are strongly $\mathbb{Z}_2$-graded rings.

Recall that for Γ-graded rings A and B, a Γ-graded ring homomorphism $f : A \to B$ is a ring homomorphism such that $f(A_\gamma) \subseteq B_\gamma$ for all $\gamma \in \Gamma$. It can easily be shown that $\ker(f)$ is a graded ideal of A and $\mathrm{im}(f)$ is a graded subring of B. It is also easy to see that f is injective (surjective/bijective) if and only if for any $\gamma \in \Gamma$, the restriction of f on A_γ is injective (surjective/bijective).

Note that if Γ is an abelian group, then the centre of a graded ring A, $C(A)$, is a graded subring of A. More generally, the centraliser of a set of homogeneous elements is a graded subring.

Example 1.1.25 THE CENTRE OF THE GRADED RING

If a group Γ is not abelian, then the centre of a Γ-graded ring may not be a graded subring. For example, let $\Gamma = S_3 = \{e, a, b, c, d, f\}$ be the symmetric group of order 3, where

$$a = (23), \quad b = (13), \quad c = (12), \quad d = (123), \quad f = (132).$$

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Let A be a ring, and consider the group ring $R = A[\Gamma]$, which is a Γ-graded ring by Example 1.1.2. Let $x = 1d + 1f \in R$, where $1 = 1_A$, and we note that x is not homogeneous in R. Then $x \in Z(R)$, but the homogeneous components of x are not in the centre of R. As x is expressed uniquely as the sum of homogeneous components, we have $x \notin \bigoplus_{\gamma\in\Gamma}(Z(R) \cap R_\gamma)$.

This example can be generalised by taking a nonabelian finite group Γ with a subgroup Ω which is normal and noncentral. Let A be a ring and consider the group ring $R = A[\Gamma]$ as above. Then $x = \sum_{\omega\in\Omega} 1\omega$ is in the centre of R, but the homogeneous components of x are not all in the centre of R.

Remark 1.1.26 Let Γ and Λ be two groups. Let A be a Γ-graded ring and B be a Λ-graded ring. Suppose $f : A \to B$ is a ring homomorphism and $g : \Gamma \to \Lambda$ a group homomorphism such that for any $\gamma \in \Gamma$, $f(A_\gamma) \subseteq B_{g(\gamma)}$. Then f is called a $\Gamma-\Lambda$-graded homomorphism. In the case $\Gamma = \Lambda$ and $g = \mathrm{id}$, we recover the usual definition of a Γ-graded homomorphism. For example, if Ω is a subgroup of Γ, then the identity map $1_A : A \to A$ is a $\Gamma-\Gamma/\Omega$-graded homomorphism, where A is considered as Γ and Γ/Ω-graded rings, respectively (see §1.1.2).

Throughout this book, we fix a given group Γ and we work with the Γ-graded category and all our considerations are within this category. (See Remark 2.3.14 for references to literature where mixed grading is studied.)

1.1.6 Graded prime and maximal ideals

A graded ideal P of Γ-graded ring A is called a *graded prime ideal* of A if $P \neq A$ and for any two graded ideals I, J, $IJ \subseteq P$, implies $I \subseteq P$ or $J \subseteq P$. If A is commutative, we obtain the familiar formulation that P is a graded prime ideal if and only if for $x, y \in A^h$, $xy \in P$ implies that $x \in P$ or $y \in P$. Note that a graded prime ideal is not necessarily a prime ideal.

A graded ideal P is called a *graded semiprime ideal* if for any graded ideal I in A, $I^2 \subseteq P$, implies $I \subseteq P$. A graded ring A is called a *graded prime (graded semiprime)* ring if the zero ideal is a graded prime (graded semiprime) ideal.

A *graded maximal ideal* of a Γ-graded ring A is defined to be a proper graded ideal of A which is maximal among the set of proper graded ideals of A. Using Zorn's lemma, one can show that graded maximal ideals exist, and it is not difficult to show that a graded maximal ideal is a graded prime. For a graded commutative ring, a graded ideal is maximal if and only if its quotient ring is a graded field. There are similar notions of graded maximal left and right ideals.

Parallel to the nongraded setting, for a Γ-graded ring A, the *graded Jacobson radical*, $J^{\mathrm{gr}}(A)$, is defined as the intersection of all graded left maximal ideals of A. This coincides with the intersection of all graded right maximal ideals and so $J^{\mathrm{gr}}(A)$ is a two-sided ideal (see [75, Proposition 2.9.1]). We denote by $J(A)$ the usual Jacobson radical. It is a theorem of G. Bergman that for a $\mathbb{Z}$-graded ring A, $J(A)$ is a graded ideal and $J(A) \subseteq J^{\mathrm{gr}}(A)$ (see [19]).

1.1.7 Graded simple rings

A nonzero graded ring A is said to be *graded simple* if the only graded two-sided ideals of A are $\{0\}$ and A. The structure of graded simple Artinian rings are known (see Remark 1.4.8). Following [52] we prove that a graded ring A is simple if and only if A is graded simple and $C(A)$, the centre of A, is a field.

For a Γ-graded ring A, recall the support Γ_A of A, from §1.1.1. For $a \in A$, writing $a = \sum_{\gamma\in\Gamma} a_\gamma$ where $a_\gamma \in A^h$, define the *support* of a to be

$$\Gamma_a = \{\, \gamma \mid a_\gamma \neq 0 \,\}.$$

We also need the notion of minimal support. A finite set X of Γ is called a *minimal support with respect to an ideal* I if $X = \Gamma_a$ for $0 \neq a \in I$ and there is no $b \in I$ such that $b \neq 0$ and $\Gamma_b \subsetneq \Gamma_a$.

We start with a lemma.

Lemma 1.1.27 *Let A be a Γ-graded simple ring and I an ideal of A. Let $0 \neq a \in I$ with $\Gamma_a = \{\gamma_1, \dots, \gamma_n\}$. Then for any $\alpha \in \Gamma_A$, there is a $0 \neq b \in I$ with $\Gamma_b \subseteq \{\gamma_1 - \gamma_n + \alpha, \dots, \gamma_n - \gamma_n + \alpha\}$.*

Proof Let $0 \neq x \in A_\alpha$, where $\alpha \in \Gamma_A$ and $0 \neq a \in I$ with $\Gamma_a = \{\gamma_1, \dots, \gamma_n\}$. Write $a = \sum_{i=1}^n a_{\gamma_i}$, where $\deg(a_{\gamma_i}) = \gamma_i$. Since A is graded simple,

$$x = \sum_l r_l a_{\gamma_n} s_l, \tag{1.8}$$

where $r_l, s_l \in A^h$. Thus there are $r_k, s_k \in A^h$ such that $r_k a_{\gamma_n} s_k \neq 0$ which implies that $b := r_k a s_k \in I$ is not zero. Comparing the degrees in Equation (1.8), it follows that $\alpha = \deg(r_k) + \deg(s_k) + \gamma_n$, or $\deg(r_k) + \deg(s_k) = \alpha - \gamma_n$. So

$$\Gamma_b \subseteq \Gamma_a + \deg(r_k) + \deg(s_k) = \{\, \gamma_1 - \gamma_n + \alpha, \dots, \gamma_n - \gamma_n + \alpha \,\}. \qquad \square$$

Theorem 1.1.28 *Let A be a Γ-graded ring. Then A is a simple ring if and only if A is a graded simple ring and $C(A)$ is a field.*

Proof One direction is straightforward.

Suppose A is graded simple and $C(A)$ is a field. We will show that A is a simple ring. Suppose I is a nontrivial ideal of A and $0 \neq a \in I$ with Γ_a a minimal support with respect to I. For any $x \in A^h$, with $\deg(x) = \alpha$ and $\gamma \in \Gamma_a$, we have

$$\Gamma_{axa_\gamma - a_\gamma xa} \subsetneq \Gamma_a + (\gamma + \alpha). \tag{1.9}$$

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Set $b = axa_\gamma - a_\gamma xa \in I$. Suppose $b \neq 0$. By (1.9),

$$\Gamma_b \subsetneq \{\gamma_1 + \gamma + \alpha, \dots, \gamma_n + \gamma + \alpha\}.$$

Applying Lemma 1.1.27 with, say, $\gamma_n + \gamma + \alpha \in \Gamma_b$ and $\gamma_n \in \Gamma_A$, we obtain a $0 \neq c \in I$ such that

$$\Gamma_c \subseteq \Gamma_b + (\gamma_n - \gamma_n - \gamma - \alpha) \subsetneq \Gamma_a.$$

This is, however, a contradiction as Γ_a was a minimal support. Thus $b = 0$, *i.e.*, $axa_\gamma = a_\gamma xa$. It follows that for any $\gamma_i \in \Gamma_a$

$$a_{\gamma_i} xa_\gamma = a_\gamma xa_{\gamma_i}. \tag{1.10}$$

Consider the R-bimodule map

$$\phi : R = \langle a_{\gamma_i} \rangle \longrightarrow \langle a_{\gamma_j} \rangle = R,$$
$$\sum_l r_l a_{\gamma_i} s_l \longmapsto \sum_l r_l a_{\gamma_j} s_l.$$

To show that ϕ is well-defined, since $\phi(t+s) = \phi(t)+\phi(s)$, it is enough to show that if $t = 0$ then $\phi(t) = 0$, where $t \in \langle a_{\gamma_i} \rangle$. Suppose $\sum_l r_l a_{\gamma_i} s_l = 0$. Then for any $x \in A^h$, using (1.10) we have

$$0 = a_{\gamma_j} x \,\Big(\sum_l r_l a_{\gamma_i} s_l\Big) = \sum_l a_{\gamma_j} x r_l a_{\gamma_i} s_l = \sum_l a_{\gamma_i} x r_l a_{\gamma_j} s_l = a_{\gamma_i} x \,\Big(\sum_l r_l a_{\gamma_j} s_l\Big).$$

Since A is graded simple, $\langle a_{\gamma_i} \rangle = 1$. It follows that $\sum_l r_l a_{\gamma_j} s_l = 0$. Thus ϕ is well-defined, injective and also clearly surjective. Then $a_{\gamma_j} = \phi(a_{\gamma_i}) = a_{\gamma_i}\phi(1)$. But $\phi(1) \in C(A)$. Thus $a = \sum_j a_{\gamma_j} = a_{\gamma_i} c$ where $c \in C(A)$. But $C(A)$ is a field, so $a_{\gamma_i} = ac^{-1} \in I$. Again, since R is graded simple, it follows that $I = R$. This finishes the proof. □

Remark 1.1.29 If the grade group is not abelian, in order for Theorem 1.1.28 to be valid, the grade group should be hyper-central; A *hyper-central group* is a group such that any nontrivial quotient has a nontrivial centre. If A is strongly graded, and the grade group is torsion-free hyper-central, then A is simple if and only if A is graded simple and $C(A) \subseteq A_0$ (see [52]).

Remark 1.1.30 GRADED SIMPLICITY IMPLYING SIMPLICITY

There are other cases that the graded simplicity of a ring implies that the ring itself is simple. For example, if a ring is graded by an ordered group (such as $\mathbb{Z}$), and has a finite support, then graded simplicity implies the simplicity of the ring [10, Theorem 3].

1.1.8 Graded local rings

Recall that a ring is a *local ring* if the set of noninvertible elements form a two-sided ideal. When A is a commutative ring, then A is local if and only if A has a unique maximal ideal.

A Γ-graded ring A is called a *graded local ring* if the two-sided ideal M generate by noninvertible homogeneous elements is a proper ideal. One can easily observe that the graded ideal M is the unique graded maximal left, right, and graded two-sided ideal of A. When A is a graded commutative ring, then A is graded local if and only if A has a unique graded maximal ideal.

If A is a graded local ring, then the graded ring A/M is a graded division ring. One can further show that A_0 is a local ring with the unique maximal ideal $A_0 \cap M$. In fact we have the following proposition.

Proposition 1.1.31 *Let A be a Γ-graded ring. Then A is a graded local ring if and only if A_0 is a local ring.*

Proof Suppose A is a graded local ring. Then by definition, the two-sided ideal M generated by noninvertible homogeneous elements is a proper ideal. Consider $m = A_0 \cap M$ which is a proper ideal of A_0. Suppose $x \in A_0 \backslash m$. Then x is a homogeneous element which is not in M. Thus x has to be invertible in A and consequently in A_0. This shows that A_0 is a local ring with the unique maximal ideal m.

Conversely, suppose A_0 is a local ring. We first show that any left or right invertible homogeneous element is a two-sided invertible element. Let a be a left invertible homogeneous element. Then there is a homogeneous element b such that $ba = 1$. If ab is not right invertible, then $ab \in m$, where m is the unique maximal ideal of the local ring A_0. Thus $1 - ab \notin m$ which implies that $1 - ab$ is invertible. But $(1 - ab)a = a - aba = a - a = 0$, and since $1 - ab$ is invertible, we get $a = 0$ which is a contradiction to the fact that a has a left inverse. Thus a has a right inverse and so is invertible. A similar argument can be written for right invertible elements. Now let M be the ideal generated by all noninvertible homogeneous elements of A. We will show that M is proper, and thus A is a graded local ring. Suppose M is not proper. Thus $1 = \sum_i r_i a_i s_i$, where a_i are noninvertible homogeneous elements and r_i, s_i are homogeneous elements such that $\deg(r_i a_i s_i) = 0$. If $r_i a_i s_i$ is invertible for some i, using the fact that right and left invertibles are invertibles, it follows that a_i is invertible, which is a contradiction. Thus $r_i a_i s_i$, for all i, are homogeneous elements of degree zero and not invertible. So they are all in m. This implies that $1 \in m$, which is a contradiction. Thus M is a proper ideal of A. □

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For more on graded local rings (graded by a cancellative monoid) see [64]. In §3.8 we determine the graded Grothendieck group of these rings.

1.1.9 Graded von Neumann regular rings

The von Neumann regular rings constitute an important class of rings. A unital ring A is *von Neumann regular* if for any $a \in A$, we have $a \in aAa$. There are several equivalent module theoretical definitions, such as A is von Neumann regular if and only if any module over A is flat. This gives a comparison with the class of division rings and semisimple rings. A ring is a division ring if and only if any module is free. A semisimple ring is characterised by the property that any module is projective. Goodearl's book [40] is devoted to the class of von Neumann regular rings. The definition extends to a nonunital ring in an obvious manner.

If a ring has a graded structure, one defines the graded version of regularity in a natural way: the graded ring A is called *graded von Neumann regular* if for any homogeneous element $a \in A$ we have $a \in aAa$. This means, for any homogeneous element $a \in A$, one can find a homogeneous element $b \in A$ such that $a = aba$. As an example, a direct sum of graded division rings is a graded von Neumann regular ring. Many of the module theoretic properties established for von Neumann regular rings can be extended to the graded setting; for example, A is graded regular if and only if any graded module is (graded) flat. We refer the reader to [74, C, I.5] for a treatment of such rings and [11, §2.2] for a concise survey. Several of the graded rings we construct in this book are graded von Neumann regular, such as Leavitt path algebras (Corollary 1.6.17) and corner skew Laurent series (Proposition 1.6.8).

In this section, we briefly give some of the properties of graded von Neumann regular rings. The following proposition is the graded version of [40, Theorem 1.1] which has a similar proof.

Proposition 1.1.32 *Let A be a Γ-graded ring. The following statements are equivalent:*

(1) *A is a graded von Neumann regular ring;*
(2) *any finitely generated right (left) graded ideal of A is generated by a homogeneous idempotent.*

Proof (1) ⇒ (2) First we show that any principal graded ideal is generated by a homogeneous idempotent. So consider the principal ideal xA, where $x \in A^h$. By the assumption, there is $y \in A^h$ such that $xyx = x$. This immediately implies $xA = xyA$. Now note that xy is homogeneous idempotent.

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Next we will prove the claim for graded ideals generated by two elements. The general case follows by an easy induction. So let $xA+yA$ be a graded ideal generated by two homogeneous elements x, y. By the previous paragraph, $xA = eA$ for a homogeneous idempotent e. Note that $y-ey \in A^h$ and $y-ey \in xA+yA$. Thus

$$xA + yA = eA + (y - ey)A. \tag{1.11}$$

Again, the previous paragraph gives us a homogeneous idempotent f such that $(y - ey)A = fA$. Let $g = f - fe \in A_0$. Notice that $ef = 0$, which implies that e and g are orthogonal idempotents. Moreover, $fg = g$ and $gf = f$. It then follows that $gA = fA = (y - ey)A$. Now from (1.11) we get

$$xA + yA = eA + gA = (e + g)A.$$

(2) ⇒ (1) Let $x \in A^h$. Then $xA = eA$ for some homogeneous idempotent e. Thus $x = ea$ and $e = xy$ for some $a, y \in A^h$. Then $x = ea = eea = ex = xyx$. □

Proposition 1.1.33 *Let A be a Γ-graded von Neumann regular ring. Then*

(1) *any graded right (left) ideal of A is idempotent;*
(2) *any graded ideal is graded semiprime;*
(3) *any finitely generated right (left) graded ideal of A is a projective module.*

Moreover, if A is a $\mathbb{Z}$-graded regular ring then

(4) $J(A) = J^{\mathrm{gr}}(A) = 0.$

Proof The proofs of (1)–(3) are similar to the nongraded case [40, Corollary 1.2]. We provide the easy proofs here.

(1) Let I be a graded right ideal. For any homogeneous element $x \in I$ there is $y \in A^h$ such that $x = xyx$. Thus $x = (xy)x \in I^2$. It follows that $I^2 = I$.

(2) This follows immediately from (1).

(3) By Proposition 1.1.32, any finitely generated right ideal is generated by a homogeneous idempotent. However, this latter ideal is a direct summand of the ring, and so is a projective module.

(4) By Bergman's observation, for a $\mathbb{Z}$-graded ring A, $J(A)$ is a graded ideal and $J(A) \subseteq J^{\mathrm{gr}}(A)$ (see [19]). By Proposition 1.1.32, $J^{\mathrm{gr}}(A)$ contains an idempotent, which then forces $J^{\mathrm{gr}}(A) = 0$. □

If the graded ring A is strongly graded then one can show that there is a one-to-one correspondence between the right ideals of A_0 and the graded right ideals of A (similarly for the left ideals) (see Remark 1.5.6). This is always the case for the graded regular rings as the following proposition shows.

Proposition 1.1.34 *Let A be a Γ-graded von Neumann regular ring. Then there is a one-to-one correspondence between the right (left) ideals of A_0 and the graded right (left) ideals of A.*

Proof Consider the following correspondences between the graded right ideals of A and the right ideals of A_0. For a graded right ideal I of A assign I_0 in A_0 and for a right ideal J in A_0 assign the graded right ideal JA in A. Note that $(JA)_0 = J$. We show that $I_0A = I$. It is enough to show that any homogeneous element a of I belongs to I_0A. Since A is graded regular, $axa = a$ for some $x \in A^h$. But $ax \in I_0$ and thus $a = axa \in I_0A$. A similar proof gives the left ideal correspondence. □

In Theorem 1.2.20 we give yet another characterisation of graded von Neumann regular rings based on the concept of divisible modules.

Later, in Corollary 1.5.10, we show that if A is a strongly graded ring, then A is graded von Neumann regular if and only if A_0 is a von Neumann regular ring. The proof uses the equivalence of suitable categories over the rings A and A_0. An element-wise proof of this fact can also be found in [96, Theorem 3].

1.2 Graded modules

1.2.1 Basic definitions

Let A be a Γ-graded ring. A *graded right A-module M* is defined to be a right A-module M with a direct sum decomposition $M = \bigoplus_{\gamma\in\Gamma} M_\gamma$, where each M_γ is an additive subgroup of M such that $M_\lambda A_\gamma \subseteq M_{\lambda+\gamma}$ for all $\gamma, \lambda \in \Gamma$.

For Γ-graded right A-modules M and N, a Γ-*graded module homomorphism* $f : M \to N$ is a module homomorphism such that $f(M_\gamma) \subseteq N_\gamma$ for all $\gamma \in \Gamma$. A graded homomorphism f is called a *graded module isomorphism* if f is bijective and, when such a graded isomorphism exists, we write $M \cong_{\mathrm{gr}} N$. Notice that if f is a graded module homomorphism which is bijective, then its inverse f^{-1} is also a graded module homomorphism.

1.2.2 Shift of modules

Let M be a graded right A-module. For $\delta \in \Gamma$, we define the δ-*suspended* or δ-*shifted* graded right A-module $M(\delta)$ as

$$\boxed{M(\delta) = \bigoplus_{\gamma\in\Gamma} M(\delta)_\gamma, \text{ where } M(\delta)_\gamma = M_{\delta+\gamma}.}$$

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This shift plays a pivotal role in the theory of graded rings. For example, if M is a $\mathbb{Z}$-graded A-module, then the following table shows how the shift like "the tick of the clock" moves the homogeneous components of M to the left.

	degrees	**-3**	**-2**	**-1**	**0**	**1**	**2**	**3**
M				M_{-1}	M_0	M_1	M_2	
$M(1)$			M_{-1}	M_0	M_1	M_2		
$M(2)$		M_{-1}	M_0	M_1	M_2			

Let M be a Γ-graded right A-module. A submodule N of M is called a *graded submodule* if

$$N = \bigoplus_{\gamma \in \Gamma} (N \cap M_\gamma).$$

Example 1.2.1 aA as a graded ideal and a graded module

Let A be a Γ-graded ring and $a \in A$ a homogeneous element of degree α. Then aA is a graded right A-module with $\gamma \in \Gamma$ homogeneous component defined as

$$(aA)_\gamma := aA_{\gamma-\alpha} \subseteq A_\gamma.$$

With this grading aA is a graded submodule (and graded right ideal) of A. Thus for $\beta \in \Gamma$, a is a homogenous element of the graded A-module $aA(\beta)$ of degree $\alpha - \beta$. This will be used throughout the book, for example in Proposition 1.2.19.

However, note that defining the grading on aA as

$$(aA)_\gamma := aA_\gamma \subseteq A_{\gamma+\alpha}$$

makes aA a graded submodule of $A(\alpha)$, which is the image of the graded homomorphism $A \to A(\alpha), r \mapsto ar$.

There are similar notions of graded left and graded bi-submodules (§1.2.5). When N is a graded submodule of M, the factor module M/N forms a graded A-module, with

$$\boxed{M/N = \bigoplus_{\gamma \in \Gamma} (M/N)_\gamma, \quad \text{where} \quad (M/N)_\gamma = (M_\gamma + N)/N.} \tag{1.12}$$

Example 1.2.2 Let A be a Γ-graded ring. Define a grading on the matrix ring $\mathbb{M}_n(A)$ as follows. For $\alpha \in \Gamma$, $\mathbb{M}_n(A)_\alpha = \mathbb{M}_n(A_\alpha)$ (for a general theory of grading on matrix rings see §1.3). Let $\mathbf{e}_{ii} \in \mathbb{M}_n(A)$, $1 \leq i \leq n$, be a *matrix unit*, *i.e.,* a matrix with 1 in the (i,i) position and zero everywhere else, and consider $\mathbf{e}_{ii}\,\mathbb{M}_n(A)$. By Example 1.2.1, $\mathbf{e}_{ii}\,\mathbb{M}_n(A)$ is a graded right $\mathbb{M}_n(A)$-module and

$$\bigoplus_{i=1}^{n} \mathbf{e}_{ii}\,\mathbb{M}_n(A) = \mathbb{M}_n(A).$$

This shows that the graded module $\mathbf{e}_{ii}\,\mathbb{M}_n(A)$ is a projective module. This is an example of a graded projective module (see §1.2.9).

Example 1.2.3 Let A be a commutative ring. Consider the matrix ring $\mathbb{M}_n(A)$ as a $\mathbb{Z}$-graded ring concentrated in degree zero. Moreover, consider $\mathbb{M}_n(A)$ as a graded $\mathbb{M}_n(A)$-module with the grading defined as follows: $\mathbb{M}_n(A)_i = \mathbf{e}_{ii}\,\mathbb{M}_n(A)$ for $1 \leq i \leq n$ and zero otherwise. Note that all nonzero homogeneous elements of this module are zero-divisors, and thus can't constitute a linear independent set. We will use this example to show that a free module which is graded is not necessarily a graded free module (§1.2.4).

Example 1.2.4 <u>Modules with no shift</u>

It is easy to construct modules whose shifts don't produce new (nonisomorphic) graded modules. Let M be a graded A-module and consider

$$N = \bigoplus_{\gamma\in\Gamma} M(\gamma).$$

We show that $N \cong_{\mathrm{gr}} N(\alpha)$ for any $\alpha \in \Gamma$. Define the map $f_\alpha : N \to N(\alpha)$ on homogeneous components as follows and extend it to N,

$$N_\beta = \bigoplus_{\gamma\in\Gamma} M_{\gamma+\beta} \longrightarrow \bigoplus_{\gamma\in\Gamma} M_{\gamma+\alpha+\beta} = N(\alpha)_\beta$$
$$\{m_\gamma\} \longmapsto \{m'_\gamma\},$$

where $m'_\gamma = m_{\gamma+\alpha}$ (*i.e.,* shift the sequence α "steps"). It is routine to see that this gives a graded A-module homomorphism with inverse homomorphism $f_{-\alpha}$. For another example, see Corollary 1.3.18.

1.2.3 The Hom groups and the category of graded modules

For graded right A-modules M and N, a *graded A-module homomorphism of degree δ* is an A-module homomorphism $f : M \to N$, such that

$$f(M_\gamma) \subseteq N_{\gamma+\delta}$$

for any $\gamma \in \Gamma$. Let $\operatorname{Hom}_A(M,N)_\delta$ denote the subgroup of $\operatorname{Hom}_A(M,N)$ consisting of all graded A-module homomorphisms of degree δ, *i.e.,*

$$\boxed{\operatorname{Hom}_A(M,N)_\delta = \{f \in \operatorname{Hom}_A(M,N) \mid f(M_\gamma) \subseteq N_{\gamma+\delta}, \gamma \in \Gamma\}.} \qquad (1.13)$$

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For graded A-modules, M, N and P, under the composition of functions, we then have

$$\mathrm{Hom}_A(N,P)_\gamma \times \mathrm{Hom}_A(M,N)_\delta \longrightarrow \mathrm{Hom}(M,P)_{\gamma+\delta}. \tag{1.14}$$

Clearly a graded module homomorphism defined in §1.2.1 is a graded homomorphism of degree 0.

By Gr-A (or Gr^Γ-A to emphasise the grade group of A), we denote a category that consists of Γ-graded right A-modules as objects and graded homomorphisms as the morphisms. Similarly, A-Gr denotes the category of graded left A-modules. Thus

$$\boxed{\mathrm{Hom}_{\text{Gr-}A}(M,N) = \mathrm{Hom}_A(M,N)_0.}$$

Moreover, for $\alpha \in \Gamma$, as *a set of functions*, one can write

$$\mathrm{Hom}_{\text{Gr-}A}\big(M(-\alpha),N\big) = \mathrm{Hom}_{\text{Gr-}A}\big(M,N(\alpha)\big) = \mathrm{Hom}_A(M,N)_\alpha. \tag{1.15}$$

A full subcategory of Gr-A consisted of all graded finitely generated A-modules is denoted by gr-A.

For $\alpha \in \Gamma$, the *α-suspension functor* or *shift functor*

$$\boxed{\begin{aligned} \mathcal{T}_\alpha : \text{Gr-}A &\longrightarrow \text{Gr-}A,\\ M &\longmapsto M(\alpha), \end{aligned}} \tag{1.16}$$

is an isomorphism with the property $\mathcal{T}_\alpha\mathcal{T}_\beta = \mathcal{T}_{\alpha+\beta}$, where $\alpha,\beta \in \Gamma$.

Remark 1.2.5 Let A be a Γ-graded ring and Ω be a subgroup of Γ such that $\Gamma_A \subseteq \Omega \subseteq \Gamma$. Then the ring A can be considered naturally as a Ω-graded ring. Similarly, if A, B are Γ-graded rings and $f : A \to B$ is a Γ-graded homomorphism and $\Gamma_A, \Gamma_B \subseteq \Omega \subseteq \Gamma$, then the homomorphism f can be naturally considered as a Ω-graded homomorphism. In this case, to make a distinction, we write Gr^Γ-A for the category of Γ-graded A-modules and Gr^Ω-A for the category of Ω-graded A-modules.

Theorem 1.2.6 *For graded right A-modules M and N, such that M is finitely generated, the abelian group* $\mathrm{Hom}_A(M,N)$ *has a natural decomposition*

$$\mathrm{Hom}_A(M,N) = \bigoplus_{\gamma\in\Gamma} \mathrm{Hom}_A(M,N)_\gamma. \tag{1.17}$$

Moreover, the endomorphism ring $\mathrm{Hom}_A(M,M)$ *is* Γ-*graded.*

Proof Let $f \in \mathrm{Hom}_A(M,N)$ and $\lambda \in \Gamma$. Define a map $f_\lambda : M \to N$ as follows: for $m \in M$,

$$f_\lambda(m) = \sum_{\gamma\in\Gamma} f(m_{\gamma-\lambda})_\gamma, \tag{1.18}$$

where $m = \sum_{\gamma \in \Gamma} m_\gamma$. One can check that $f_\lambda \in \operatorname{Hom}_A(M, N)$.

Now let $m \in M_\alpha$, $\alpha \in \Gamma$. Then (1.18) reduces to

$$f_\lambda(m) = f(m)_{\alpha+\lambda} \subseteq M_{\alpha+\lambda}.$$

This shows that $f_\lambda \in \operatorname{Hom}_A(M, N)_\lambda$. Moreover, $f_\lambda(m)$ is zero for all but a finite number of $\lambda \in \Gamma$ and

$$\sum_\lambda f_\lambda(m) = \sum_\lambda f(m)_{\alpha+\lambda} = f(m).$$

Now since M is finitely generated, there are a finite number of homogeneous elements which generate any element $m \in M$. The above argument shows that only a finite number of the $f_\lambda(m)$ are nonzero and $f = \sum_\lambda f_\lambda$. This in turn shows that

$$\operatorname{Hom}_A(M, N) = \sum_{\gamma \in \Gamma} \operatorname{Hom}_A(M, N)_\gamma.$$

Finally, it is easy to see that $\operatorname{Hom}_A(M, N)_\gamma$, $\gamma \in \Gamma$ constitutes a direct sum.

For the second part, replacing N by M in (1.17), we get

$$\operatorname{Hom}_A(M, M) = \bigoplus_{\gamma \in \Gamma} \operatorname{Hom}_A(M, M)_\gamma.$$

Moreover, by (1.14) if $f \in \operatorname{Hom}_A(M, M)_\gamma$ and $g \in \operatorname{Hom}_A(M, M)_\lambda$ then

$$fg \in \operatorname{Hom}_A(M, M)_{\gamma+\lambda}.$$

This shows that when M is finitely generated $\operatorname{Hom}_A(M, M)$ is a Γ-graded ring. □

Let M be a graded finitely generated right A-module. Then the usual *dual* of M, *i.e.*, $M^* = \operatorname{Hom}_A(M, A)$, is a left A-module. Moreover, using Theorem 1.2.6, one can check that M^* is a graded left A-module. Since

$$\operatorname{Hom}_A(M, N)(\alpha) = \operatorname{Hom}_A(M(-\alpha), N) = \operatorname{Hom}_A(M, N(\alpha)),$$

we have

$$M(\alpha)^* = M^*(-\alpha). \tag{1.19}$$

This should also make sense: the dual of "pushing forward" M by α, is the same as "pulling back" the dual M^* by α.

Note that although $\operatorname{Hom}_A(M, N)$ is defined in the category Mod-A, the graded structures of M and N are intrinsic in the grading defined on $\operatorname{Hom}_A(M, N)$. Thus if M is isomorphic to N as a nongraded A-module, then $\operatorname{End}_A(M)$ is not necessarily graded isomorphic to $\operatorname{End}_A(N)$. However if $M \cong_{\mathrm{gr}} N(\alpha)$, $\alpha \in \Gamma$, then one can observe that $\operatorname{End}_A(M) \cong_{\mathrm{gr}} \operatorname{End}_A(N)$ as graded rings.

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When M is a free module, $\mathrm{Hom}_A(M, M)$ can be represented as a matrix ring over A. Next we define graded free modules. In §1.3 we will see that if M is a graded free module, the graded ring $\mathrm{Hom}_A(M, M)$ can be represented as a matrix ring over A with a very concrete grading.

Example 1.2.7 THE VERONESE SUBMODULE

For a Γ-graded ring A, recall the construction of nth-Veronese subring

$$A^{(n)} = \bigoplus_{\gamma \in \Gamma} A_{n\gamma}$$

(Example 1.1.19). In a similar fashion, for a graded A-module M and $n \in \mathbb{Z}$, define the nth-*Veronese module* of M as

$$M^{(n)} = \bigoplus_{\gamma \in \Gamma} M_{n\gamma}.$$

This is a Γ-graded $A^{(n)}$-module. Clearly there is a natural "forgetful" functor

$$U : \text{Gr-}A \longrightarrow \text{Gr-}A^{(n)},$$

which commutes with suspension functors as follows $\mathcal{T}_\alpha U = U\mathcal{T}_{n\alpha}$, *i.e.,*

$$M^{(n)}(\alpha) = M(n\alpha)^{(n)},$$

for $\alpha \in \Gamma$ and $n \in \mathbb{Z}$ (see §1.2.7 for more on forgetful functors).

1.2.4 Graded free modules

A Γ-graded (right) A-module F is called a *graded free A-module* if F is a free right A-module with a homogeneous base. Clearly a graded free module is a free module but the converse is not correct, *i.e.,* a free module which is graded is not necessarily a graded free module. As an example, for $A = \mathbb{R}[x]$ considered as a $\mathbb{Z}$-graded ring, $A \oplus A(1)$ is not a graded free $A \oplus A$-module, whereas $A \oplus A$ is a free $A \oplus A$-module (see also Example 1.2.3). The definition of free given here is consistent with the categorical definition of free objects over a set of homogeneous elements in the category of graded modules ([50, I, §7]).

Consider a Γ-graded A-module $\bigoplus_{i \in I} A(\delta_i)$, where I is an indexing set and $\delta_i \in \Gamma$. Note that for each $i \in I$, the element e_i of the standard basis (*i.e.,* 1 in the ith component and zero elsewhere) is homogeneous of degree $-\delta_i$. The set $\{e_i\}_{i \in I}$ forms a base for $\bigoplus_{i \in I} A(\delta_i)$, which by definition makes this a graded free A-module. On the other hand, a graded free A-module F with a homogeneous base $\{b_i\}_{i \in I}$, where $\deg(b_i) = -\delta_i$ is graded isomorphic to $\bigoplus_{i \in I} A(\delta_i)$. Indeed

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one can easily observe that the map induced by

$$\varphi : \bigoplus_{i\in I} A(\delta_i) \longrightarrow F \qquad (1.20)$$
$$e_i \longmapsto b_i$$

is a graded A-module isomorphism.

If the indexing set I is finite, say $I = \{1, \dots, n\}$, then

$$\bigoplus_{i\in I} A(\delta_i) = A(\delta_1) \oplus \cdots \oplus A(\delta_n),$$

is also denoted by $A^n(\delta_1, \dots, \delta_n)$ or $A^n(\overline{\delta})$, where $\overline{\delta} = (\delta_1, \dots, \delta_n)$.

In §1.3.4, we consider the situation when the graded free right A-modules $A^n(\overline{\delta})$ and $A^m(\overline{\alpha})$, where $\overline{\delta} = (\delta_1, \dots, \delta_n)$ and $\overline{\alpha} = (\alpha_1, \dots, \alpha_m)$, are isomorphic. In §1.7, we will also consider the concept of graded rings with the graded invariant basis numbers.

1.2.5 Graded bimodules

The notion of graded *left* A-modules is developed similarly. The category of graded left A-modules with graded homomorphisms is denoted by A-Gr. In a similar manner for Γ-graded rings A and B, we can consider the *graded* $A-B$-*bimodule* M. That is, M is a $A-B$-bimodule and additionally $M = \bigoplus_{\gamma\in\Gamma} M_\gamma$ is a graded left A-module and a graded right B-module, *i.e.,*

$$A_\alpha M_\gamma B_\beta \subseteq M_{\alpha+\gamma+\beta},$$

where $\alpha, \gamma, \beta \in \Gamma$. The category of graded A-bimodules is denoted by Gr-A-Gr.

Remark 1.2.8 SHIFT OF NONABELIAN GROUP GRADED MODULES

If the grade group Γ is not abelian, then in order that the shift of components matches, for a graded left A-module M one needs to define

$$M(\delta)_\gamma = M_{\gamma\delta},$$

whereas for the graded right M-module A, shift is defined by

$$M(\delta)_\gamma = M_{\delta\gamma}.$$

With these definitions, for $\mathcal{T}_\alpha, \mathcal{T}_\beta : \text{Gr-}A \to \text{Gr-}A$, we have $\mathcal{T}_\alpha\mathcal{T}_\beta = \mathcal{T}_{\beta\alpha}$, whereas for $\mathcal{T}_\alpha, \mathcal{T}_\beta : A\text{-Gr} \to A\text{-Gr}$, we have $\mathcal{T}_\alpha\mathcal{T}_\beta = \mathcal{T}_{\alpha\beta}$. For this reason, in the nonabelian grade group setting, several books choose to work with the graded left modules as opposed to the graded right modules we have adopted in this book.

1.2.6 Tensor product of graded modules

Let A be a Γ-graded ring and M_i, $i \in I$, be a direct system of Γ-graded A-modules, *i.e.,* I is a directed partially ordered set and for $i \leq j$, there is a graded A-homomorphism $\phi_{ij} : M_i \to M_j$ which is compatible with the ordering. Then $M := \varinjlim M_i$ is a Γ-graded A-module with homogeneous components $M_\alpha = \varinjlim M_{i\alpha}$ (see Example 1.1.9 for the similar construction for rings).

In particular, let $\{M_i \mid i \in I\}$ be Γ-graded right A-modules. Then $\bigoplus_{i\in I} M_i$ has a natural graded A-module given by $(\bigoplus_{i\in I} M_i)_\alpha = \bigoplus_{i\in I} M_{i\alpha}$, $\alpha \in \Gamma$.

Let M be a graded right A-module and N be a graded left A-module. We will observe that the tensor product $M \otimes_A N$ has a natural Γ-graded $\mathbb{Z}$-module structure. Since each of M_γ, $\gamma \in \Gamma$, is a right A_0-module and similarly N_γ, $\gamma \in \Gamma$, is a left A_0-module, then $M \otimes_{A_0} N$ can be decomposed as a direct sum

$$\boxed{M \otimes_{A_0} N = \bigoplus_{\gamma\in\Gamma} (M \otimes N)_\gamma,}$$

where

$$\boxed{(M \otimes N)_\gamma = \Big\{ \sum_i m_i \otimes n_i \mid m_i \in M^h, n_i \in N^h, \deg(m_i) + \deg(n_i) = \gamma \Big\}.}$$

Now note that $M \otimes_A N \cong (M \otimes_{A_0} N)/J$, where J is a subgroup of $M \otimes_{A_0} N$ generated by the homogeneous elements

$$\{ma \otimes n - m \otimes an \mid m \in M^h, n \in N^h, a \in A^h\}.$$

This shows that $M \otimes_A N$ is also a graded module. It is easy to check that, for example, if N is a graded A-bimodule, then $M \otimes_A N$ is a graded right A-module. It follows from the definition that

$$M \otimes N(\alpha) = M(\alpha) \otimes N = (M \otimes N)(\alpha). \tag{1.21}$$

Observe that for a graded right A-module M, the map

$$\begin{aligned} M \otimes_A A(\alpha) &\longrightarrow M(\alpha), \\ m \otimes a &\longmapsto ma, \end{aligned} \tag{1.22}$$

is a graded isomorphism. In particular, for any $\alpha, \beta \in \Gamma$, there is a graded A-bimodule isomorphism

$$\boxed{A(\alpha) \otimes_A A(\beta) \cong_{\text{gr}} A(\alpha + \beta).} \tag{1.23}$$

Example 1.2.9 GRADED FORMAL MATRIX RINGS

The construction of formal matrix rings (Example 1.1.4) can be carried over

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to the graded setting as follows. Let R and S be Γ-graded rings, M be a graded R–S-bimodule and N be a graded S–R-bimodule. Suppose that there are graded bimodule homomorphisms $\phi : M \otimes_S N \to R$ and $\psi : N \otimes_R M \to S$ such that $(mn)m' = n(nm')$, where we denote $\phi(m,n) = mn$ and $\psi(n,m) = nm$. Consider the ring

$$T = \begin{pmatrix} R & M \\ N & S \end{pmatrix},$$

and define, for any $\gamma \in \Gamma$,

$$T_\gamma = \begin{pmatrix} R_\gamma & M_\gamma \\ N_\gamma & S_\gamma \end{pmatrix}.$$

One checks that T is a Γ-graded ring, called a *graded formal matrix ring*. One specific type of such rings is a Morita ring which appears in graded Morita theory (§2.3).

1.2.7 Forgetting the grading

Most forgetful functors in algebra tend to have left adjoints, which have a "free" construction. One such example is the forgetful functor from the category of abelian groups to abelian monoids that we will study in Chapter 3 in relation to Grothendieck groups. However, some of the forgetful functors in the graded setting naturally have right adjoints, as we will see below.

Consider the *forgetful* functor

$$U : \text{Gr-}A \longrightarrow \text{Mod-}A, \tag{1.24}$$

which simply assigns to any graded module M in Gr-A its underlying module M in Mod-A, ignoring the grading. Similarly, the graded homomorphisms are sent to the same homomorphisms, disregarding their graded compatibilities.

There is a functor $F : \text{Mod-}A \to \text{Gr-}A$ which is a right adjoint to U. The construction is as follows: let M be an A-module. Consider the abelian group $F(M) := \bigoplus_{\gamma\in\Gamma} M_\gamma$, where M_γ is a copy of M. Moreover, for $a \in A_\alpha$ and $m \in M_\gamma$ define $m.a = ma \in M_{\alpha+\gamma}$. This defines a graded A-module structure on $F(M)$ and makes F an exact functor from Mod-A to Gr-A. One can prove that for any $M \in \text{Gr-}A$ and $N \in \text{Mod-}A$, we have a bijective map

$$\begin{aligned} \operatorname{Hom}_{\text{Mod-}A}(U(M), N) &\stackrel{\phi}{\longrightarrow} \operatorname{Hom}_{\text{Gr-}A}(M, F(N)), \\ f &\longmapsto \phi_f, \end{aligned}$$

where $\phi_f(m_\alpha) = f(m_\alpha) \in N_\alpha$.

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Remark 1.2.10 It is not difficult to observe that for any $M \in \text{Gr-}A$,

$$FU(M) \cong_{\text{gr}} \bigoplus_{\gamma \in \Gamma} M(\gamma).$$

By Example 1.2.4, we have $FU(M) \cong_{\text{gr}} FU(M)(\alpha)$ for any $\alpha \in \Gamma$. We also note that if Γ is finite, then F is also a left adjoint functor of U. Further, if U has a left adjoint functor, then one can prove that Γ is finite (see [75, §2.5] for details).

1.2.8 Partitioning graded modules

Let $f : \Gamma \to \Delta$ be a group homomorphism. Recall from §1.1.2 that there is a functor from the category of Γ-graded rings to the category of Δ-graded rings which gives the natural forgetful functor when $\Delta = 0$. This functor has a right adjoint functor (see [75, Proposition 1.2.2] for the case of $\Delta = 0$). The homomorphism f induces a forgetful functor on the level of module categories. We describe this here.

Let A be a Γ-graded ring and consider the corresponding Δ-graded structure induced by the homomorphism $f : \Gamma \to \Delta$ (§1.1.2). Then one can construct a functor $U_f : \text{Gr}^{\Gamma}\text{-}A \to \text{Gr}^{\Delta}\text{-}A$ which has a right adjoint. In particular, for a subgroup Ω of Γ, we have the following canonical "forgetful" functor (a *block* functor or a *coarsening* functor)

$$U : \text{Gr}^{\Gamma}\text{-}A \to \text{Gr}^{\Gamma/\Omega}\text{-}A,$$

such that when $\Omega = \Gamma$, it gives the functor (1.24). The construction is as follows. Let $M = \bigoplus_{\alpha \in \Gamma} M_\alpha$ be a Γ-graded A-module. Write

$$M = \bigoplus_{\Omega+\alpha \in \Gamma/\Omega} M_{\Omega+\alpha}, \tag{1.25}$$

where

$$M_{\Omega+\alpha} := \bigoplus_{\omega \in \Omega} M_{\omega+\alpha}. \tag{1.26}$$

One can easily check that M is a Γ/Ω-graded A-module. Moreover,

$$U(M(\alpha)) = M(\Omega + \alpha).$$

We will use this functor to relate the Grothendieck groups of these categories in Examples 3.1.10 and 3.1.11.

In a similar manner we have the following functor

$$(-)_\Omega : \text{Gr}^{\Gamma}\text{-}A \longrightarrow \text{Gr}^{\Omega}\text{-}A_\Omega,$$

where $\mathrm{Gr}^{\Omega}\text{-}A$ is the category of Ω-graded (right) A_{Ω}-modules, where $\Omega = \ker(f)$.

The above construction motivates the following which will establish a relation between the categories $\mathrm{Gr}^{\Gamma}\text{-}A$ and $\mathrm{Gr}^{\Omega}\text{-}A_{\Omega}$.

Consider the quotient group Γ/Ω and fix a complete set of coset representatives $\{\alpha_i\}_{i\in I}$. Let $\beta \in \Gamma$ and consider the permutation map ρ_β:

$$\begin{aligned}\rho_\beta : \Gamma/\Omega &\longrightarrow \Gamma/\Omega,\\ \Omega + \alpha_i &\longmapsto \Omega + \alpha_i + \beta = \Omega + \alpha_j.\end{aligned}$$

This defines a bijective map (called ρ_β again) $\rho_\beta : \{\alpha_i\}_{i\in I} \to \{\alpha_i\}_{i\in I}$. Moreover, for any α_i, since

$$\Omega + \alpha_i + \beta = \Omega + \alpha_j = \Omega + \rho_\beta(\alpha_i),$$

there is a unique $w_i \in \Omega$ such that

$$\alpha_i + \beta = \omega_i + \rho_\beta(\alpha_i). \tag{1.27}$$

Recall that if $\mathcal{C}$ is an additive category, $\bigoplus_I \mathcal{C}$, where I is a nonempty index set, is defined in the obvious manner, with objects $\bigoplus_{i\in I} M_i$, where M_i are objects of $\mathcal{C}$ and morphisms accordingly.

Define the functor

$$\boxed{\begin{aligned}\mathcal{P} : \mathrm{Gr}^{\Gamma}\text{-}A &\longrightarrow \bigoplus_{\Gamma/\Omega} \mathrm{Gr}^{\Omega}\text{-}A_{\Omega},\\ M &\longmapsto \bigoplus_{\Omega+\alpha_i\in\Gamma/\Omega} M_{\Omega+\alpha_i},\end{aligned}} \tag{1.28}$$

where

$$\boxed{M_{\Omega+\alpha_i} = \bigoplus_{\omega\in\Omega} M_{\omega+\alpha_i}.}$$

Since $M_{\Omega+\alpha}$, $\alpha \in \Gamma$, as defined in (1.26), can be naturally considered as an Ω-graded A_{Ω}-module, where

$$(M_{\Omega+\alpha})_{\omega} = M_{\omega+\alpha}, \tag{1.29}$$

it follows that the functor $\mathcal{P}$ defined in (1.28) is well-defined. Note that the homogeneous components defined in (1.29) depend on the coset representation, thus choosing another complete set of coset representatives gives a different functor between these categories.

For any $\beta \in \Gamma$, define a shift functor

$$\overline{\rho}_\beta : \bigoplus_{\Gamma/\Omega} \mathrm{Gr}^{\Omega}\text{-}A_\Omega \longrightarrow \bigoplus_{\Gamma/\Omega} \mathrm{Gr}^{\Omega}\text{-}A_\Omega,$$
$$\bigoplus_{\Omega+\alpha_i \in \Gamma/\Omega} M_{\Omega+\alpha_i} \longmapsto \bigoplus_{\Omega+\alpha_i \in \Gamma/\Omega} M(\omega_i)_{\Omega+\rho_\beta(\alpha_i)}, \tag{1.30}$$

where $\rho_\beta(\alpha_i)$ and ω_i are defined in (1.27). The action of $\overline{\rho}_\beta$ on morphisms are defined accordingly. Note that in the left hand side of (1.30) the graded A_Ω-module which appears in $\Omega + \alpha_i$ component is denoted by $M_{\Omega+\alpha_i}$. When M is a Γ-graded module, then $M_{\Omega+\alpha_i}$ has a Ω-structure as described in (1.26).

We are in a position to prove the next theorem.

Theorem 1.2.11 *Let A be a Γ-graded ring and Ω a subgroup of Γ. Then for any $\beta \in \Gamma$, the following diagram is commutative,*

$$\begin{array}{ccc} \mathrm{Gr}^{\Gamma}\text{-}A & \xrightarrow{\ \mathcal{P}\ } & \bigoplus_{\Gamma/\Omega} \mathrm{Gr}^{\Omega}\text{-}A_\Omega \\ \downarrow{\scriptstyle \mathcal{T}_\beta} & & \downarrow{\scriptstyle \overline{\rho}_\beta} \\ \mathrm{Gr}^{\Gamma}\text{-}A & \xrightarrow{\ \mathcal{P}\ } & \bigoplus_{\Gamma/\Omega} \mathrm{Gr}^{\Omega}\text{-}A_\Omega, \end{array} \tag{1.31}$$

where the functors $\mathcal{P}$ and $\overline{\rho}_\beta$ are defined in (1.28) *and* (1.30)*, respectively. Moreover, if $\Gamma_A \subseteq \Omega$, then the functor $\mathcal{P}$ induces an equivalence of categories.*

Proof We first show that Diagram (1.31) is commutative. Let $\beta \in \Gamma$ and M be a Γ-graded A-module. As in (1.27), let $\{\alpha_i\}$ be a fixed complete set of coset representative and

$$\alpha_i + \beta = \omega_i + \rho_\beta(\alpha_i).$$

Then

$$\mathcal{P}(\mathcal{T}_\beta(M)) = \mathcal{P}(M(\beta)) = \bigoplus_{\Omega+\alpha_i \in \Gamma/\Omega} M(\beta)_{\Omega+\alpha_i}. \tag{1.32}$$

But

$$M(\beta)_{\Omega+\alpha_i} = \bigoplus_{\omega\in\Omega} M(\beta)_{\omega+\alpha_i} = \bigoplus_{\omega\in\Omega} M_{\omega+\alpha_i+\beta} = \bigoplus_{\omega\in\Omega} M_{\omega+\omega_i+\rho_\beta(\alpha_i)} =$$
$$\bigoplus_{\omega\in\Omega} M(\omega_i)_{\omega+\rho_\beta(\alpha_i)} = M(\omega_i)_{\Omega+\rho_\beta(\alpha_i)}.$$

Replacing this into Equation (1.32) we have

$$\mathcal{P}(\mathcal{T}_\beta(M)) = \bigoplus_{\Omega+\alpha_i \in \Gamma/\Omega} M(\omega_i)_{\Omega+\rho_\beta(\alpha_i)}. \tag{1.33}$$

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On the other hand, by (1.30),

$$\overline{\rho}_\beta \mathcal{P}(M) = \overline{\rho}_\beta\Big(\bigoplus_{\Omega+\alpha_i\in\Gamma/\Omega} M_{\Omega+\alpha_i}\Big) = \bigoplus_{\Omega+\alpha_i\in\Gamma/\Omega} M(\omega_i)_{\Omega+\rho_\beta(\alpha_i)}. \tag{1.34}$$

Comparing (1.33) and (1.34) shows that Diagram (1.31) is commutative.

For the last part of the theorem, suppose $\Gamma_A \subseteq \Omega$. We construct a functor

$$\mathcal{P}' : \bigoplus_{\Gamma/\Omega} \mathrm{Gr}^{\Omega}\text{-}A_\Omega \longrightarrow \mathrm{Gr}^{\Gamma}\text{-}A,$$

which depends on the coset representative $\{\alpha_i\}_{i\in I}$ of Γ/Ω. First note that any $\alpha \in \Gamma$ can be written uniquely as $\alpha = \alpha_i + \omega$, for some $i \in I$ and $\omega \in \Omega$. Now let

$$\bigoplus_{\Omega+\alpha_i\in\Gamma/\Omega} N_{\Omega+\alpha_i} \in \bigoplus_{\Gamma/\Omega} \mathrm{Gr}^{\Omega}\text{-}A_\Omega,$$

where $N_{\Omega+\alpha_i}$ is an Ω-graded A_Ω-module. Define a Γ-graded A-module N as follows: $N = \bigoplus_{\alpha\in\Gamma} N_\alpha$, where $N_\alpha := (N_{\Omega+\alpha_i})_\omega$ and $\alpha = \alpha_i + \omega$. We check that N is a Γ-graded A-module, *i.e.,* $N_\alpha A_\gamma \subseteq N_{\alpha+\gamma}$, for $\alpha, \gamma \in \Gamma$. If $\gamma \notin \Omega$, since $\Gamma_A \subseteq \Omega$, $A_\gamma = 0$ and thus $0 = N_\alpha A_\gamma \subseteq N_{\alpha+\gamma}$. Let $\gamma \in \Omega$. Then

$$N_\alpha A_\gamma = (N_{\Omega+\alpha_i})_\omega A_\gamma \subseteq (N_{\Omega+\alpha_i})_{\omega+\gamma} = N_{\alpha+\gamma},$$

as $\alpha + \gamma = \alpha_i + \omega + \gamma$.

We define $\mathcal{P}'(\bigoplus_{\Omega+\alpha_i\in\Gamma/\Omega} N_{\Omega+\alpha_i}) = N$ for the objects and similarly for the morphisms. It is now not difficult to check that $\mathcal{P}'$ is an inverse of the functor $\mathcal{P}$. This finishes the proof. □

The above theorem will be used to compare the graded K-theories with respect to Γ and Ω (see Example 3.1.11).

Corollary 1.2.12 *Let A be a Γ-graded ring concentrated in degree zero. Then*

$$\mathrm{Gr}\text{-}A \approx \bigoplus_{\Gamma} \mathrm{Mod}\text{-}A.$$

The action of Γ on $\bigoplus_\Gamma$ Mod-A described in (1.30) *reduces to the following: for $\beta \in \Gamma$,*

$$\overline{\rho}_\beta\Big(\bigoplus_{\alpha\in\Gamma} M_\alpha\Big) = \bigoplus_{\alpha\in\Gamma} M(\beta)_\alpha = \bigoplus_{\alpha\in\Gamma} M_{\alpha+\beta}. \tag{1.35}$$

Proof This follows by replacing Ω by a trivial group in Theorem 1.2.11. □

The following corollary, which is a more general case of Corollary 1.2.12 with a similar proof, will be used in the proof of Lemma 6.1.6.

Corollary 1.2.13 *Let A be a $\Gamma \times \Omega$ graded ring which is concentrated in Ω. Then*

$$\mathrm{Gr}^{\Gamma\times\Omega}\text{-}A \cong \bigoplus_{\Gamma} \mathrm{Gr}^{\Omega}\text{-}A.$$

The action of $\Gamma\times\Omega$ on $\bigoplus_\Gamma \mathrm{Gr}^\Omega\text{-}A$ described in (1.30) *reduces to the following: for $(\beta, \omega) \in \Gamma \times \Omega$,*

$$\overline{\rho}_{(\beta,\omega)}\Big(\bigoplus_{\alpha\in\Gamma} M_\alpha\Big) = \bigoplus_{\alpha\in\Gamma} M_{\alpha+\beta}(\omega).$$

1.2.9 Graded projective modules

Graded projective modules play a crucial role in this book. They will appear in the graded Morita theory in Chapter 2 and will be used to define the graded Grothendieck groups in Chapter 3. Moreover, the graded higher K-theory is constructed from the exact category consisting of graded finitely generated projective modules (see Chapter 6). In this section we define the graded projective modules and give several equivalent criteria for a module to be graded projective. As before, unless stated otherwise, we work in the category of (graded) right modules.

A graded A-module P is called a *graded projective module* if it is a projective object in the abelian category Gr-A. More concretely, P is graded projective if for any diagram of graded modules and graded A-module homomorphisms

$$\begin{array}{ccccc} & & P & & \\ & \overset{h}{\swarrow} & \downarrow{\scriptstyle j} & & \\ M & \xrightarrow[g]{} & N & \longrightarrow & 0, \end{array} \tag{1.36}$$

there is a graded A-module homomorphism $h : P \to M$ with $gh = j$.

In Proposition 1.2.15 we give some equivalent characterisations of graded projective modules, including the one that shows an A-module is graded projective if and only if it is graded and projective as an A-module. By Pgr-A (or Pgr^Γ-A to emphasise the grade group of A) we denote a full subcategory of Gr-A, consisting of graded finitely generated projective right A-modules. This is the primary category we are interested in. The graded Grothendieck group (Chapter 3) and higher K-groups (Chapter 6) are constructed from this exact category (see Definition 3.12.1).

We need the following lemma, which says if a graded map factors into two maps, with one being graded, then we can replace the other one with a graded map as well.

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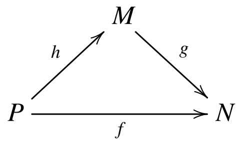
Lemma 1.2.14 *Let P, M, N be graded A-modules, with A-module homomorphisms f, g, h*



such that $f = gh$, where f is a graded A-module homomorphism. If g (resp. h) is a graded A-homomorphism then there exists a graded homomorphism $h' : P \to M$ (resp. $g' : M \to N$) such that $f = gh'$ (resp. $f = g'h$).

Proof Suppose $g : M \to N$ is a graded A-module homomorphism. Define $h' : P \to M$ as follows: for $p \in P_\alpha$, $\alpha \in \Gamma$, let $h'(p) = h(p)_\alpha$ and extend this linearly to all elements of P, *i.e.,* for $p \in P$ with $p = \sum_{\alpha\in\Gamma} p_\alpha$,

$$h(p) = \sum_{\alpha\in\Gamma} h(p_\alpha)_\alpha.$$

One can easily see that $h' : P \to M$ is a graded A-module homomorphism. Moreover, for $p \in P_\alpha$, $\alpha \in \Gamma$, we have

$$f(p) = gh(p) = g\Big(\sum_{\gamma\in\Gamma} h(p)_\gamma\Big) = \sum_{\gamma\in\Gamma} g\big(h(p)_\gamma\big).$$

Since f and g are graded homomorphisms, comparing the degrees of the homogeneous elements of each side of the equation, we get

$$f(p) = g(h(p)_\alpha) = gh'(p).$$

Using the linearity of f, g, h' it follows that $f = gh'$. This proves the lemma for the case g. The other case is similar. □

We are in a position to give equivalent characterisations of graded projective modules.

Proposition 1.2.15 *Let A be a Γ-graded ring and P be a graded A-module. Then the following are equivalent:*

(1) *P is graded and projective;*
(2) *P is graded projective;*
(3) $\operatorname{Hom}_{\text{Gr-}A}(P, -)$ *is an exact functor in* Gr-A*;*
(4) *every short exact sequence of graded A-module homomorphisms*

$$0 \longrightarrow L \xrightarrow{f} M \xrightarrow{g} P \longrightarrow 0$$

splits via a (graded) map;

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(5) *P is graded isomorphic to a direct summand of a graded free A-module.*

Proof (1) ⇒ (2) Consider the diagram

$$\begin{array}{ccccc} & & P & & \\ & & \big\downarrow{\scriptstyle j} & & \\ M & \xrightarrow[g]{} & N & \longrightarrow & 0, \end{array}$$

where M and N are graded modules, g and j are graded homomorphisms and g is surjective. Since P is projective, there is an A-module homomorphism $h : P \to M$ with $gh = j$. By Lemma 1.2.14, there is a graded A-module homomorphism $h' : P \to M$ with $gh' = j$. This gives that P is a graded projective module.

(2) ⇒ (3) In exactly the same way as the nongraded setting, we can show (with no assumption on P) that $\operatorname{Hom}_{\text{Gr-}A}(P, -)$ is left exact (see [50, §IV, Theorem 4.2]). The right exactness follows immediately from the definition of graded projective modules that any diagram of the form (1.36) can be completed.

(3) ⇒ (4) Let

$$0 \longrightarrow L \stackrel{f}{\longrightarrow} M \stackrel{g}{\longrightarrow} P \longrightarrow 0 \tag{1.37}$$

be a short exact sequence. Since $\operatorname{Hom}_{\text{Gr-}A}(P, -)$ is exact,

$$\operatorname{Hom}_{\text{Gr-}A}(P, M) \longrightarrow \operatorname{Hom}_{\text{Gr-}A}(P, P)$$
$$h \longmapsto gh$$

is an epimorphism. In particular, there is a graded homomorphism h such that $gh = 1$, *i.e.,* the short exact sequence (1.37) is spilt.

(4) ⇒ (5) First note that P is a homomorphic image of a graded free A-module as follows: Let $\{p_i\}_{i\in I}$ be a homogeneous generating set for P, where $\deg(p_i) = \delta_i$. Let $\bigoplus_{i\in I} A(-\delta_i)$ be the graded free A-module with standard homogeneous basis $\{e_i\}_{i\in I}$ where $\deg(e_i) = \delta_i$. Then there is an exact sequence

$$0 \longrightarrow \ker(g) \stackrel{\subseteq}{\longrightarrow} \bigoplus_{i\in I} A(-\delta_i) \stackrel{g}{\longrightarrow} P \longrightarrow 0, \tag{1.38}$$

as the map

$$g : \bigoplus_{i\in I} A(-\delta_i) \longrightarrow P,$$
$$e_i \longmapsto p_i,$$

is a surjective graded A-module homomorphism. By the assumption, there is

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a A-module homomorphism $h : P \to \bigoplus_{i\in I} A(-\delta_i)$ such that $gh = \mathrm{id}_P$. By Lemma 1.2.14 one can assume h is a graded homomorphism.

Since the exact sequence (1.38) is, in particular, a split exact sequence of A-modules, we know from the nongraded setting [67, Proposition 2.5] that there is an A-module isomorphism

$$\theta : P \oplus \ker(g) \longrightarrow \bigoplus_{i\in I} A(-\delta_i)$$
$$(p, q) \longmapsto h(p) + q.$$

Clearly this map is also a graded A-module homomorphism, so

$$P \oplus \ker(g) \cong_{\mathrm{gr}} \bigoplus_{i\in I} A(-\delta_i).$$

(5) ⇒ (1) Graded free modules are free, so P is isomorphic to a direct summand of a free A-module. From the nongraded setting, we know that P is projective. □

The proof of Proposition 1.2.15 (see in particular (4) ⇒ (5) and (5) ⇒ (1)) shows that a graded A-module P is a graded finitely generated projective A-module if and only if

$$\boxed{P \oplus Q \cong_{\mathrm{gr}} A^n(\overline{\alpha})} \tag{1.39}$$

for some $\overline{\alpha} = (\alpha_1, \ldots, \alpha_n)$, $\alpha_i \in \Gamma$. This fact will be used frequently throughout this book.

Remark 1.2.16 Recall the functor $\mathcal{P}$ from (1.28). It is easy to see that if M is a Γ-graded projective A-module, then $M_{\Omega+\alpha}$ is a Ω-graded projective A_Ω-graded module. Thus the functor $\mathcal{P}$ restricts to

$$\boxed{\begin{aligned} \mathcal{P} : \mathrm{Pgr}^{\Gamma}\text{-}A &\longrightarrow \bigoplus_{\Gamma/\Omega} \mathrm{Pgr}^{\Omega}\text{-}A_\Omega, \\ M &\longmapsto \bigoplus_{\Omega+\alpha_i\in\Gamma/\Omega} M_{\Omega+\alpha_i}. \end{aligned}} \tag{1.40}$$

This will be used later in Examples 3.1.5, 3.1.11 and Lemma 6.1.6.

Theorem 1.2.17 (The dual basis lemma) *Let A be a Γ-graded ring and P be a graded A-module. Then P is graded projective if and only if there exists $p_i \in P^h$ with* $\deg(p_i) = \delta_i$ *and* $f_i \in \mathrm{Hom}_{\mathrm{Gr}\text{-}A}(P, A(-\delta_i))$, *for some indexing set I, such that*

(1) *for every $p \in P$, $f_i(p) = 0$ for all but a finite subset of $i \in I$,*
(2) *for every $p \in P$, $\sum_{i\in I} f_i(p)p_i = p$.*

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Proof Since P is graded projective, by Proposition 1.2.15(5), there is a graded module Q such that $P \oplus Q \cong_{\mathrm{gr}} \bigoplus_i A(-\delta_i)$. This gives two graded maps

$$\phi : P \to \bigoplus_i A(-\delta_i) \text{ and } \pi : \bigoplus_i A(-\delta_i) \to P,$$

such that $\pi\phi = 1_P$. Let

$$\pi_i : \bigoplus_i A(-\delta_i) \longrightarrow A(-\delta_i),$$
$$\{a_i\}_{i\in I} \longmapsto a_i$$

be the projection on the ith component. So if

$$a = \{a_i\}_{i\in I} \in \bigoplus_i A(-\delta_i),$$

then

$$\sum_i \pi_i(a) e_i = a,$$

where $\{e_i\}_{i\in I}$ is the standard homogeneous basis of $\bigoplus_i A(-\delta_i)$. Now let $p_i = \pi(e_i)$ and $f_i = \pi_i\phi$. Note that $\deg(p_i) = \delta_i$ and

$$f_i \in \mathrm{Hom}_{\mathrm{Gr}\text{-}A}(P, A(-\delta_i)).$$

Clearly $f_i(p) = \pi_i\phi(p)$ is zero for all but a finite number of $i \in I$. This gives (1). Moreover,

$$\sum_i p_i f_i(p) = \sum_i p_i \pi_i \phi(p) = \sum \pi(e_i)\pi_i\phi(p) = \pi\Big(\sum_i e_i \pi_i \phi(p)\Big) = \pi\phi(p) = p.$$

This gives (2).

Conversely, suppose that there exists a dual basis $\{p_i, f_i \mid i \in I\}$. Consider the maps

$$\phi : P \longrightarrow \bigoplus_i A(-\delta_i),$$
$$p \longmapsto \{f_i(p)\}_{i\in I}$$

and

$$\pi : \bigoplus_i A(-\delta_i) \longrightarrow P,$$
$$\{a_i\}_{i\in I} \longmapsto \sum_i p_i a_i.$$

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One sees easily that ϕ and π are graded right A-module homomorphisms and $\pi\phi = 1_P$. Therefore the exact sequence

$$0 \longrightarrow \ker(\pi) \longrightarrow \bigoplus_i A(-\delta_i) \stackrel{\pi}{\longrightarrow} P \longrightarrow 0$$

splits. Thus P is a direct summand of the graded free module $\bigoplus_i A(-\delta_i)$. By Proposition 1.2.15, P is a graded projective. □

Remark 1.2.18 <u>Graded injective modules</u>

Proposition 1.2.15 shows that an A-module P is graded projective if and only if P is graded and projective. However, the similar statement is not valid for graded injective modules. A graded A-module I is called a *graded injective module* if for any diagram of graded modules and graded A-module homomorphisms

$$\begin{array}{ccccc} 0 & \longrightarrow & N & \stackrel{g}{\longrightarrow} & M \\ & & {\scriptstyle j}\downarrow & \swarrow {\scriptstyle h} & \\ & & I & & \end{array}$$

there is a graded A-module homomorphism $h : M \to I$ with $hg = j$.

Using Lemma 1.2.14 one can show that if a graded module is injective, then it is also graded injective. However, a graded injective module is not necessarily injective. The reason for this difference between projective and injective behaviour is that the forgetful functor U : Gr-A → Mod-A is a left adjoint functor (see Remark 1.2.7). In detail, consider a graded projective module P and the diagram

$$\begin{array}{ccccc} & & P & & \\ & & \downarrow{\scriptstyle j} & & \\ M & \underset{g}{\longrightarrow} & N & \longrightarrow & 0, \end{array}$$

where M and N are A-modules. Since the diagram below is commutative

$$\begin{array}{ccc} \operatorname{Hom}_{\text{Mod-}A}(U(P), M) & \stackrel{\cong}{\longrightarrow} & \operatorname{Hom}_{\text{Gr-}A}(P, F(M)) \\ \downarrow & & \downarrow \\ \operatorname{Hom}_{\text{Mod-}A}(U(P), N) & \stackrel{\cong}{\longrightarrow} & \operatorname{Hom}_{\text{Gr-}A}(P, F(N)) \end{array}$$

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and there is a graded homomorphism $h' : P \to F(M)$ such that the diagram

$$\begin{array}{ccccc} & & P & & \\ & \overset{h'}{\swarrow} & \downarrow j' & & \\ F(M) & \xrightarrow[F(g)]{} & F(N) & \longrightarrow & 0, \end{array}$$

is commutative, there is a homomorphism $h : P \to M$ such that $gh = j$. So P is projective (see Proposition 1.2.15 for another proof).

If the grade group is finite, then the forgetful functor is right adjoint as well (see Remark 1.2.7) and a similar argument as above shows that a graded injective module is injective.

1.2.10 Graded divisible modules

Here we define the notion of graded divisible modules and we give yet another characterisation of graded von Neumann regular rings (see §1.1.9).

Let A be a Γ-graded ring and M a graded right A-module. We say $m \in M^h$ (a homogeneous element of M) is divisible by $a \in A^h$ if $m \in Ma$, *i.e.,* there is a homogeneous element $n \in M$ such that $m = na$. We say that M is a *graded divisible module* if for any $m \in M^h$ and any $a \in A^h$, where $\operatorname{ann}_r(a) \subseteq \operatorname{ann}(m)$, we have that m is divisible by a. Note that for $m \in M^h$, *the annihilator* of m,

$$\operatorname{ann}(m) := \{ a \in A \mid ma = 0 \}$$

is a graded ideal of A.

Proposition 1.2.19 *Let A be a Γ-graded ring and M a graded right A-module. Then the following are equivalent:*

(1) *M is a graded divisible module;*
(2) *for any $a \in A^h$, $\gamma \in \Gamma$, and any graded A-module homomorphism $f : aA(\gamma) \to M$, the following diagram can be completed:*

$$\begin{array}{ccccc} 0 & \longrightarrow & aA(\gamma) & \xrightarrow{\subseteq} & A(\gamma) \\ & & f \downarrow & \overset{\overline{f}}{\swarrow} & \\ & & M. & & \end{array}$$

Proof (1) $\Rightarrow$ (2) Let $f : aA(\gamma) \to M$ be a graded A-homomorphism. Set $m := f(a)$. Note that $\deg(m) = \alpha - \gamma$, where $\deg(a) = \alpha$ (see Example 1.2.1). If $x \in \operatorname{ann}_r(a)$ then $0 = f(ax) = f(a)x = mx$, thus $x \in \operatorname{ann}(m)$. Since M is graded divisible, m is divisible by a, *i.e.,* $m = na$ for some $n \in M$. It follows

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that $\deg(n) = -\gamma$. Define $\overline{f} : A(\gamma) \to M$ by $\overline{f}(1) = n$ and extend it to $A(\gamma)$. Thisis a graded A-module. Since $\overline{f}(a) = \overline{f}(1)a = na = m$, $\overline{f}$ extends f.

(2) $\Rightarrow$ (1) Let $m \in M^h$ and $a \in A^h$, where $\operatorname{ann}_r(a) \subseteq \operatorname{ann}(m)$. Suppose $\deg(a) = \alpha$ and $\deg(m) = \beta$. Let $\gamma = \alpha - \beta$ and define the map $f : aA(\gamma) \to M$, by $f(a) = m$ and extend it to $aA(\gamma)$. The following shows that f is in fact a graded A-homomorphism:

$$f\Big((aA(\gamma))_\delta\Big) \subseteq f((aA)_{\gamma+\delta}) \subseteq f(aA_{\gamma+\delta-\alpha}) = f(aA_{\delta-\beta}) = mA_{\delta-\beta} \subseteq M_\delta.$$

Thus there is an $\overline{f} : A(\gamma) \to M$ which extends f. So $m = f(a) = \overline{f}(a) = \overline{f}(1)a = na$, where $f(1) = n$. This means m is divisible by a and the proof is complete. □

Theorem 1.2.20 *Let A be a Γ-graded ring. Then A is graded von Neumann regular if and only if any graded right A-module is divisible.*

Proof Let A be a graded regular ring. Consider the exact sequence of graded right A-modules

$$0 \longrightarrow aA(\gamma) \xrightarrow{\subseteq} A(\gamma) \longrightarrow A(\gamma)/aA(\gamma) \longrightarrow 0.$$

Define

$$\begin{aligned} f : A(\gamma) &\longrightarrow aA(\gamma),\\ x &\longmapsto abx. \end{aligned}$$

Since $\deg(ab) = 0$ this gives a split graded homomorphism for the exact sequence above. Thus $aA(\gamma)$ is a direct summand of $A(\gamma)$. This shows that any graded A-module homomorphism $f : aA(\gamma) \to M$ can be extend to $A(\gamma)$.

Conversely, consider the diagram

$$\begin{array}{ccccc} 0 & \longrightarrow & aA & \xrightarrow{\subseteq} & A \\ & & {\scriptstyle\cap}\big\downarrow & \swarrow{\scriptstyle \overline{f}} & \\ & & aA. & & \end{array}$$

Since aA is divisible, there is an $\overline{f}$ which completes this diagram. Set $\overline{f}(1) = ab$, where $b \in A^h$. We then have $a = \overline{f}(a) = \overline{f}(1)a = aba$. Thus A is a graded regular ring. □

Example 1.2.21 <u>Graded rings associated with filter rings</u>

A ring A with identity is called a *filtered ring* if there is an ascending family $\{A_i \mid i \in \mathbb{Z}\}$ of additive subgroups of A such that $1 \in A_0$ and $A_iA_j \subseteq A_{i+j}$, for all

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$i, j \in \mathbb{Z}$. Let A be a filtered ring and M be a right A-module. M is called a *filtered module* if there is an ascending family $\{M_i \mid i \in \mathbb{Z}\}$ of additive subgroups of M such that $M_i A_j \subseteq M_{i+j}$, for all $i, j \in \mathbb{Z}$. An A-module homomorphism $f : M \to N$ of filtered modules M and N is called a *filtered homomorphism* if $f(M_i) \subseteq N_i$ for $i \in \mathbb{Z}$. A category consisting of filtered right A-modules for objects and filtered homomorphisms as morphisms is denoted by Filt-A. If M is a filtered A-module then

$$\mathrm{gr}(M) := \bigoplus_{i \in \mathbb{Z}} M_i / M_{i-1}$$

is a $\mathbb{Z}$-graded $\mathrm{gr}(A) := \bigoplus_{i \in \mathbb{Z}} A_i / A_{i-1}$-module. The operations here are defined naturally. This gives a functor $\mathrm{gr} : \text{Filt-}A \to \text{Gr-gr}(A)$. In Example 1.4.7 we use a variation of this construction to associate a graded division algebra with a valued division algebra.

In the theory of filtered rings, one defines the concepts of filtered free and projective modules and under certain conditions the functor gr sends these objects to the corresponding objects in the category Gr-gr(A). For a comprehensive study of filtered rings see [73].

1.3 Grading on matrices

Let A be an arbitrary ring and Γ an arbitrary group. Then one can consider Γ-gradings on the matrix ring $\mathbb{M}_n(A)$ which, at first glance, might look somewhat artificial. However, these types of gradings on matrices appear quite naturally in the graded rings arising from graphs. In this section we study the grading on matrices. We then include a section on graph algebras (including path algebras and Leavitt path algebras, §1.6). These algebras give us a wealth of examples of graded rings and graded matrix rings.

For a free right A-module V of dimension n, there is a ring isomorphism $\mathrm{End}_A(V) \cong \mathbb{M}_n(A)$. When A is a Γ-graded ring and V is a graded free module of finite rank, by Theorem 1.2.6, $\mathrm{End}_A(V)$ has a natural Γ-grading. This induces a graded structure on the matrix ring $\mathbb{M}_n(A)$. In this section we study this grading on matrices. For an n-tuple $(\delta_1, \dots, \delta_n)$, $\delta_i \in \Gamma$, we construct a grading on the matrix ring $\mathbb{M}_n(A)$, denoted by $\mathbb{M}_n(A)(\delta_1, \dots, \delta_n)$, and we show that

$$\mathrm{End}_A\left(A(-\delta_1) \oplus A(-\delta_2) \oplus \cdots \oplus A(-\delta_n)\right) \cong_{\mathrm{gr}} \mathbb{M}_n(A)(\delta_1, \dots, \delta_n).$$

We will see that these graded structures on matrices appear very naturally when studying the graded structure of path algebras in §1.6.

1.3.1 Graded calculus on matrices

Let A be a Γ-graded ring and let $M = M_1 \oplus \cdots \oplus M_n$, where M_i are graded finitely generated right A-modules. Then M is also a graded right A-module (see §1.2.6). Let

$$\big(\operatorname{Hom}(M_j, M_i)\big)_{1\le i,j\le n} := \begin{pmatrix} \operatorname{Hom}_A(M_1,M_1) & \operatorname{Hom}_A(M_2,M_1) & \cdots & \operatorname{Hom}_A(M_n,M_1) \\ \operatorname{Hom}_A(M_1,M_2) & \operatorname{Hom}_A(M_2,M_2) & \cdots & \operatorname{Hom}_A(M_n,M_2) \\ \vdots & \vdots & \ddots & \vdots \\ \operatorname{Hom}_A(M_1,M_n) & \operatorname{Hom}_A(M_2,M_n) & \cdots & \operatorname{Hom}_A(M_n,M_n) \end{pmatrix}.$$

It is easy to observe that $\big(\operatorname{Hom}(M_j, M_i)\big)_{1\le i,j\le n}$ forms a ring with componentwise addition and matrix multiplication. Moreover, for $\lambda \in \Gamma$, assigning the additive subgroup

$$\begin{pmatrix} \operatorname{Hom}_A(M_1,M_1)_\lambda & \operatorname{Hom}_A(M_2,M_1)_\lambda & \cdots & \operatorname{Hom}_A(M_n,M_1)_\lambda \\ \operatorname{Hom}_A(M_1,M_2)_\lambda & \operatorname{Hom}_A(M_2,M_2)_\lambda & \cdots & \operatorname{Hom}_A(M_n,M_2)_\lambda \\ \vdots & \vdots & \ddots & \vdots \\ \operatorname{Hom}_A(M_1,M_n)_\lambda & \operatorname{Hom}_A(M_2,M_n)_\lambda & \cdots & \operatorname{Hom}_A(M_n,M_n)_\lambda \end{pmatrix} \tag{1.41}$$

as a λ-homogeneous component of $\big(\operatorname{Hom}(M_j, M_i)\big)_{1\le i,j\le n}$, using Theorem 1.2.6 and (1.14), it follows that $\big(\operatorname{Hom}(M_j, M_i)\big)_{1\le i,j\le n}$ is a Γ-graded ring.

Let $\pi_j : M \to M_j$ and $\kappa_j : M_j \to M$ be the (graded) projection and injection homomorphisms. For the next theorem, we need the following identities:

$$\sum_{i=1}^{n} \kappa_i\pi_i = \mathrm{id}_M \quad \text{and} \quad \pi_i\kappa_j = \delta_{ij}\,\mathrm{id}_{M_j}, \tag{1.42}$$

where δ_{ij} is the Kronecker delta.

Theorem 1.3.1 *Let A be a Γ-graded ring and $M = M_1 \oplus \cdots \oplus M_n$, where M_i are graded finitely generated right A-modules. Then there is a graded ring isomorphism*

$$\Phi : \operatorname{End}_A(M) \longrightarrow \big(\operatorname{Hom}(M_j, M_i)\big)_{1\le i,j\le n}$$

defined by $f \mapsto (\pi_i f \kappa_j)$, $1 \le i, j \le n$.

Proof The map Φ is clearly well-defined. Since for $f, g \in \operatorname{End}_A(M)$,

$$\begin{aligned} \Phi(f+g) &= (\pi_i(f+g)\kappa_j)_{1\le i,j\le n} = (\pi_i f\kappa_j + \pi_i g\kappa_j)_{1\le i,j\le n} \\ &= (\pi_i f\kappa_j)_{1\le i,j\le n} + (\pi_i g\kappa_j)_{1\le i,j\le n} = \Phi(f) + \Phi(g) \end{aligned}$$

and

$$\Phi(fg) = (\pi_i f g \kappa_j)_{1\leq i,j\leq n} = \Big(\pi_i f (\sum_{l=1}^{n} \kappa_l \pi_l) g \kappa_j\Big)_{1\leq i,j\leq n}$$
$$= \Big(\sum_{l=1}^{n} (\pi_i f \kappa_l)(\pi_l g \kappa_j)\Big)_{1\leq i,j\leq n} = \Phi(f)\Phi(g),$$

Φ is a ring homomorphism. Moreover, if $f \in \operatorname{End}_A(M)_\lambda$, $\lambda \in \Gamma$, then

$$\pi_i f \kappa_j \in \operatorname{Hom}_A(M_j, M_i)_\lambda,$$

for $1 \leq i, j \leq n$. This (see (1.41)) shows that Φ is a graded ring homomorphism. Define the map

$$\Psi : \big(\operatorname{Hom}(M_j, M_i)\big)_{1\leq i,j\leq n} \longrightarrow \operatorname{End}_A(M),$$
$$(g_{ij})_{1\leq i,j\leq n} \longmapsto \sum_{1\leq i,j\leq n} \kappa_i g_{ij} \pi_j.$$

Using the identities (1.42), one can observe that the compositions $\Psi\Phi$ and $\Phi\Psi$ give the identity maps of the corresponding rings. Thus Φ is an isomorphism. □

For a graded ring A, consider $A(\delta_i)$, $1 \leq i \leq n$, as graded right A-modules and observe that

$$\Phi_{\delta_j,\delta_i} : \operatorname{Hom}_A\big(A(\delta_i), A(\delta_j)\big) \cong_{\text{gr}} A(\delta_j - \delta_i), \tag{1.43}$$

as graded left A-modules such that

$$\Phi_{\delta_k,\delta_i}(gf) = \Phi_{\delta_k,\delta_j}(g)\Phi_{\delta_j,\delta_i}(f),$$

where $f \in \operatorname{Hom}(A(\delta_i), A(\delta_j))$ and $g \in \operatorname{Hom}(A(\delta_j), A(\delta_k))$ (see (1.17)). If

$$V = A(-\delta_1) \oplus A(-\delta_2) \oplus \cdots \oplus A(-\delta_n),$$

then by Theorem 1.3.1,

$$\operatorname{End}_A(V) \cong_{\text{gr}} \Big(\operatorname{Hom}\big(A(-\delta_j), A(-\delta_i)\big)\Big)_{1\leq i,j\leq n}.$$

Applying Φ_{δ_j,δ_i} defined in (1.43) to each entry, we have

$$\operatorname{End}_A(V) \cong_{\text{gr}} \Big(\operatorname{Hom}\big(A(-\delta_j), A(-\delta_i)\big)\Big)_{1\leq i,j\leq n} \cong_{\text{gr}} \big(A(\delta_j - \delta_i)\big)_{1\leq i,j\leq n}.$$

Denoting the graded matrix ring $\big(A(\delta_j - \delta_i)\big)_{1\leq i,j\leq n}$ by $\mathbb{M}_n(A)(\delta_1, \dots, \delta_n)$, we have

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$$\mathbb{M}_n(A)(\delta_1,\dots,\delta_n) = \begin{pmatrix} A(\delta_1-\delta_1) & A(\delta_2-\delta_1) & \cdots & A(\delta_n-\delta_1) \\ A(\delta_1-\delta_2) & A(\delta_2-\delta_2) & \cdots & A(\delta_n-\delta_2) \\ \vdots & \vdots & \ddots & \vdots \\ A(\delta_1-\delta_n) & A(\delta_2-\delta_n) & \cdots & A(\delta_n-\delta_n) \end{pmatrix}. \quad (1.44)$$

By (1.41), $\mathbb{M}_n(A)(\delta_1,\dots,\delta_n)_\lambda$, the λ-homogeneous elements, are the $n \times n$-matrices over A with the degree shifted (suspended) as follows:

$$\boxed{\mathbb{M}_n(A)(\delta_1,\dots,\delta_n)_\lambda = \begin{pmatrix} A_{\lambda+\delta_1-\delta_1} & A_{\lambda+\delta_2-\delta_1} & \cdots & A_{\lambda+\delta_n-\delta_1} \\ A_{\lambda+\delta_1-\delta_2} & A_{\lambda+\delta_2-\delta_2} & \cdots & A_{\lambda+\delta_n-\delta_2} \\ \vdots & \vdots & \ddots & \vdots \\ A_{\lambda+\delta_1-\delta_n} & A_{\lambda+\delta_2-\delta_n} & \cdots & A_{\lambda+\delta_n-\delta_n} \end{pmatrix}.} \quad (1.45)$$

This also shows that for $x \in A^h$,

$$\deg(\mathbf{e}_{ij}(x)) = \deg(x) + \delta_i - \delta_j, \quad (1.46)$$

where $\mathbf{e}_{ij}(x)$ is a matrix with x in the ij-position and zero elsewhere.

In particular the zero homogeneous component (which is a ring) is of the form

$$\boxed{\mathbb{M}_n(A)(\delta_1,\dots,\delta_n)_0 = \begin{pmatrix} A_0 & A_{\delta_2-\delta_1} & \cdots & A_{\delta_n-\delta_1} \\ A_{\delta_1-\delta_2} & A_0 & \cdots & A_{\delta_n-\delta_2} \\ \vdots & \vdots & \ddots & \vdots \\ A_{\delta_1-\delta_n} & A_{\delta_2-\delta_n} & \cdots & A_0 \end{pmatrix}.} \quad (1.47)$$

Setting $\overline{\delta} = (\delta_1,\dots,\delta_n) \in \Gamma^n$, one denotes the graded matrix ring (1.44) by $\mathbb{M}_n(A)(\overline{\delta})$. To summarise, we have shown that there is a graded ring isomorphism

$$\boxed{\operatorname{End}_A\left(A(-\delta_1)\oplus A(-\delta_2)\oplus\cdots\oplus A(-\delta_n)\right) \cong_{\mathrm{gr}} \mathbb{M}_n(A)(\delta_1,\dots,\delta_n).} \quad (1.48)$$

Remark 1.3.2 MATRIX RINGS OF A NONABELIAN GROUP GRADING

If the grade group Γ is nonabelian, the homogeneous components of the matrix ring take the following form:

$$\mathbb{M}_n(A)(\delta_1\dots,\delta_n)_\lambda = \begin{pmatrix} A_{\delta_1\lambda\delta_1^{-1}} & A_{\delta_1\lambda\delta_2^{-1}} & \cdots & A_{\delta_1\lambda\delta_n^{-1}} \\ A_{\delta_2\lambda\delta_1^{-1}} & A_{\delta_2\lambda\delta_2^{-1}} & \cdots & A_{\delta_2\lambda\delta_n^{-1}} \\ \vdots & \vdots & \ddots & \vdots \\ A_{\delta_n\lambda\delta_1^{-1}} & A_{\delta_n\lambda\delta_2^{-1}} & \cdots & A_{\delta_n\lambda\delta_n^{-1}} \end{pmatrix}.$$

Consider the graded A-bimodule $A^n(\overline{\delta}) = A(\delta_1) \oplus \cdots \oplus A(\delta_n)$. Then one can check that $A^n(\overline{\delta})$ is a graded right $\mathbb{M}_n(A)(\overline{\delta})$-module and $A^n(-\overline{\delta})$ is a graded left $\mathbb{M}_n(A)(\overline{\delta})$-module, where $-\overline{\delta} = (-\delta_1, \dots, -\delta_n)$. These will be used in the graded Morita theory (see Proposition 2.1.1).

One can easily check the graded ring $R = \mathbb{M}_n(A)(\overline{\delta})$, where $\overline{\delta} = (\delta_1, \dots \delta_n)$, $\delta_i \in \Gamma$, has the support

$$\Gamma_R = \bigcup_{1 \le i,j \le n} \Gamma_A + \delta_i - \delta_j. \tag{1.49}$$

One can rearrange the shift, without changing the graded matrix ring, as the following theorem shows (see also [75, pp. 60–61]).

Theorem 1.3.3 *Let A be a Γ-graded ring and $\delta_i \in \Gamma$, $1 \le i \le n$.*

(1) *If $\alpha \in \Gamma$, and $\pi \in S_n$ is a permutation then*

$$\mathbb{M}_n(A)(\delta_1, \dots, \delta_n) \cong_{\mathrm{gr}} \mathbb{M}_n(A)(\delta_{\pi(1)} + \alpha, \dots, \delta_{\pi(n)} + \alpha). \tag{1.50}$$

(2) *If $\tau_1, \dots, \tau_n \in \Gamma_A^*$, then*

$$\mathbb{M}_n(A)(\delta_1, \dots, \delta_n) \cong_{\mathrm{gr}} \mathbb{M}_n(A)(\delta_1 + \tau_1, \dots, \delta_n + \tau_n). \tag{1.51}$$

Proof (1) Let V be a graded free module over A with a homogeneous basis $v_1, \dots, v_n$ of degree $\lambda_1, \dots, \lambda_n$, respectively. It is easy to see that ((1.20))

$$V \cong_{\mathrm{gr}} A(-\lambda_1) \oplus \cdots \oplus A(-\lambda_n),$$

and thus $\mathrm{End}_A(V) \cong_{\mathrm{gr}} \mathbb{M}_n(A)(\lambda_1, \dots, \lambda_n)$ (see (1.48)). Now let $\pi \in S_n$. Rearranging the homogeneous basis as $v_{\pi(1)}, \dots, v_{\pi(n)}$ and defining the A-graded isomorphism $\phi : V \to V$ by $\phi(v_i) = v_{\pi^{-1}(i)}$, we get a graded isomorphism in the level of endomorphism rings, called ϕ again

$$\mathbb{M}_n(A)(\lambda_1, \dots, \lambda_n) \cong_{\mathrm{gr}} \mathrm{End}_A(V) \xrightarrow{\phi} \mathrm{End}_A(V) \cong_{\mathrm{gr}} \mathbb{M}_n(A)(\lambda_{\pi(1)}, \dots, \lambda_{\pi(n)}). \tag{1.52}$$

Moreover, (1.45) shows that it does not make any difference adding a fixed $\alpha \in \Gamma$ to each of the entries in the shift. This gives us (1.50).

In fact, the isomorphism ϕ in (1.52) is defined as $\phi(M) = P_\pi M P_\pi^{-1}$, where P_π is the $n \times n$ permutation matrix with entries at $(i, \pi(i))$, $1 \le i \le n$, being 1 and zero elsewhere.

(2) For (1.51), let $\tau_i \in \Gamma_A^*$, $1 \le i \le n$, that is, $\tau_i = \deg(u_i)$ for some units $u_i \in A^h$. Consider the basis $v_i u_i$, $1 \le i \le n$ for V. With this basis,

$$\mathrm{End}_A(V) \cong_{\mathrm{gr}} \mathbb{M}_n(A)(\delta_1 + \tau_1, \dots, \delta_n + \tau_n).$$

Consider the A-graded isomorphism id : $V \to V$, by $\mathrm{id}(v_i) = (v_i u_i) u_i^{-1}$. A

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similar argument as Part (1) now gives (1.51). The isomorphism is given by $\phi(M) = P^{-1}MP$, where $P = D[u_1, \dots, u_n]$ is the diagonal matrix. □

Note that if A has a trivial Γ-grading, *i.e.,* $A = \bigoplus_{\gamma\in\Gamma} A_\gamma$, where $A_0 = A$ and $A_\gamma = 0$, for $0 \neq \gamma \in \Gamma$, this construction induces a *good grading* on $\mathbb{M}_n(A)$. By definition, this is a grading on $\mathbb{M}_n(A)$ such that the matrix unit $\mathbf{e}_{ij}$, the matrix with 1 in the ij-position and zero everywhere else, is homogeneous, for $1 \leq i, j \leq n$. This particular group gradings on matrix rings have been studied by Dăscălescu et al. [34] (see Remark 1.3.9). Therefore, for $x \in A$,

$$\deg(\mathbf{e}_{ij}(x)) = \delta_i - \delta_j. \tag{1.53}$$

One can easily check that for a ring A with trivial Γ-grading, the graded ring $\mathbb{M}_n(A)(\overline{\delta})$, where $\overline{\delta} = (\delta_1, \dots \delta_n)$, $\delta_i \in \Gamma$, has the support $\{\, \delta_i - \delta_j \mid 1 \leq i, j \leq n \,\}$. (This follows also immediately from (1.49).)

The grading on matrices appears quite naturally in the graded rings arising from graphs. We will show that the graded structure Leavitt path algebras of acyclic and comet graphs are in effect the graded matrix rings as constructed above (see §1.6, in particular, Theorems 1.6.19 and 1.6.21).

Example 1.3.4 Let A be a ring, Γ a group and A graded trivially by Γ, *i.e., A* is concentrated in degree zero (see §1.1.1). Consider the Γ-graded matrix ring

$$R = \mathbb{M}_n(A)(0, -1, \dots, -n+1),$$

where $n \in \mathbb{N}$. By (1.49) the support of R is the set $\{-n+1, -n+2, \dots, n-2, n-1\}$. By (1.45), for $k \in \mathbb{Z}$ we have the following arrangements for the homogeneous elements of R:

$$R_k = \begin{pmatrix} A_k & A_{k-1} & \dots & A_{k+1-n} \\ A_{k+1} & A_k & \dots & A_{k+2-n} \\ \vdots & \vdots & \ddots & \vdots \\ A_{k+n-1} & A_{k+n-2} & \dots & A_k \end{pmatrix}.$$

Thus the 0-component ring is

$$R_0 = \begin{pmatrix} A & 0 & \dots & 0 \\ 0 & A & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & A \end{pmatrix}$$

and

$$R_{-1} = \begin{pmatrix} 0 & 0 & \dots & 0 \\ A & 0 & \dots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \dots & A & 0 \end{pmatrix}, \dots, R_{-n+1} = \begin{pmatrix} 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ A & 0 & \dots & 0 \end{pmatrix}$$

$$R_1 = \begin{pmatrix} 0 & A & \dots & 0 \\ 0 & 0 & \ddots & \vdots \\ \vdots & \vdots & \ddots & A \\ 0 & 0 & \dots & 0 \end{pmatrix}, \dots, \quad R_{n-1} = \begin{pmatrix} 0 & 0 & \dots & A \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{pmatrix}.$$

In Chapter 2, we will see that R is graded Morita equivalent to the trivially graded ring A.

Example 1.3.5 Let S be a ring, $S[x, x^{-1}]$ the $\mathbb{Z}$-graded Laurent polynomial ring and $A = S[x^3, x^{-3}]$ the $\mathbb{Z}$-graded subring with support $3\mathbb{Z}$ (see Example 1.1.19). Consider the $\mathbb{Z}$-graded matrix ring

$$\mathbb{M}_6(A)(0, 1, 1, 2, 2, 3).$$

By (1.45), the homogeneous elements of degree 1 have the form

$$\begin{pmatrix} A_1 & A_0 & A_0 & A_{-1} & A_{-1} & A_{-2} \\ A_2 & A_1 & A_1 & A_0 & A_0 & A_{-1} \\ A_2 & A_1 & A_1 & A_0 & A_0 & A_{-1} \\ A_3 & A_2 & A_2 & A_1 & A_1 & A_0 \\ A_3 & A_2 & A_2 & A_1 & A_1 & A_0 \\ A_4 & A_3 & A_3 & A_2 & A_2 & A_1 \end{pmatrix} = \begin{pmatrix} 0 & S & S & 0 & 0 & 0 \\ 0 & 0 & 0 & S & S & 0 \\ 0 & 0 & 0 & S & S & 0 \\ Sx^3 & 0 & 0 & 0 & 0 & S \\ Sx^3 & 0 & 0 & 0 & 0 & S \\ 0 & Sx^3 & Sx^3 & 0 & 0 & 0 \end{pmatrix}.$$

Example 1.3.6 Let K be a field. Consider the $\mathbb{Z}$-graded ring

$$R = \mathbb{M}_5(K)(0, 1, 2, 2, 3).$$

Then the support of this ring is $\{0, \pm 1, \pm 2\}$ and by (1.47) the zero homogeneous component (which is a ring) is

$$R_0 = \begin{pmatrix} K & 0 & 0 & 0 & 0 \\ 0 & K & 0 & 0 & 0 \\ 0 & 0 & K & K & 0 \\ 0 & 0 & K & K & 0 \\ 0 & 0 & 0 & 0 & K \end{pmatrix} \cong K \oplus K \oplus \mathbb{M}_2(K) \oplus K.$$

Example 1.3.7 $\mathbb{M}_n(A)(\delta_1, \dots, \delta_n)$ WITH $\Gamma = \{\delta_1, \dots, \delta_n\}$ IS A SKEW GROUP RING

Let A be a Γ-graded ring, where $\Gamma = \{\delta_1, \dots, \delta_n\}$ is a finite group. Consider $\mathbb{M}_n(A)(\delta_1, \dots, \delta_n)$, which is a Γ-graded ring with its homogeneous components described by (1.45). We will show that this graded ring is the skew group ring $\mathbb{M}_n(A)_0 \star \Gamma$. In particular, by Proposition 1.1.15(3), it is a strongly graded ring. Consider the matrix $u_\alpha \in \mathbb{M}_n(A)(\delta_1, \dots, \delta_n)_\alpha$, where in each row i, we have 1 in the (i, j) position, where $\delta_j - \delta_i + \alpha = 0$, and zero everywhere else. One can easily see that u_α is a permutation matrix with exactly one 1 in each row and column. Moreover, for $\alpha, \beta \in \Gamma$, $u_\alpha u_\beta = u_{\alpha+\beta}$. Indeed, consider the ith row of u_α, with the only 1 in the jth column where $\delta_j - \delta_i + \alpha = 0$. Now, consider the jth row of u_β with a kth column such that $\delta_k - \delta_j + \beta = 0$ and so with 1 in the (j, k) row. Thus, multiplying u_α with u_β, we have zero everywhere in the ith row except in the (i, k)th position. On the other hand, since $\delta_k - \delta_i + \alpha + \beta = 0$, in the ith row of $u_{\alpha+\beta}$ we have zero except in the (i, k)th position. Repeating this argument for each row of u_α shows that $u_\alpha u_\beta = u_{\alpha+\beta}$.

Now defining $\phi : \Gamma \to \mathrm{Aut}(\mathbb{M}_n(A)_0)$ by $\phi(\alpha)(a) = u_\alpha a {u_\alpha}^{-1}$, and setting the 2-cocycle ψ trivial, by §1.1.4, $R = \mathbb{M}_n(A)_0 \star_\phi \Gamma$.

This was observed in [76], where it was proved that $\mathbb{M}_n(A)(\delta_1, \dots, \delta_n)_0$ is isomorphic to the smash product of Cohen and Montgomery [29] (see Remark 2.3.13).

Example 1.3.8 The following examples (from [34, Example 1.3]) provide two instances of $\mathbb{Z}_2$-grading on $\mathbb{M}_2(K)$, where K is a field. The first grading is a good grading, whereas the second one is not a good grading.

1 Let $R = \mathbb{M}_2(K)$ with the $\mathbb{Z}_2$-grading defined by

$$R_0 = \left\{ \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \mid a, b \in K \right\} \quad \text{and} \quad R_1 = \left\{ \begin{pmatrix} 0 & c \\ d & 0 \end{pmatrix} \mid c, d \in K \right\}.$$

Since $e_{11}, e_{22} \in R_0$ and $e_{12}, e_{21} \in R_1$, by definition, this is a good grading. Note that $R = \mathbb{M}_2(K)(0, 1)$.

2 Let $S = \mathbb{M}_2(K)$ with the $\mathbb{Z}_2$-grading defined by

$$S_0 = \left\{ \begin{pmatrix} a & b-a \\ 0 & b \end{pmatrix} \mid a, b \in K \right\} \quad \text{and} \quad S_1 = \left\{ \begin{pmatrix} d & c \\ d & -d \end{pmatrix} \mid c, d \in K \right\}.$$

Then S is a graded ring, such that the $\mathbb{Z}_2$-grading is not a good grading, since e_{11} is not homogeneous. Moreover, comparing S_0 with (1.47), shows that the grading on S does not come from the construction given by (1.44).

Consider the map

$$f : R \longrightarrow S; \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \longmapsto \begin{pmatrix} a+c & b+d-a-c \\ c & d-c \end{pmatrix}.$$

This map is in fact a graded ring isomorphism, and so $R \cong_{\mathrm{gr}} S$. This shows that the good grading is not preserved under graded isomorphisms.

Remark 1.3.9 GOOD GRADINGS ON MATRIX ALGEBRAS

Let K be a field and Γ be an abelian group. One can put a Γ-grading on the ring $\mathbb{M}_n(K)$, by assigning a degree (an element of the group Γ) to each matrix unit $\mathbf{e}_{ij}$, $1 \leq i, j \leq n$. This is called a *good grading* or an *elementary grading*. This grading has been studied in [34]. In particular it has been shown that a grading on $\mathbb{M}_n(K)$ is good if and only if it can be described as $\mathbb{M}_n(K)(\delta_1, \dots, \delta_n)$ for some $\delta_i \in \Gamma$. Moreover, any grading on $\mathbb{M}_n(K)$ is a good grading if Γ is torsion free. It has also been shown that if $R = \mathbb{M}_n(K)$ has a Γ-grading such that $\mathbf{e}_{ij}$ is a homogeneous for some $1 \leq i, j \leq n$, then there exists a good grading on $S = \mathbb{M}_n(K)$ with a graded isomorphism $R \cong S$. It is shown that if Γ is finite, then the number of good gradings on $\mathbb{M}_n(K)$ is $|\Gamma|^{n-1}$. Moreover, (for a finite Γ) the class of strongly graded and crossed product good gradings of $\mathbb{M}_n(K)$ have been classified.

Remark 1.3.10 Let A be a Γ-graded ring and Ω a subgroup of Γ. Then A can be considered as Γ/Ω-graded ring. Recall that this gives the forgetful functor $U : \mathcal{R}^{\Gamma} \to \mathcal{R}^{\Gamma/\Omega}$ (§1.1.2). Similarly, on the level of modules, one has (again) the forgetful functor $U : \mathrm{Gr}^{\Gamma}\text{-}A \to \mathrm{Gr}^{\Gamma/\Omega}\text{-}A$ (§1.2.8).

If M is a finitely generated A-module, then by Theorem 1.2.6, $\mathrm{End}(M)$ is a Γ-graded ring. One can observe that

$$U(\mathrm{End}(M)) = \mathrm{End}(U(M)).$$

In particular, $\mathbb{M}_n(A)(\delta_1, \dots, \delta_n)$ as a Γ/Ω-graded ring coincides with

$$\mathbb{M}_n(A)(\Omega + \delta_1, \dots, \Omega + \delta_n),$$

where A here in the latter case is considered as a Γ/Ω ring.

Remark 1.3.11 GRADING ON MATRIX RINGS WITH INFINITE ROWS AND COLUMNS

Let A be a Γ-graded ring and I an index set which can be uncountable. Denote by $\mathbb{M}_I(A)$ the matrix ring with entries indexed by $I \times I$, namely, $a_{ij} \in A$, where $i, j \in I$, which are all but a finite number nonzero. For $i \in I$, choose $\delta_i \in \Gamma$ and following the grading on usual matrix rings (see (1.46)) for $a \in A^h$, define

$$\deg(a_{ij}) = \deg(a) + \delta_i - \delta_j. \tag{1.54}$$

This makes $\mathbb{M}_I(A)$ a Γ-graded ring. Clearly if I is finite, then this graded ring coincides with $\mathbb{M}_n(A)(\delta_1, \dots, \delta_n)$, where $|I| = n$.

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1.3.2 Homogeneous idempotents calculus

The idempotents naturally arise in relation to the decomposition of rings and modules. The following facts about idempotents are well known in the non-graded setting and one can check that they translate into the graded setting with similar proofs (see [60, §21]). Let P_i, $1 \leq i \leq l$, be graded right ideals of A such that $A = P_1 \oplus \cdots \oplus P_l$. Then there are homogeneous orthogonal idempotents e_i (hence of degree zero) such that $1 = e_1 + \cdots + e_l$ and $e_i A = P_i$.

Let e and f be homogeneous idempotent elements in the graded ring A. (Note that, in general, there are nonhomogeneous idempotents in a graded ring.) Let $\theta : eA \to fA$ be a right A-module homomorphism. Then $\theta(e) = \theta(e^2) = \theta(e)e = fae$ for some $a \in A$ and for $b \in eA$, $\theta(b) = \theta(eb) = \theta(e)b$. This shows that there is a map

$$\begin{aligned} \operatorname{Hom}_A(eA, fA) &\to fAe, \\ \theta &\mapsto \theta(e) \end{aligned} \tag{1.55}$$

and one can easily check this is a group isomorphism. We have

$$fAe = \bigoplus_{\gamma \in \Gamma} fA_\gamma e$$

and by Theorem 1.2.6,

$$\operatorname{Hom}_A(eA, fA) \cong \bigoplus_{\gamma \in \Gamma} \operatorname{Hom}_A(eA, fA)_\gamma.$$

Then one can see that the homomorphism (1.55) respects the graded decomposition.

Now if $\theta : eA \to fA(\alpha)$, where $\alpha \in \Gamma$, is a graded A-isomorphism, then $x = \theta(e) \in fA_\alpha e$ and $y = \theta^{-1}(f) \in eA_{-\alpha} f$, where x and y are homogeneous of degrees α and $-\alpha$, respectively, such that $yx = e$ and $xy = f$.

Finally, for $f = 1$, the map (1.55) gives that

$$\operatorname{Hom}_A(eA, A) \to Ae$$

is a graded left A-module isomorphism and for $f = e$,

$$\operatorname{End}_A(eA) \to eAe$$

is a graded ring isomorphism. In particular, we have a ring isomorphism

$$\operatorname{End}_A(eA)_0 = \operatorname{End}_{\text{Gr-}A}(eA) \cong eA_0 e.$$

These facts will be used later in Theorem 5.1.3.

1.3.3 Graded matrix units

Let A be a Γ-graded ring. Modelling on the properties of the matrix units $\mathbf{e}_{ij}$, we call a set of homogeneous elements $\{ e_{ij} \in A \mid 1 \le i, j \le n \}$, a set of *graded matrix units* if

$$e_{ij} e_{kl} = \delta_{jk} e_{il}, \tag{1.56}$$

where δ_{jk} are the Kronecker deltas. Let $\deg(e_{i1}) = \delta_i$. From (1.56) it follows that $\deg(e_{ii}) = 0$, $\deg(e_{1i}) = -\delta_i$ and

$$\deg(e_{ij}) = \delta_i - \delta_j. \tag{1.57}$$

The above set is called a *full set of graded matrix units* if $\sum_{i=1}^{n} e_{ii} = 1$. If a graded ring contains a full set of graded matrix units, then the ring is of the form of a matrix ring over an appropriate graded ring (Lemma 1.3.12). We can use this to characterise the two-sided ideals of graded matrix rings (Corollary 1.3.14). For this we adopt Lam's presentation [61, §17A] to the graded setting.

Lemma 1.3.12 *Let R be a Γ-graded ring. Then $R = \mathbb{M}_n(A)(\delta_1, \dots, \delta_n)$ for some graded ring A if and only if R has a full set of graded matrix units $\{ e_{ij} \in R \mid 1 \le i, j \le n \}$.*

Proof One direction is obvious. Suppose $\{e_{ij} \in R \mid 1 \le i, j \le n\}$ is a full set of graded matrix units in R and A is its centraliser in R which is a graded subring of R. We show that R is a graded free A-module with the basis $\{e_{ij}\}$. Let $x \in R$ and set

$$a_{ij} = \sum_{k=1}^{n} e_{ki} x e_{jk} \in R.$$

Since $a_{ij} e_{uv} = e_{ui} x e_{jv} = e_{uv} a_{ij}$, it follows that $a_{ij} \in A$. Let $u = i, v = j$. Then $a_{ij} e_{ij} = e_{ii} x e_{jj}$, and since $\{e_{ij}\}$ is full, $\sum_{i,j} a_{ij} e_{ij} = \sum_{ij} e_{ii} x e_{jj} = x$. This shows that $\{e_{ij}\}$ generates R as an A-module. It is easy to see that $\{e_{ij} \mid 1 \le i, j \le n\}$ is linearly independent as well. Let $\deg(e_{i1}) = \delta_i$. Then (1.57) shows that the map

$$\begin{aligned} R &\longrightarrow \mathbb{M}_n(A)(\delta_1, \dots, \delta_n),\\ a e_{ij} &\longmapsto \mathbf{e}_{ij}(a) \end{aligned}$$

induces a graded isomorphism. □

Corollary 1.3.13 *Let A, R and S be Γ-graded rings. Suppose*

$$R = \mathbb{M}_n(A)(\delta_1, \dots, \delta_n)$$

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and there is a graded ring homomorphism $f : R \to S$. Then

$$S = \mathbb{M}_n(B)(\delta_1, \dots, \delta_n),$$

for a graded ring B and f is induced by a graded homomorphism $f_0 : A \to B$.

Proof Consider the standard full graded matrix units $\{\mathbf{e}_{ij} \mid 1 \leq i, j \leq n\}$ in R. Then $\{f(\mathbf{e}_{ij})\}$ is a set of full graded matrix units in S. Since f is a graded homomorphism, $\deg(f(\mathbf{e}_{ij})) = \deg(e_{ij}) = \delta_i - \delta_j$. Let B be the centraliser of this set in S. By Lemma 1.3.12 (and its proof),

$$S = \mathbb{M}_n(B)(0, \delta_2 - \delta_1, \dots, \delta_n - \delta_1) = \mathbb{M}_n(B)(\delta_1, \delta_2, \dots, \delta_n).$$

Since A is the centraliser of $\{\mathbf{e}_{ij}\}$, f sends A to B and thus induces the map on the matrix algebras. □

Corollary 1.3.14 *Let A be a Γ-graded ring, $R = \mathbb{M}_n(A)(\delta_1, \dots, \delta_n)$ and I be a graded ideal of R. Then $I = \mathbb{M}_n(I_0)(\delta_1, \dots, \delta_n)$, where I_0 is a graded ideal of A.*

Proof Consider the canonical graded quotient homomorphism $f : R \to R/I$. Set $I_0 = \ker(f|_A)$. One can easily see $\mathbb{M}_n(I_0)(\delta_1, \dots, \delta_n) \subseteq I$. By Lemma 1.3.12, $R/I = \mathbb{M}_n(B)(\delta_1, \dots, \delta_n)$, where B is the centraliser of the set $\{f(\mathbf{e}_{ij})\}$. Since A is the centraliser of $\{\mathbf{e}_{ij}\}$, $f(A) \subseteq B$. Now for $x \in I$, write $x = \sum_{i,j} a_{ij}\mathbf{e}_{ij}$, $a_{ij} \in A$. Then $0 = f(x) = \sum_{i,j} f(a_{ij})f(\mathbf{e}_{ij})$, which implies $f(a_{ij}) = 0$ as $f(\mathbf{e}_{ij})$ are linear independent (see the proof of Lemma 1.3.12). Thus $a_{i,j} \in I_0$. This shows $I \subseteq \mathbb{M}_n(I_0)(\delta_1, \dots, \delta_n)$, which finishes the proof. □

Corollary 1.3.14 shows that there is a one-to-one inclusion preserving correspondence between the graded ideals of A and the graded ideals of $\mathbb{M}_n(A)(\overline{\delta})$, where $\overline{\delta} = (\delta_1, \dots, \delta_n)$.

1.3.4 Mixed shift

For a Γ-graded ring A, $\overline{\alpha} = (\alpha_1, \dots \alpha_m) \in \Gamma^m$ and $\overline{\delta} = (\delta_1, \dots, \delta_n) \in \Gamma^n$, set

$$\boxed{\mathbb{M}_{m\times n}(A)[\overline{\alpha}][\overline{\delta}] := \begin{pmatrix} A_{\alpha_1-\delta_1} & A_{\alpha_1-\delta_2} & \cdots & A_{\alpha_1-\delta_n} \\ A_{\alpha_2-\delta_1} & A_{\alpha_2-\delta_2} & \cdots & A_{\alpha_2-\delta_n} \\ \vdots & \vdots & \ddots & \vdots \\ A_{\alpha_m-\delta_1} & A_{\alpha_m-\delta_2} & \cdots & A_{\alpha_m-\delta_n} \end{pmatrix}.}$$

So $\mathbb{M}_{m\times n}(A)[\overline{\alpha}][\overline{\delta}]$ consists of matrices with the ij-entry in $A_{\alpha_i-\delta_j}$.

https://doi.org/10.1017/CBO9781316717134.002 Published online by Cambridge University Press

If $a \in \mathbb{M}_{m\times n}(A)[\overline{\alpha}][\overline{\delta}]$, then one can easily check that multiplying a from the left induces a graded right A-module homomorphism

$$\phi_a : A^n(\overline{\delta}) \longrightarrow A^m(\overline{\alpha}), \tag{1.58}$$
$$\begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} \longmapsto a \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}.$$

Conversely, suppose $\phi : A^n(\overline{\delta}) \to A^m(\overline{\alpha})$ is graded right A-module homomorphism. Let e_j denote the standard basis element of $A^n(\overline{\delta})$ with 1 in the j-th entry and zeros elsewhere. Let $\phi(e_j) = (a_{1j}, a_{2j}, \dots, a_{mj})$, $1 \leq j \leq n$. Since ϕ is a graded map, comparing the grading of both sides, one can observe that $\deg(a_{ij}) = \alpha_i - \delta_j$. So that the map ϕ is represented by the left multiplication with the matrix $a = (a_{ij})_{m\times n} \in \mathbb{M}_{m\times n}(A)[\overline{\alpha}][\overline{\delta}]$.

In particular $\mathbb{M}_{m\times m}(A)[\overline{\alpha}][\overline{\alpha}]$ represents $\operatorname{End}(A^m(\alpha), A^m(\alpha))_0$. Combining this with (1.48), we get

$$\mathbb{M}_{m\times m}(A)[\overline{\alpha}][\overline{\alpha}] = \mathbb{M}_m(A)(-\overline{\alpha})_0. \tag{1.59}$$

The mixed shift will be used in §3.2 to describe graded Grothendieck groups by idempotent matrices. The following simple lemma comes in handy.

Lemma 1.3.15 *Let $a \in \mathbb{M}_{m\times n}(A)[\overline{\alpha}][\overline{\delta}]$ and $b \in \mathbb{M}_{n\times k}(A)[\overline{\delta}][\overline{\beta}]$. Then $ab \in \mathbb{M}_{m\times k}(A)[\overline{\alpha}][\overline{\beta}]$.*

Proof Let $\phi_a : A^n(\overline{\delta}) \to A^m(\overline{\alpha})$ and $\phi_b : A^k(\overline{\beta}) \to A^n(\overline{\delta})$ be the graded right A-module homomorphisms induced by multiplications with a and b, respectively (see 1.58). Then

$$\phi_{ab} = \phi_a\phi_b : A^k(\overline{\beta}) \longrightarrow A^m(\overline{\alpha}).$$

This shows that $ab \in \mathbb{M}_{m\times k}(A)[\overline{\alpha}][\overline{\beta}]$. (This can also be checked directly, by multiplying the matrices a and b and taking into account the shift arrangements.) □

Proposition 1.3.16 *Let A be a Γ-graded ring and let $\overline{\alpha} = (\alpha_1, \dots, \alpha_m) \in \Gamma^m$, $\overline{\delta} = (\delta_1, \dots, \delta_n) \in \Gamma^n$. Then the following are equivalent:*

(1) *$A^m(\overline{\alpha}) \cong_{\text{gr}} A^n(\overline{\delta})$ as graded right A-modules;*
(2) *$A^m(-\overline{\alpha}) \cong_{\text{gr}} A^n(-\overline{\delta})$ as graded left A-modules;*
(3) *there exist $a = (a_{ij}) \in \mathbb{M}_{n\times m}(A)[\overline{\delta}][\overline{\alpha}]$ and $b = (b_{ij}) \in \mathbb{M}_{m\times n}(A)[\overline{\alpha}][\overline{\delta}]$ such that $ab = \mathbb{I}_n$ and $ba = \mathbb{I}_m$.*

Proof (1) ⇒ (3) Let $\phi : A^m(\overline{\alpha}) \to A^n(\overline{\delta})$ and $\psi : A^n(\overline{\delta}) \to A^m(\overline{\alpha})$ be graded right A-module isomorphisms such that $\phi\psi = 1$ and $\psi\phi = 1$. The paragraph prior to Lemma 1.3.15 shows that the map ϕ is represented by the left multiplication with a matrix $a = (a_{ij})_{n\times m} \in \mathbb{M}_{n\times m}(A)[\overline{\delta}][\overline{\alpha}]$. In the same way one can construct $b \in \mathbb{M}_{m\times n}(A)[\overline{\alpha}][\overline{\delta}]$ which induces ψ. Now $\phi\psi = 1$ and $\psi\phi = 1$ translate to $ab = \mathbb{I}_n$ and $ba = \mathbb{I}_m$.

(3) ⇒ (1) If $a \in \mathbb{M}_{n\times m}(A)[\overline{\delta}][\overline{\alpha}]$, then multiplication from the left induces a graded right A-module homomorphism $\phi_a : A^m(\overline{\alpha}) \longrightarrow A^n(\overline{\delta})$. Similarly b induces $\psi_b : A^n(\overline{\delta}) \longrightarrow A^m(\overline{\alpha})$. Now $ab = \mathbb{I}_n$ and $ba = \mathbb{I}_m$ translate to $\phi_a\psi_b = 1$ and $\psi_b\phi_a = 1$.

(2) ⟺ (3) This part is similar to the previous cases by considering the matrix multiplication from the right. Specifically, the graded left A-module homomorphism $\phi : A^m(-\overline{\alpha}) \to A^n(-\overline{\delta})$ represented by a matrix multiplication from the right of the form $\mathbb{M}_{m\times n}(A)[\overline{\alpha}][\overline{\delta}]$ and similarly ψ gives a matrix in $\mathbb{M}_{n\times m}(A)[\overline{\delta}][\overline{\alpha}]$. The rest follows easily. □

The following corollary shows that $A(\alpha) \cong_{\mathrm{gr}} A$ as graded right A-modules if and only if $\alpha \in \Gamma_A^*$. In fact, replacing $m = n = 1$ in Proposition 1.3.16 we obtain the following.

Corollary 1.3.17 *Let A be a Γ-graded ring and $\alpha \in \Gamma$. Then the following are equivalent:*

(1) *$A(\alpha) \cong_{\mathrm{gr}} A$ as graded right A-modules;*
(2) *$A(-\alpha) \cong_{\mathrm{gr}} A$ as graded right A-modules;*
(3) *$A(\alpha) \cong_{\mathrm{gr}} A$ as graded left A-modules;*
(4) *$A(-\alpha) \cong_{\mathrm{gr}} A$ as graded left A-modules;*
(5) *there is an invertible homogeneous element of degree α;*
(6) *there is an invertible homogeneous element of degree $-\alpha$.*

Proof This follows from Proposition 1.3.16. □

Corollary 1.3.18 *Let A be a Γ-graded ring. Then the following are equivalent:*

(1) *A is crossed product;*
(2) *$A(\alpha) \cong_{\mathrm{gr}} A$, as graded right A-modules, for all $\alpha \in \Gamma$;*
(3) *$A(\alpha) \cong_{\mathrm{gr}} A$, as graded left A-modules, for all $\alpha \in \Gamma$;*
(4) *the shift functor $\mathcal{T}_\alpha$: Gr-$A \to$ Gr-A is isomorphic to identity functor, for all $\alpha \in \Gamma$.*

Proof This follows from Corollary 1.3.17, (1.22) and the definition of the crossed product rings (§1.1.3). □

The Corollary above will be used to show that the action of Γ on the graded Grothendieck group of a crossed product algebra is trivial (see Example 3.1.9).

Example 1.3.19 THE LEAVITT ALGEBRA $\mathcal{L}(1, n)$

In [63] Leavitt considered the free associative ring A with coefficient in $\mathbb{Z}$ generated by symbols $\{x_i, y_i \mid 1 \leq i \leq n\}$ subject to the relations

$$\begin{aligned} x_i y_j &= \delta_{ij}, \text{ for all } 1 \leq i, j \leq n, \\ \sum_{i=1}^{n} y_i x_i &= 1, \end{aligned} \tag{1.60}$$

where $n \geq 2$ and δ_{ij} is the Kronecker delta. The relations guarantee the right A-module homomorphism

$$\begin{aligned} \phi : A &\longrightarrow A^n \\ a &\mapsto (x_1 a, x_2 a, \dots, x_n a) \end{aligned} \tag{1.61}$$

has an inverse

$$\begin{aligned} \psi : A^n &\longrightarrow A \\ (a_1, \dots, a_n) &\mapsto y_1 a_1 + \cdots + y_n a_n, \end{aligned} \tag{1.62}$$

so $A \cong A^n$ as right A-modules. He showed that A is universal with respect to this property, of type $(1, n-1)$ (see §1.7) and it is a simple ring.

Leavitt's algebra constructed in (1.60) has a natural grading; assigning 1 to y_i and -1 to x_i, $1 \leq i \leq n$, since the relations are homogeneous (of degree zero), the ring A is a $\mathbb{Z}$-graded ring (see §1.6.1 for a general construction of graded rings from free algebras). The isomorphism (1.61) induces a graded isomorphism

$$\begin{aligned} \phi : A &\longrightarrow A(-1)^n \\ a &\longmapsto (x_1 a, x_2 a, \dots, x_n a), \end{aligned} \tag{1.63}$$

where $A(-1)$ is the suspension of A by -1. In fact, letting

$$y = (y_1, \dots, y_n) \text{ and}$$

$$x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix},$$

we have $y \in \mathbb{M}_{1\times n}(A)[\overline{\alpha}][\overline{\delta}]$ and $x \in \mathbb{M}_{n\times 1}(A)[\overline{\delta}][\overline{\alpha}]$, where $\overline{\alpha} = (0)$ and $\overline{\delta} = (-1, \dots, -1)$. Thus by Proposition 1.3.16, $A \cong_{\text{gr}} A(-1)^n$.

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Motivated by this algebra, the Leavitt path algebras were introduced in [2, 5], which associate with a direct graph a certain algebra. When the graph has one vertex and n loops, the algebra corresponds to this graph is the Leavitt algebra constructed in (1.60) and is denoted by $\mathcal{L}(1, n)$ or $\mathcal{L}_n$. The Leavitt path algebras will provide a vast array of examples of graded algebras. We will study these algebras in §1.6.4.

1.4 Graded division rings

Graded fields and their noncommutative version, *i.e.,* graded division rings, are among the simplest graded rings. With a little effort, we can completely compute the invariants of these algebras which we are interested in, namely, the graded Grothendieck groups (§3.7) and the graded Picard groups (Chapter 4).

Recall from §1.1.4 that a Γ-graded ring $A = \bigoplus_{\gamma \in \Gamma} A_\gamma$ is called a graded division ring if every nonzero homogeneous element has a multiplicative inverse. Throughout this section we consider graded right modules over graded division rings. Note that we work with the abelian grade groups, however, all the results are valid for nonabelian grading as well. We first show that for graded modules over a graded division ring, there is well-defined notion of dimension. The proofs follow the standard proofs in the nongraded setting (see [50, §IV, Theorem 2.4, 2.7, 2.13]), or the graded setting (see [75, Proposition 4.6.1], [90, Chapter 2]).

Proposition 1.4.1 *Let A be a Γ-graded ring. Then A is a graded division ring if and only if any graded A-module is graded free. If M is a graded module over graded division ring A, then any linearly independent subset of M consisting of homogeneous elements can be extended to a homogeneous basis of M.*

Proof Suppose any graded (right) module is graded free. Let I be a right ideal of A. Consider A/I as a right A-module, which is graded free by assumption. Thus $I = \operatorname{ann}(A/I) = 0$. This shows that the only graded right ideal of A is the zero ideal. This gives that A is a graded division ring.

For the converse, note that if A is a graded division ring (*i.e.,* all homogeneous elements are invertible), then for any $m \in M^h$, $\{m\}$ is a linearly independent subset of M. This immediately gives the converse of the statement of the theorem as a consequence of the second part of the theorem.

Fix a linearly independent subset X of M consisting of homogeneous elements. Let

$$F = \{Q \subseteq M^h \mid X \subseteq Q \text{ and } Q \text{ is } A\text{-linearly independent}\}.$$

This is a nonempty partially ordered set with inclusion, and every chain $Q_1 \subseteq Q_2 \subseteq \dots$ in F has an upper bound $\bigcup Q_i \in F$. By Zorn's lemma, F has a maximal element, which we denote by P. If $\langle P \rangle \neq M$, then there is a homogeneous element $m \in M^h \setminus \langle P \rangle$. We will show that $P \cup \{m\}$ is a linearly independent set containing X, contradicting the maximality of P.

Suppose $ma + \sum p_i a_i = 0$, where $a, a_i \in A$, $p_i \in P$ with $a \neq 0$. Then there is a homogeneous component of a, say a_λ, which is also nonzero. Considering the $\lambda + \deg(m)$-homogeneous component of this sum, we have

$$m = ma_\lambda a_\lambda^{-1} = -\sum p_i a'_i a_\lambda^{-1}$$

for a'_i homogeneous, which contradicts the assumption $m \in M^h \setminus \langle P \rangle$. Hence $a = 0$, which implies each $a_i = 0$. This gives the required contradiction, so $M = \langle P \rangle$, completing the proof. □

The following proposition shows in particular that a graded division ring has graded invariant basis number (we discuss this type of ring in §1.7).

Proposition 1.4.2 *Let A be a Γ-graded division ring and M be a graded A-module. Then any two homogeneous bases of M over A have the same cardinality.*

Proof By [50, §IV, Theorem 2.6], if a module M has an infinite basis over a ring, then any other basis of M has the same cardinality. This proves the proposition in the case where the homogeneous basis is infinite.

Now suppose that M has two finite homogeneous bases X and Y. Then $X = \{x_1, \dots, x_n\}$ and $Y = \{y_1, \dots, y_m\}$, for $x_i, y_i \in M^h \setminus 0$. As X is a basis for M, we can write

$$y_m = x_1 a_1 + \cdots + x_n a_n,$$

for some $a_i \in A^h$, where $\deg(y_m) = \deg(a_i) + \deg(x_i)$ for each $1 \leq i \leq n$. Since $y_m \neq 0$, we have at least one $a_i \neq 0$. Let a_k be the first nonzero a_i, and we note that a_k is invertible as it is nonzero and homogeneous in A. Then

$$x_k = y_m a_k^{-1} - x_{k+1} a_{k+1} a_k^{-1} - \cdots - x_n a_n a_k^{-1},$$

and the set $X' = \{ y_m, x_1, \dots, x_{k-1}, x_{k+1}, \dots, x_n \}$ spans M since X spans M. So

$$y_{m-1} = y_m b_m + x_1 c_1 + \cdots + x_{k-1} c_{k-1} + x_{k+1} c_{k+1} + \cdots + x_n c_n,$$

for $b_m, c_i \in A^h$. There is at least one nonzero c_i, since if all the c_i are zero, then either y_m and y_{m-1} are linearly dependent or y_{m-1} is zero, which are not the case. Let c_j denote the first nonzero c_i. Then x_j can be written as a linear combination of y_{m-1}, y_m and those x_i with $i \neq j, k$. Therefore the set

$$X'' = \{ y_{m-1}, y_m \} \cup \{ x_i : i \neq j, k \}$$

spans M since X' spans M.

Continuing this process of adding a y and removing an x gives, after the kth step, a set which spans M consisting of $y_m, y_{m-1}, \ldots, y_{m-k+1}$ and $n-k$ of the x_i. If $n < m$, then after the nth step, we would have that the set $\{y_m, \ldots, y_{m-n+1}\}$ spans M. But if $n < m$, then $m - n + 1 \geq 2$, so this set does not contain y_1, and therefore y_1 can be written as a linear combination of the elements of this set. This contradicts the linear independence of Y, so we must have $m \leq n$. Repeating a similar argument with X and Y interchanged gives $n \leq m$, so $n = m$. □

The Propositions 1.4.1 and 1.4.2 above show that for a graded module M over a graded division ring A, M has a homogeneous basis and any two homogeneous bases of M have the same cardinality. The cardinal number of any homogeneous basis of M is called the *dimension* of M over A, and it is denoted by $\dim_A(M)$ or $[M : A]$.

Proposition 1.4.3 *Let A be a Γ-graded division ring and M be a graded A-module. If N is a graded submodule of M, then*

$$\dim_A(N) + \dim_A(M/N) = \dim_A(M).$$

Proof By Proposition 1.4.1, the submodule N is a graded free A-module with a homogeneous basis Y which can be extended to a homogeneous basis X of M. We will show that $U = \{x + N \mid x \in X \setminus Y\}$ is a homogeneous basis of M/N. Note that by (1.12), U consists of homogeneous elements. Let $t \in (M/N)^h$. Again by (1.12), $t = m + N$, where $m \in M^h$ and $m = \sum x_i a_i + \sum y_j b_j$ where a_i, $b_j \in A$, $y_j \in Y$ and $x_i \in X \setminus Y$. So $m + N = \sum (x_i + N)a_i$, which shows that U spans M/N. If $\sum (x_i + N)a_i = 0$, for $a_i \in A$, $x_i \in X \setminus Y$, then $\sum x_i a_i \in N$ which implies that $\sum x_i a_i = \sum y_k b_k$ for $b_k \in A$ and $y_k \in Y$, which implies that $a_i = 0$ and $b_k = 0$ for all i, k. Therefore U is a homogeneous basis for M/N and as we can construct a bijective map $X \setminus Y \to U$, we have $|U| = |X \setminus Y|$. Then

$$\dim_A M = |X| = |Y| + |X \setminus Y| = |Y| + |U| = \dim_A N + \dim_A(M/N). \qquad \square$$

The following statement is the graded version of a similar statement on simple rings (see [50, §IX.1]). This is required for the proof of Theorem 1.4.5.

Proposition 1.4.4 *Let A and B be Γ-graded division rings. If*

$$\mathbb{M}_n(A)(\lambda_1, \ldots, \lambda_n) \cong_{\mathrm{gr}} \mathbb{M}_m(B)(\gamma_1, \ldots, \gamma_m)$$

as graded rings, where $\lambda_i, \gamma_j \in \Gamma$, $1 \le i \le n$, $1 \le j \le m$, *then* $n = m$ *and* $A \cong_{\mathrm{gr}} B$.

Proof The proof follows the nongraded case (see [50, §IX.1]) with an extra attention given to the grading. We refer the reader to [71, §4.3] for a detailed proof. □

We can further determine the relations between the shift $(\lambda_1, \dots, \lambda_n)$ and $(\gamma_1, \dots, \gamma_m)$ in the above proposition. For this we need to extend [27, Theorem 2.1] (see also [75, Theorem 9.2.18]) from fields (with trivial grading) to graded division algebras. The following theorem states that two graded matrix algebras over a graded division ring with two shifts are isomorphic if and only if one can obtain one shift from the other by applying (1.50) and (1.51).

Theorem 1.4.5 *Let* A *be a* Γ*-graded division ring. Then for* $\lambda_i, \gamma_j \in \Gamma$, $1 \le i \le n$, $1 \le j \le m$,

$$\mathbb{M}_n(A)(\lambda_1, \dots, \lambda_n) \cong_{\mathrm{gr}} \mathbb{M}_m(A)(\gamma_1, \dots, \gamma_m) \tag{1.64}$$

if and only if $n = m$ *and for a suitable permutation* $\pi \in S_n$, *we have* $\lambda_i = \gamma_{\pi(i)} + \tau_i + \sigma$, $1 \le i \le n$, *where* $\tau_i \in \Gamma_A$ *and a fixed* $\sigma \in \Gamma$, i.e., $(\lambda_1, \dots, \lambda_n)$ *is obtained from* $(\gamma_1, \dots, \gamma_m)$ *by applying* (1.50) *and* (1.51).

Proof One direction is Theorem 1.3.3, noting that since A is a graded division ring, $\Gamma_A = \Gamma_A^*$.

We now prove the converse. That $n = m$ follows from Proposition 1.4.4. By (1.4.1) one can find $\epsilon = (\varepsilon_1, \dots, \varepsilon_1, \varepsilon_2, \dots, \varepsilon_2, \dots, \varepsilon_k, \dots, \varepsilon_k)$ in Γ such that $\mathbb{M}_n(A)(\lambda_1, \dots, \lambda_n) \cong_{\mathrm{gr}} \mathbb{M}_n(A)(\epsilon)$ as in (1.68). Now set

$$V = A(-\varepsilon_1) \times \cdots \times A(-\varepsilon_1) \times \cdots \times A(-\varepsilon_k) \times \cdots \times A(-\varepsilon_k)$$

and pick the (standard) homogeneous basis e_i, $1 \le i \le n$ and define $E_{ij} \in \operatorname{End}_A(V)$ by $E_{ij}(e_t) = \delta_{j,t} e_i$, $1 \le i, j, t \le n$. One can easily see that E_{ij} is a A-graded homomorphism of degree $\varepsilon_{s_i} - \varepsilon_{s_j}$ where ε_{s_i} and ε_{s_j} are ith and jth elements in ϵ. Moreover, $\operatorname{End}_A(V) \cong_{\mathrm{gr}} \mathbb{M}_n(A)(\epsilon)$ and E_{ij} corresponds to the matrix e_{ij} in $\mathbb{M}_n(A)(\epsilon)$. In a similar manner, one can find

$$\epsilon' = (\varepsilon'_1, \dots, \varepsilon'_1, \varepsilon'_2, \dots, \varepsilon'_2, \dots, \varepsilon'_{k'}, \dots, \varepsilon'_{k'})$$

and a graded A-vector space W such that

$$\mathbb{M}_n(A)(\gamma_1, \dots, \gamma_n) \cong_{\mathrm{gr}} \mathbb{M}_n(A)(\epsilon'),$$

and $\operatorname{End}_A(W) \cong_{\mathrm{gr}} \mathbb{M}_n(A)(\epsilon')$. Therefore (1.64) provides a graded ring isomorphism

$$\theta : \operatorname{End}_A(V) \to \operatorname{End}_A(W).$$

Define $E'_{ij} := \theta(E_{ij})$ and $E'_{ii}(W) = Q_i$, for $1 \leq i, j \leq n$. Since $\{E_{ii} \mid 1 \leq i \leq n\}$ is a complete system of orthogonal idempotents, so is $\{E'_{ii} \mid 1 \leq i \leq n\}$. It follows that

$$W \cong_{\text{gr}} \bigoplus_{1 \leq j \leq n} Q_j.$$

Moreover, $E'_{ij}E'_{tr} = \delta_{j,t}E'_{ir}$ and E'_{ii} acts as identity on Q_i. These relations show that restricting E'_{ij} on Q_j induces an A-graded isomorphism $E'_{ij} : Q_j \to Q_i$ of degree $\varepsilon_{s_i} - \varepsilon_{s_j}$ (same degree as E_{ij}). So $Q_j \cong_{\text{gr}} Q_1(\varepsilon_{s_1} - \varepsilon_{s_j})$ for any $1 \leq j \leq n$. Therefore

$$W \cong_{\text{gr}} \bigoplus_{1 \leq j \leq n} Q_1(\varepsilon_{s_1} - \varepsilon_{s_j}).$$

By dimension count (see Proposition 1.4.3), it follows that $\dim_A Q_1 = 1$.

A similar argument for the identity map $\text{id} : \text{End}_A(V) \to \text{End}_A(V)$ produces

$$V \cong_{\text{gr}} \bigoplus_{1 \leq j \leq n} P_1(\varepsilon_{s_1} - \varepsilon_{s_j}),$$

where $P_1 = E_{11}(V)$, with $\dim_A P_1 = 1$.

Since P_1 and Q_1 are A-graded vector spaces of dimension 1, there is $\sigma \in \Gamma$, such that $Q_1 \cong_{\text{gr}} P_1(\sigma)$. Using the fact that for an A-graded module P and $\alpha, \beta \in \Gamma$, $P(\alpha)(\beta) = P(\alpha + \beta) = P(\beta)(\alpha)$, we have

$$\begin{aligned} W \cong_{\text{gr}} \bigoplus_{1 \leq j \leq n} Q_1(\varepsilon_{s_1} - \varepsilon_{s_j}) &\cong_{\text{gr}} \bigoplus_{1 \leq j \leq n} P_1(\sigma)(\varepsilon_{s_1} - \varepsilon_{s_j}) \\ &\cong_{\text{gr}} \bigoplus_{1 \leq j \leq n} P_1(\varepsilon_{s_1} - \varepsilon_{s_j})(\sigma) \cong_{\text{gr}} V(\sigma). \quad (1.65) \end{aligned}$$

We denote this A-graded isomorphism with $\phi : W \to V(\sigma)$. Let e'_i, $1 \leq i \leq n$ be a (standard) homogeneous basis of degree ε'_{s_i} in W. Then $\phi(e'_i) = \sum_{1 \leq j \leq n} e_j a_j$, where $a_j \in A^h$ and e_j are homogeneous of degree $\varepsilon_{s_j} - \sigma$ in $V(\sigma)$. Since $\deg(\phi(e'_i)) = \varepsilon'_{s_i}$, any e_j with nonzero a_j in the sum has the same degree. For if $\varepsilon_{s_j} - \sigma = \deg(e_j) \neq \deg(e_l) = \varepsilon_{s_l} - \sigma$, then since $\deg(e_j a_j) = \deg(e_l a_l) = \varepsilon'_{s_i}$ it follows that $\varepsilon_{s_j} - \varepsilon_{s_l} \in \Gamma_A$ which is a contradiction as $\Gamma_A + \varepsilon_{s_j}$ and $\Gamma_A + \varepsilon_{s_l}$ are distinct. Thus $\varepsilon'_{s_i} = \varepsilon_{s_j} + \tau_j - \sigma$, where $\tau_j = \deg(a_j) \in \Gamma_A$. In the same manner one can show that $\varepsilon'_{s_i} = \varepsilon'_{s_{i'}}$ in ϵ' if and only if ε_{s_j} and $\varepsilon_{s_{j'}}$ assigned to them by the previous argument are also equal. This shows that ϵ' can be obtained from ϵ by applying (1.50) and (1.51). Since ϵ' and ϵ are also obtained from $\gamma_1, \ldots, \gamma_n$ and $\lambda_1, \ldots, \lambda_n$, respectively, by applying (1.50) and (1.51), putting these together shows that $\lambda_1, \ldots, \lambda_n$ and $\gamma_1, \ldots, \gamma_n$ have similar relations, *i.e.*, $\lambda_i = \gamma_{\pi(i)} + \tau_i + \sigma$, $1 \leq i \leq n$, where $\tau_i \in \Gamma_A$ and a fixed $\sigma \in \Gamma$. □

A *graded division algebra A* is defined to be a graded division ring with centre R such that $[A : R] < \infty$. Note that since R is a graded field, by Propositions 1.4.1 and 1.4.2, A has a well-defined dimension over R. A graded division algebra A over its centre R is said to be *unramified* if $\Gamma_A = \Gamma_R$ and *totally ramified* if $A_0 = R_0$.

Let A be a graded division ring and let R be a graded subfield of A which is contained in the centre of A. It is clear that $R_0 = R \cap A_0$ is a field and A_0 is a division ring. The group of invertible homogeneous elements of A is denoted by A^{h*}, which is equal to $A^h \setminus 0$. Considering A as a graded R-module, since R is a graded field, there is a uniquely defined dimension $[A : R]$ by Theorem 1.4.1. The proposition below has been proven in [90, Chapter 5] for two graded fields $R \subseteq S$ with a torsion-free abelian grade group.

Proposition 1.4.6 *Let A be a graded division ring and let R be a graded subfield of A which is contained in the centre of A. Then*

$$[A : R] = [A_0 : R_0]|\Gamma_A : \Gamma_R|.$$

Proof Since A is a graded division ring, A_0 is a division ring. Moreover, R_0 is a field. Let $\{x_i\}_{i\in I}$ be a basis for A_0 over R_0. Consider the cosets of Γ_A over Γ_R and let $\{\delta_j\}_{j\in J}$ be a coset representative, where $\delta_j \in \Gamma_A$. Take $\{y_j\}_{j\in J} \subseteq A^{h*}$ such that $\deg(y_j) = \delta_j$ for each j. We will show that $\{x_i y_j\}$ is a basis for A over F.

Consider the map

$$\begin{aligned}\psi : A^{h*} &\longrightarrow \Gamma_A/\Gamma_R,\\ a &\longmapsto \deg(a) + \Gamma_R.\end{aligned}$$

This is a group homomorphism with kernel $A_0 R^{h*}$, since for any $a \in \ker(\psi)$ there is some $r \in R^{h*}$ with $ar^{-1} \in A_0$. For $a \in A$, $a = \sum_{\gamma\in\Gamma} a_\gamma$, where $a_\gamma \in A_\gamma$ and $\psi(a_\gamma) = \gamma + \Gamma_R = \delta_j + \Gamma_F$ for some δ_j in the coset representative of Γ_A over Γ_R. Then there is some y_j with $\deg(y_j) = \delta_j$ and $a_\gamma y_j^{-1} \in \ker(\psi) = A_0 R^{h*}$. So

$$a_\gamma y_j^{-1} = (\sum_i x_i r_i) g$$

for $g \in R^{h*}$ and $r_i \in R_0$. Since R is in the centre of A, it follows that $a_\gamma = \sum_i x_i y_j r_i g$. So a can be written as an R-linear combination of the elements of $\{x_i y_j\}$.

To show linear independence, suppose

$$\sum_{i=1}^{n} x_i y_i r_i = 0, \tag{1.66}$$

for $r_i \in R$. Write r_i as the sum of its homogeneous components, and then consider a homogeneous component of the sum (1.66), say $\sum_{k=1}^{m} x_k y_k r'_k$, where $\deg(x_k y_k r'_k) = \alpha$. Since $x_k \in A_0$, $\deg(r'_k) + \deg(y_k) = \alpha$ for all k, so all of the y_k

are the same. This implies that $\sum_k x_k r'_k = 0$, where all of the r'_k have the same degree. If $r'_k = 0$ for all k then $r_i = 0$ for all i we are done. Otherwise, for some $r'_l \neq 0$, we have $\sum_k x_k(r'_k {r'}_l^{-1}) = 0$. Since $\{x_i\}$ forms a basis for A_0 over R_0, this implies $r'_k = 0$ for all k and thus $r_i = 0$ for all $1 \leq i \leq n$. □

Example 1.4.7 GRADED DIVISION ALGEBRAS FROM VALUED DIVISION ALGEBRAS

Let D be a division algebra with a valuation. To this one associates a graded division algebra $\operatorname{gr}(D) = \bigoplus_{\gamma \in \Gamma_D} \operatorname{gr}(D)_\gamma$, where Γ_D is the value group of D and the summands $\operatorname{gr}(D)_\gamma$ arise from the filtration on D induced by the valuation (see details below and also Example 1.2.21). As is illustrated in [90], even though computations in the graded setting are often easier than working directly with D, it seems that not much is lost in passage from D to its corresponding graded division algebra $\operatorname{gr}(D)$. This has provided motivation to systematically study this correspondence, notably by Hwang, Tignol and Wadsworth [90], and to compare certain functors defined on these objects, notably the Brauer group [90, Chapter 6] and the reduced Whitehead group SK_1 [90, Chapter 11]. We introduce this correspondence here and in Chapter 3 we calculate their graded Grothendieck groups (Example 3.7.5).

Let D be a division algebra finite dimensional over its centre F, with a *valuation* $v : D^* \to \Gamma$. So Γ is a totally ordered abelian group, and for any $a, b \in D^*$, v satisfies the following conditions:

(i) $v(ab) = v(a) + v(b)$;
(ii) $v(a + b) \geq \min\{ v(a), v(b) \} \quad (b \neq -a)$.

Let

$V_D = \{ a \in D^* : v(a) \geq 0 \} \cup \{0\}$, the valuation ring of v;
$M_D = \{ a \in D^* : v(a) > 0 \} \cup \{0\}$, the unique maximal left and right ideal of V_D;
$\overline{D} = V_D/M_D$, the residue division ring of v on D; and
$\Gamma_D = \operatorname{im}(v)$, the value group of the valuation.

For background on valued division algebras, see [90, Chapter 1]. One associates to D a graded division algebra as follows. For each $\gamma \in \Gamma_D$, let

$D^{\geq\gamma} = \{ d \in D^* : v(d) \geq \gamma \} \cup \{0\}$, an additive subgroup of D;
$D^{>\gamma} = \{ d \in D^* : v(d) > \gamma \} \cup \{0\}$, a subgroup of $D^{\geq\gamma}$; and
$\operatorname{gr}(D)_\gamma = D^{\geq\gamma}/D^{>\gamma}$.

Then define

$$\operatorname{gr}(D) = \bigoplus_{\gamma \in \Gamma_D} \operatorname{gr}(D)_\gamma.$$

https://doi.org/10.1017/CBO9781316717134.002 Published online by Cambridge University Press

Because $D^{>\gamma}D^{\geq\delta} + D^{\geq\gamma}D^{>\delta} \subseteq D^{>(\gamma+\delta)}$ for all $\gamma, \delta \in \Gamma_D$, the multiplication on $\operatorname{gr}(D)$ induced by multiplication on D is well-defined, giving that $\operatorname{gr}(D)$ is a Γ-graded ring, called the *associated graded ring* of D. The multiplicative property (i) of the valuation v implies that $\operatorname{gr}(D)$ is a graded division ring. Clearly, we have $\operatorname{gr}(D)_0 = \overline{D}$, and $\Gamma_{\operatorname{gr}(D)} = \Gamma_D$. For $d \in D^*$, we write $\widetilde{d}$ for the image $d + D^{>v(d)}$ of d in $\operatorname{gr}(D)_{v(d)}$. Thus, the map given by $d \mapsto \widetilde{d}$ is a group epimorphism $D^* \to \operatorname{gr}(D)^*$ with kernel $1 + M_D$.

The restriction $v|_F$ of the valuation on D to its centre F is a valuation on F, which induces a corresponding graded field $\operatorname{gr}(F)$. Then it is clear that $\operatorname{gr}(D)$ is a graded $\operatorname{gr}(F)$-algebra, and one can prove that for

$$[\operatorname{gr}(D) : \operatorname{gr}(F)] = [\overline{D} : \overline{F}]\,|\Gamma_D : \Gamma_F| \;\leq\; [D : F] \;< \infty.$$

Now let F be a field with a *henselian* valuation v, *i.e.,* the valuation v has a unique extension to any algebraic extension of F. It was proved that (see [90, Chapter 1]) the valuation v extends uniquely to D as well. With respect to this valuation, D is said to be *tame* if $Z(\overline{D})$ is separable over $\overline{F}$ and

$$\operatorname{char}(\overline{F}) \nmid \operatorname{ind}(D)/(\operatorname{ind}(\overline{D})[Z(\overline{D}) : \overline{F}]).$$

It is known ([90, Chapter 8]) that D is tame if and only if

$$[\operatorname{gr}(D) : \operatorname{gr}(F)] = [D : F]$$

and $Z(\operatorname{gr}(D)) = \operatorname{gr}(F)$.

We compute the graded Grothendieck group and the graded Picard group of these division algebras in Examples 3.7.5 and 4.2.6.

1.4.1 The zero component ring of a graded central simple ring

Let A be a Γ-graded division ring and $\mathbb{M}_n(A)(\lambda_1, \dots, \lambda_n)$ be a graded simple ring, where $\lambda_i \in \Gamma$, $1 \leq i \leq n$. Since A is a graded division ring, Γ_A is a subgroup of Γ. Consider the quotient group Γ/Γ_A and let $\Gamma_A + \varepsilon_1, \dots, \Gamma_A + \varepsilon_k$ be the distinct elements in Γ/Γ_A representing the cosets $\Gamma_A + \lambda_i$, $1 \leq i \leq n$, and for each ε_l, let r_l be the number of i with $\Gamma_A + \lambda_i = \Gamma_A + \varepsilon_l$. It was observed in [90, Chapter 2] that

$$\boxed{\mathbb{M}_n(A)(\lambda_1, \dots, \lambda_n)_0 \cong \mathbb{M}_{r_1}(A_0) \times \cdots \times \mathbb{M}_{r_k}(A_0)} \tag{1.67}$$

and in particular, $\mathbb{M}_n(A)(\lambda_1, \dots, \lambda_n)_0$ is a simple ring if and only if $k = 1$. Indeed, using (1.50) and (1.51) we get

$$\mathbb{M}_n(A)(\lambda_1, \dots, \lambda_n) \cong_{\operatorname{gr}} \mathbb{M}_n(A)(\varepsilon_1, \dots, \varepsilon_1, \varepsilon_2, \dots, \varepsilon_2, \dots, \varepsilon_k, \dots, \varepsilon_k), \tag{1.68}$$

with each ε_l occurring r_l times. Now (1.45) for $\lambda = 0$ and

$$(\delta_1, \dots, \delta_n) = (\varepsilon_1, \dots \varepsilon_1, \varepsilon_2, \dots, \varepsilon_2, \dots, \varepsilon_k, \dots, \varepsilon_k)$$

immediately gives (1.67).

Remark 1.4.8 The graded Artin–Wedderburn structure theorem

The Artin–Wedderburn theorem shows that division rings are the basic "building blocks" of ring theory, *i.e.,* if a ring A satisfies some finite condition, for example A is right Artinian, then $A/J(A)$ is isomorphic to a finite product of matrix rings over division rings. A graded version of the Artin–Wedderburn structure theorem also holds. We state the statement here without proof. We refer the reader to [75, 90] for proofs of these statements.

A Γ-graded ring B is isomorphic to $\mathbb{M}_n(A)(\lambda_1, \dots, \lambda_n)$, where A is a Γ-graded division ring and $\lambda_i \in \Gamma$, $1 \leq i \leq n$, if and only if B is *graded right Artinian* (*i.e.,* a decreasing chain of graded right ideals becomes stationary) and graded simple.

A Γ-graded ring B is isomorphic to a finite product of matrix rings overs graded division rings (with suitable shifts) if and only if B is graded right Artinian and *graded primitive* (*i.e.,* $J^{\text{gr}}(B) = 0$).

1.5 Strongly graded rings and Dade's theorem

Let A be a Γ-graded ring and Ω be a subgroup of Γ. Recall from §1.1.2 that A has a natural Γ/Ω-graded structure and $A_\Omega = \bigoplus_{\gamma\in\Omega} A_\gamma$ is a Ω-graded ring. If A is a Γ/Ω-strongly graded ring, then one can show that the category of Γ-graded A-modules, $\text{Gr}^\Gamma\text{-}A$, is equivalent to the category of Ω-graded A_Ω-modules, $\text{Gr}^\Omega\text{-}A_\Omega$. In fact, the equivalence

$$\text{Gr}^\Gamma\text{-}A \approx \text{Gr}^\Omega\text{-}A_\Omega,$$

under the given natural functors (see Theorem 1.5.7) implies that A is a Γ/Ω-strongly graded ring. This was first proved by Dade [32] in the case of $\Omega = 0$, *i.e.,* when $\text{Gr}^\Gamma\text{-}A \approx \text{Mod-}A_0$. We prove Dade's theorem (Theorem 1.5.1) and then state this more general case in Theorem 1.5.7.

Let A be a Γ-graded ring. For any right A_0-module N and any $\gamma \in \Gamma$, we identify the right A_0-module $N \otimes_{A_0} A_\gamma$ with its image in $N \otimes_{A_0} A$. Since $A = \bigoplus_{\gamma\in\Gamma} A_\gamma$ and A_γ are A_0-bimodules, $N \otimes_{A_0} A$ is a Γ-graded right A-module, with

$$N \otimes_{A_0} A = \bigoplus_{\gamma\in\Gamma} (N \otimes_{A_0} A_\gamma). \tag{1.69}$$

Consider the *restriction functor*

$$\begin{aligned} \mathcal{G} := (-)_0 : \text{Gr-}A &\longrightarrow \text{Mod-}A_0 \\ M &\longmapsto M_0 \\ \psi &\longmapsto \psi|_{M_0} \end{aligned}$$

and the *induction functor* defined by

$$\begin{aligned} \mathcal{I} := - \otimes_{A_0} A : \text{Mod-}A_0 &\longrightarrow \text{Gr-}A \\ N &\longmapsto N \otimes_{A_0} A \\ \phi &\longmapsto \phi \otimes \text{id}_A . \end{aligned}$$

One can easily check that $\mathcal{G} \circ \mathcal{I} \cong \text{id}_{A_0}$ with the natural transformation

$$\begin{aligned} \mathcal{G}\,\mathcal{I}(N) = \mathcal{G}(N \otimes_{A_0} A) = N \otimes_{A_0} A_0 &\longrightarrow N, \\ n \otimes a &\mapsto na. \end{aligned} \tag{1.70}$$

On the other hand, there is a natural transformation

$$\begin{aligned} \mathcal{I}\,\mathcal{G}(M) = \mathcal{I}(M_0) = M_0 \otimes_{A_0} A &\longrightarrow M, \\ m \otimes a &\mapsto ma. \end{aligned} \tag{1.71}$$

The theorem below shows that $\mathcal{I} \circ \mathcal{G} \cong \text{id}_A$ (under (1.71)), if and only if A is a strongly graded ring. Theorem 1.5.1 was proved by Dade [32, Theorem 2.8] (see also [75, Theorem 3.1.1]).

Theorem 1.5.1 (Dade's theorem) *Let A be a Γ-graded ring. Then A is strongly graded if and only if the functors*

$$(-)_0 : \text{Gr-}A \to \text{Mod-}A_0$$

and

$$- \otimes_{A_0} A : \text{Mod-}A_0 \to \text{Gr-}A$$

form mutually inverse equivalences of categories.

Proof One can easily check that (without using the assumption that A is strongly graded) $\mathcal{G} \circ \mathcal{I} \cong \text{id}_{A_0}$ (see (1.70)). Suppose A is strongly graded. We show that $\mathcal{I} \circ \mathcal{G} \cong \text{id}_A$.

For a graded A-module M, we have $\mathcal{I} \circ \mathcal{G}(M) = M_0 \otimes_{A_0} A$. We show that the natural homomorphism

$$\begin{aligned} \phi : M_0 \otimes_{A_0} A &\to M, \\ m \otimes a &\mapsto ma, \end{aligned}$$

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is a Γ-graded A-module isomorphism. The map ϕ is clearly graded (see (1.69)). Since A is strongly graded, it follows that for $\gamma, \delta \in \Gamma$,

$$M_{\gamma+\delta} = M_{\gamma+\delta}A_0 = M_{\gamma+\delta}A_{-\gamma}A_\gamma \subseteq M_\delta A_\gamma \subseteq M_{\gamma+\delta}. \tag{1.72}$$

Thus $M_\delta A_\gamma = M_{\gamma+\delta}$. Therefore, $\phi(M_0 \otimes_{A_0} A_\gamma) = M_0 A_\gamma = M_\gamma$, which implies that ϕ is surjective.

Let $N = \ker(\phi)$, which is a graded A-submodule of $M_0 \otimes_{A_0} A$, so $N_0 = N \cap (M_0 \otimes_{A_0} A_0)$. However, the restriction of ϕ to $M_0 \otimes_{A_0} A_0 \to M_0$ is the canonical isomorphism, so $N_0 = 0$. Since N is a graded A-module, a similar argument as (1.72) shows $N_\gamma = N_0 A_\gamma = 0$ for all $\gamma \in \Gamma$. It follows that ϕ is injective. Thus $\mathcal{I} \circ \mathcal{G}(M) = M_0 \otimes_{A_0} A \cong M$. Since all the homomorphisms involved are natural, this shows that $\mathcal{I} \circ \mathcal{G} \cong \mathrm{id}_A$.

For the converse, suppose $\mathcal{I}$ and $\mathcal{G}$ are mutually inverse (under (1.71) and (1.70)). For any graded A-module M, $\mathcal{I} \circ \mathcal{G}(M) \cong_{\mathrm{gr}} M$, which gives that the map

$$M_0 \otimes_{A_0} A_\alpha \longrightarrow M_\alpha,$$
$$m \otimes a \mapsto ma$$

is bijective, where $\alpha \in \Gamma$. This immediately implies

$$M_0 A_\alpha = M_\alpha. \tag{1.73}$$

Now for any $\beta \in \Gamma$, consider the graded A-module $A(\beta)$. Replacing M by $A(\beta)$ in (1.73), we get $A(\beta)_0 A_\alpha = A(\beta)_\alpha$, *i.e.*, $A_\beta A_\alpha = A_{\beta+\alpha}$. This shows that A is strongly graded. □

Corollary 1.5.2 *Let A be a Γ-graded ring and Ω a subgroup of Γ such that A is a Γ/Ω-strongly graded ring. Then the functors*

$$(-)_0 : \mathrm{Gr}^{\Gamma/\Omega}\text{-}A \longrightarrow \mathrm{Mod}\text{-}A_\Omega$$

and

$$- \otimes_{A_\Omega} A : \mathrm{Mod}\text{-}A_\Omega \longrightarrow \mathrm{Gr}^{\Gamma/\Omega}\text{-}A$$

form mutually inverse equivalences of categories.

Proof The result follows from Theorem 1.5.1. □

Remark 1.5.3 Recall that gr-A denotes the category of graded finitely generated right A-modules and Pgr-A denotes the category of graded finitely generated projective right A-modules. Note that in general the restriction functor $(-)_0 : \mathrm{Gr}\text{-}A \to \mathrm{Mod}\text{-}A_0$ does not induce a functor $(-)_0 : \mathrm{Pgr}\text{-}A \to \mathrm{Pr}\text{-}A_0$. In fact, one can easily produce a graded finitely generated projective A-module

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P such that P_0 is not a projective A_0-module. As an example, consider the $\mathbb{Z}$-graded ring T of Example 1.1.5. Then $T(1)$ is clearly a graded finitely generated projective T-module. However $T(1)_0 = M$ is not $T_0 = R$-module.

Remark 1.5.4 The proof of Theorem 1.5.1 also shows that A is strongly graded if and only if gr-$A \cong$ mod-A_0, if and only if Pgr-$A \cong$ Pr-A_0, (see Remark 1.5.3) via the same functors $(-)_0$ and $- \otimes_{A_0} A$ of the Theorem 1.5.1.

Remark 1.5.5 STRONGLY GRADED MODULES

Let A be a Γ-graded ring and M be a graded A-module. Then M is called a *strongly graded* A-module if

$$M_\alpha A_\beta = M_{\alpha+\beta}, \tag{1.74}$$

for any $\alpha, \beta \in \Gamma$. The proof of Theorem 1.5.1 shows that A is strongly graded if and only if any graded A-module is strongly graded. Indeed, if A is strongly graded then (1.72) shows that any graded A-module is strongly graded. Conversely, if any graded module is strongly graded, then considering A as a graded A-module, (1.74) for $M = A$, shows that $A_\alpha A_\beta = A_{\alpha+\beta}$ for any $\alpha, \beta \in \Gamma$.

Remark 1.5.6 IDEALS CORRESPONDENCE BETWEEN A_0 AND A

The proof of Theorem 1.5.1 shows that there is a one-to-one correspondence between the right ideals of A_0 and the graded right ideals of A (similarly for the left ideals). However, this correspondence does not hold between two-sided ideals. As an example, $A = \mathbb{M}_2(K[x^2, x^{-2}])(0, 1)$, where K is a field, is a strongly $\mathbb{Z}$-graded simple ring, whereas $A_0 \cong K \otimes K$ is not a simple ring. (See §1.4.1. Also see Proposition 4.2.9 for a relation between simplicity of A_0 and A.)

In the same way, the equivalence Gr-$A \approx$ Mod-A_0 of Theorem 1.5.1 gives a correspondence between several (one-sided) properties of graded objects in A with objects over A_0. For example, one can easily show that A is graded right (left) Noetherian if and only if A_0 is right (left) Noetherian (see also Corollary 1.5.10).

Using Theorem 1.5.1, we will see that the graded Grothendieck group of a strongly graded ring coincides with the (classical) Grothendieck group of its 0-component ring (see §3.1.3).

We need a more general version of grading defined in (1.69) in order to extend Dade's theorem. Let A be a Γ-graded ring and Ω a subgroup of Γ. Let N be a Ω-graded right A_Ω-module. Then $N \otimes_{A_\Omega} A$ is a Γ-graded right A-module, with the grading defined by

$$(N \otimes_{A_\Omega} A)_\gamma = \Big\{ \sum_i n_i \otimes a_i \mid n_i \in N^h, a_i \in A^h, \deg(n_i) + \deg(a_i) = \gamma \Big\}.$$

A similar argument as in §1.2.6 for tensor products will show that this grading is well-defined. Note that with this grading,

$$(N \otimes_{A_\Omega} A)_\Omega = N \otimes_{A_\Omega} A_\Omega \cong N,$$

as graded right A_Ω-modules.

Theorem 1.5.7 *Let A be a Γ-graded ring and Ω be a subgroup of Γ. Consider A as a Γ/Ω-graded ring. Then A is a Γ/Ω-strongly graded ring if and only if*

$$\begin{aligned}(-)_\Omega : \mathrm{Gr}^\Gamma\text{-}A &\longrightarrow \mathrm{Gr}^\Omega\text{-}A_\Omega\\ M &\longmapsto M_\Omega\\ \psi &\longmapsto \psi|_{M_\Omega}\end{aligned}$$

and

$$\begin{aligned}- \otimes_{A_0} A : \mathrm{Gr}^\Omega\text{-}A_\Omega &\longrightarrow \mathrm{Gr}^\Gamma\text{-}A\\ N &\longmapsto N \otimes_{A_\Omega} A\\ \phi &\longmapsto \phi \otimes \mathrm{id}_A\end{aligned}$$

form mutually inverse equivalences of categories.

Proof The proof is similar to the proof of Theorem 1.5.1 and it is omitted. □

Remark 1.5.8 Compare Theorem 1.5.7, with the following statement. Let A be a Γ-graded ring and Ω be a subgroup of Γ. Then A is Γ-strongly graded ring if and only if

$$\begin{aligned}(-)_\Omega : \mathrm{Gr}^\Gamma\text{-}A &\longrightarrow \mathrm{Gr}^\Omega\text{-}A_\Omega\\ M &\longmapsto M_\Omega\\ \psi &\longmapsto \psi|_{M_\Omega}\end{aligned}$$

and

$$\begin{aligned}- \otimes_{A_0} A : \mathrm{Gr}^\Omega\text{-}A_\Omega &\longrightarrow \mathrm{Gr}^\Gamma\text{-}A\\ N &\longmapsto N_0 \otimes_{A_0} A\\ \phi &\longmapsto \phi_0 \otimes \mathrm{id}_A\end{aligned}$$

form mutually inverse equivalences of categories.

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Example 1.5.9 Let A be a $\Gamma \times \Omega$-graded ring such that $1 \in A_{(\alpha,\Omega)}A_{(-\alpha,\Omega)}$ for any $\alpha \in \Gamma$, where $A_{(\alpha,\Omega)} = \bigoplus_{\omega\in\Omega} A_{(\alpha,\omega)}$. Then by Theorem 1.5.7

$$\mathrm{Gr}^{\Gamma\times\Omega}\text{-}A \approx \mathrm{Gr}^{\Omega}\text{-}A_{(0,\Omega)}.$$

This example will be used in §6.4. Compare this also with Corollary 1.2.13.

Another application of Theorem 1.5.1 is to provide a condition when a strongly graded ring is a graded von Neumann ring (§1.1.9). This will be used later in Corollary 1.6.17 to show that the Leavitt path algebras are von Neumann regular rings.

Corollary 1.5.10 *Let A be a strongly graded ring. Then A a is graded von Neumann regular ring if and only if A_0 is a von Neumann regular ring.*

Sketch of proof Since any (graded) flat module is a direct limit of (graded) projective modules, from the equivalence of categories $\mathrm{Gr}\text{-}A \approx_{\mathrm{gr}} \mathrm{Mod}\text{-}A_0$ (Theorem 1.5.1), it follows that A is graded von Neumann regular if and only if A_0 is von Neumann regular. □

Remark 1.5.11 An element-wise proof of Corollary 1.5.10 can also be found in [96, Theorem 3].

For a Γ-graded ring A, and $\alpha, \beta \in \Gamma$, one has an A_0-bimodule homomorphism

$$\begin{aligned} \phi_{\alpha,\beta} : A_\alpha \otimes_{A_0} A_\beta &\longrightarrow A_{\alpha+\beta} \\ a \otimes b &\longmapsto ab. \end{aligned} \tag{1.75}$$

The following theorem gives another characterisation for strongly graded rings.

Theorem 1.5.12 *Let A be a Γ-graded ring. Then A is a strongly graded ring if and only if for any $\gamma \in \Gamma$, the homomorphism*

$$\begin{aligned} \phi_{\gamma,-\gamma} : A_\gamma \otimes_{A_0} A_{-\gamma} &\longrightarrow A_0, \\ a \otimes b &\longmapsto ab \end{aligned}$$

is an isomorphism. In particular, if A is strongly graded, then the homogeneous components A_γ, $\gamma \in \Gamma$, are finitely generated projective A_0-modules.

Proof Suppose that for any $\gamma \in \Gamma$, the map $\phi_{\gamma,-\gamma} : A_\gamma \otimes A_{-\gamma} \to A_0$ is an isomorphism. Thus there are $a_i \in A_\gamma$, $b_i \in A_{-\gamma}$ such that

$$\sum_i a_i b_i = \phi_{\gamma,-\gamma}\Big(\sum_i a_i \otimes b_i\Big) = 1.$$

So $1 \in A_\gamma A_{-\gamma}$. Now by Proposition 1.1.15(1) A is strongly graded.

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Conversely, suppose A is a strongly graded ring. We prove that the homomorphism (1.75) is an isomorphism. The definition of strongly graded implies that $\phi_{\alpha,\beta}$ is surjective. Suppose

$$\phi_{\alpha,\beta}\Big(\sum_i a_i \otimes b_i\Big) = \sum_i a_i b_i = 0. \tag{1.76}$$

Using Proposition 1.1.15(1), write $1 = \sum_j x_j y_j$, where $x_j \in A_{-\beta}$ and $y_j \in A_\beta$. Then

$$\begin{aligned}\sum_i a_i \otimes b_i &= \Big(\sum_i a_i \otimes b_i\Big)\Big(\sum_j x_j y_j\Big) = \sum_i \Big(a_i \otimes \sum_j b_i x_j y_j\Big)\\ &= \sum_i \Big(\sum_j (a_i b_i x_j \otimes y_j)\Big) = \sum_j \sum_i \big(a_i b_i x_j \otimes y_j\big)\\ &= \sum_j \Big(\sum_i (a_i b_i) x_j \otimes y_j)\Big) = 0.\end{aligned}$$

This shows that $\phi_{\alpha,\beta}$ is injective. Now setting $\alpha = \gamma$ and $\beta = -\gamma$ finishes the proof.

Finally, if A is strongly graded, the above argument shows that the homogeneous components A_γ, $\gamma \in \Gamma$, are invertible A_0-modules, which in turn implies that A_α are finitely generated projective A_0-modules. □

1.5.1 Invertible components of strongly graded rings

Let A and B be rings and P be an A–B-bimodule. Then P is called an *invertible* A–B-bimodule if there is a B–A-bimodule Q such that $P \otimes_B Q \cong A$ as A–A-bimodules and $Q \otimes_A P \cong B$ as B–B-bimodules and the following diagrams are commutative:

$$\begin{array}{ccc} P \otimes_B Q \otimes_A P & \longrightarrow & A \otimes_A P \\ \downarrow & & \downarrow \\ P \otimes_B B & \longrightarrow & P \end{array} \qquad \begin{array}{ccc} Q \otimes_A P \otimes_B Q & \longrightarrow & B \otimes_B Q \\ \downarrow & & \downarrow \\ Q \otimes_A A & \longrightarrow & Q \end{array}$$

One can prove that P is a finitely generated projective A and B-module.

Now Theorem 1.5.12 shows that for a strongly Γ-graded ring A, the A_0-bimodules A_γ, $\gamma \in \Gamma$, are invertible modules and thus are finitely generated projective A_0-modules. This in return implies that A is a projective A_0-module. Note that, in general, one can easily construct a graded ring A where A is not projective over A_0 (see Example 1.1.5) and A_γ is not a finitely generated A_0-module, such as the $\mathbb{Z}$-graded ring $\mathbb{Z}[x_i \mid i \in \mathbb{N}]$ of Example 1.1.9.

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Remark 1.5.13 OTHER TERMINOLOGIES FOR STRONGLY GRADED RINGS

The term "strongly graded" for such rings was coined by E. Dade in [32] and is now commonly in use. Other terms for these rings are *fully graded* and *generalised crossed products*. See [33] for a history of the development of such rings in literature.

1.6 Grading on graph algebras

1.6.1 Grading on free rings

Let X be a nonempty set of symbols and Γ be a group. (As always we assume the groups are abelian, although the entire theory can be written for an arbitrary group.) Let $d : X \to \Gamma$ be a map. One can extend d in a natural way to a map from the set of finite words on X to Γ, which is called d again. For example if $x, y, z \in X$ and xyz is a word, then $d(xyz) = d(x) + d(y) + d(z)$. One can easily see that if w_1, w_2 are two words, then $d(w_1w_2) = d(w_1) + d(w_2)$. If we allow an empty word, which will be the identity element in the free ring, then we assign the identity of Γ to this word.

Let R be a ring and $R(X)$ be the free ring (with or without identity) on a set X with coefficients in R. The elements of $R(X)$ are of the form $\sum_w r_w w$, where $r_w \in R$ and w stands for a word on X. The multiplication is defined by convolution, *i.e.,*

$$\Big(\sum_w r_w w\Big)\Big(\sum_v r_v v\Big) = \sum_z \Big(\sum_{\{w,v|z=wv,\, r_w,r_v\neq 0\}} r_w r_v\Big) z.$$

In order to make $R(X)$ into a graded ring, define

$$R(X)_\gamma = \Big\{ \sum_w r_w w \mid d(w) = \gamma \Big\}.$$

One can check that $R(X) = \bigoplus_{\gamma\in\Gamma} R(X)_\gamma$. Thus $R(X)$ is a Γ-graded ring. Note that if we don't allow the empty word in the construction, then $R(X)$ is a graded ring without identity (see Remark 1.1.14). It is easy to see that $R(X)$ is never a strongly graded ring.

Example 1.6.1 Let R be a ring and $R(X)$ be the free ring on a set X with a graded structure induced by a map $d : X \to \Gamma$. Let Ω be a subgroup of Γ and consider the map

$$\overline{d} : X \longrightarrow \Gamma/\Omega,$$
$$x \longmapsto \Omega + d(x).$$

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The map $\overline{d}$ induces a Γ/Ω-graded structure on $R(X)$ which coincides with the general construction of quotient grading given in §1.1.2.

Example 1.6.2 Let $X = \{x\}$ be a set of symbols with one element and $\mathbb{Z}_n$ be the cyclic group with n elements. Assign $1 \in \mathbb{Z}_n$ to x and generate the free ring with identity on X with coefficients in a field F. This ring is the usual polynomial ring $F[x]$ which, by the above construction, is equipped with a $\mathbb{Z}_n$-grading. Namely,

$$F[x] = \bigoplus_{k\in\mathbb{Z}_n}\Big(\sum_{\substack{l\in\mathbb{N},\\ \bar{l}=k}} Fx^l\Big),$$

where $\bar{l}$ is the image of l in the group $\mathbb{Z}_n$. For $a \in F$, since the polynomial $x^n - a$ is a homogeneous element of degree zero, the ideal $\langle x^n - a\rangle$ is a graded ideal and thus the quotient ring $F[x]/\langle x^n - a\rangle$ is also a $\mathbb{Z}_n$-graded ring (see §1.1.5). In particular, if $x^n - a$ is an irreducible polynomial in $F[x]$, then the field $F[x]/\langle x^n - a\rangle$ is a $\mathbb{Z}_n$-graded field as well.

Example 1.6.3 Let $\{x, y\}$ be a set of symbols. Assign $1 \in \mathbb{Z}_2$ to x and y and consider the graded free ring $\mathbb{R}(x, y)$. The ideal generated by homogeneous elements $\{x^2 + 1, y^2 + 1, xy + yx\}$ is graded and thus we retrieve the $\mathbb{Z}_2$-graded Hamilton quaternion algebra of Example 1.1.20 as follows:

$$\mathbb{H} \cong \mathbb{R}(x, y)/\langle x^2 + 1, y^2 + 1, xy + yx\rangle.$$

Moreover, assigning $(1, 0) \in \mathbb{Z}_2 \times \mathbb{Z}_2$ to x and $(0, 1) \in \mathbb{Z}_2 \times \mathbb{Z}_2$ to y we obtained the $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded quaternion algebra of Example 1.1.20.

Example 1.6.4 THE WEYL ALGEBRA

For a (commutative) ring R, the *Weyl algebra* $R(x, y)/\langle xy - yx - 1\rangle$ can be considered as a $\mathbb{Z}$-graded ring by assigning 1 to x and -1 to y.

Example 1.6.5 THE LEAVITT ALGEBRA $\mathcal{L}(n, k + 1)$

Let K be a field, n and k positive integers and A be the free associative K-algebra with identity generated by symbols $\{x_{ij}, y_{ji} \mid 1 \leq i \leq n + k, 1 \leq j \leq n\}$ subject to relations (coming from)

$$Y \cdot X = I_{n,n} \qquad \text{and} \qquad X \cdot Y = I_{n+k,n+k},$$

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where

$$Y = \begin{pmatrix} y_{11} & y_{12} & \dots & y_{1,n+k} \\ y_{21} & y_{22} & \dots & y_{2,n+k} \\ \vdots & \vdots & \ddots & \vdots \\ y_{n,1} & y_{n,2} & \dots & y_{n,n+k} \end{pmatrix}, \quad X = \begin{pmatrix} x_{11} & x_{12} & \dots & x_{1,n} \\ x_{21} & x_{22} & \dots & x_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ x_{n+k,1} & x_{n+k,2} & \dots & x_{n+k,n} \end{pmatrix}. \tag{1.77}$$

To be concrete, the relations are

$$\sum_{j=1}^{n+k} y_{ij}x_{jl} = \delta_{i,l}, \qquad 1 \le i, l \le n,$$
$$\sum_{j=1}^{n} x_{ij}y_{jl} = \delta_{i,l}, \qquad 1 \le i, l \le n + k.$$

In Example 1.3.19 we studied a special case of this algebra when $n = 1$ and $k = n - 1$. This algebra was studied by Leavitt in relation with its type in [63, p.130] where it is shown that for arbitrary n and k the algebra is of type (n, k) (see §1.7) and when $n \ge 2$ they are domains. We denote this algebra by $\mathcal{L}(n, k + 1)$. (Cohn's notation in [28] for this algebra is $V_{n,n+k}$.)

Assigning

$$\deg(y_{ji}) = (0, \dots, 0, 1, 0 \dots, 0),$$
$$\deg(x_{ij}) = (0, \dots, 0, -1, 0 \dots, 0),$$

for $1 \le i \le n + k$, $1 \le j \le n$, in $\bigoplus_n \mathbb{Z}$, where 1 and -1 are in the jth entries respectively, makes the free algebra generated by x_{ij} and y_{ji} a graded ring. Moreover, one can easily observe that the relations coming from (1.77) are all homogeneous with respect to this grading, so that the Leavitt algebra $\mathcal{L}(n, k + 1)$ is a $\bigoplus_n \mathbb{Z}$-graded ring. In particular, $\mathcal{L}(1, k)$ is a $\mathbb{Z}$-graded ring (Example 1.3.19).

1.6.2 Corner skew Laurent polynomial rings

Let R be a ring with identity and p an idempotent of R. Let $\phi : R \to pRp$ be a *corner* isomorphism, i.e, a ring isomorphism such that $\phi(1) = p$. A *corner skew Laurent polynomial ring* with coefficients in R, denoted by $R[t_+, t_-, \phi]$, is a unital ring which is constructed as follows: The elements of $R[t_+, t_-, \phi]$ are the formal expressions

$$t_-^j r_{-j} + t_-^{j-1} r_{-j+1} + \dots + t_- r_{-1} + r_0 + r_1 t_+ + \dots + r_i t_+^i,$$

where $r_{-n} \in p_n R$ and $r_n \in Rp_n$, for all $n \geq 0$, where $p_0 = 1$ and $p_n = \phi^n(p_0)$. The addition is component-wise, and the multiplication is determined by the distribution law and the following rules:

$$t_- t_+ = 1, \qquad t_+ t_- = p, \qquad rt_- = t_-\phi(r), \qquad t_+ r = \phi(r) t_+. \tag{1.78}$$

The corner skew Laurent polynomial rings are studied in [6], where their K_1-groups are calculated. This construction is a special case of the so-called fractional skew monoid rings constructed in [7]. Assigning -1 to t_- and 1 to t_+ makes $A := R[t_+, t_-, \phi]$ a $\mathbb{Z}$-graded ring with $A = \bigoplus_{i \in \mathbb{Z}} A_i$, where

$$\begin{aligned} A_i &= Rp_i t_+^i, \text{ for } i > 0, \\ A_i &= t_-^i p_{-i} R, \text{ for } i < 0, \\ A_0 &= R, \end{aligned}$$

(see [7, Proposition 1.6]). Clearly, when $p = 1$ and ϕ is the identity map, then $R[t_+, t_-, \phi]$ reduces to the familiar ring $R[t, t^{-1}]$.

In the next three propositions we will characterise those corner skew Laurent polynomials which are strongly graded rings (§1.1.3), crossed products (§1.1.4) and graded von Neumann regular rings (§1.1.9).

Recall that an idempotent element p of the ring R is called a *full idempotent* if $RpR = R$.

Proposition 1.6.6 *Let R be a ring with identity and $A = R[t_+, t_-, \phi]$ a corner skew Laurent polynomial ring. Then A is strongly graded if and only if $\phi(1)$ is a full idempotent.*

Proof First note that $A_1 = R\phi(1)t_+$ and $A_{-1} = t_-\phi(1)R$. Moreover, since $\phi(1) = p$, we have

$$r_1\phi(1)t_+t_-\phi(1)r_2 = r_1\phi(1)p\phi(1)r_2 = r_1pppr_2 = r_1\phi(1)r_2.$$

Suppose A is strongly graded. Then $1 \in A_1A_{-1}$. That is

$$1 = \sum_i \Big(r_i\phi(1)t_+\Big)\Big(t_-\phi(1)r_i'\Big) = \sum_i r_i\phi(1)r_i', \tag{1.79}$$

where $r_i, r_i' \in R$. So $R\phi(1)R = R$, that is $\phi(1)$ is a full idempotent.

On the other hand suppose $\phi(1)$ is a full idempotent. Since $\mathbb{Z}$ is generated by 1, in order to prove that A is strongly graded, it is enough to show that $1 \in A_1A_{-1}$ and $1 \in A_{-1}A_1$ (see §1.1.3). But

$$t_-\phi(1)\phi(1)t_+ = t_{-1}\phi(1)t_+ = 1t_-t_+ = 1,$$

which shows that $1 \in A_{-1}A_1$. Since $\phi(1)$ is a full idempotent, there are $r_i, r_i' \in R$, $i \in I$ such that $\sum r_i\phi(1)r_i' = 1$. Then Equation (1.79) shows that $1 \in A_1A_{-1}$. □

Recall that a ring R is called *Dedekind finite* or *directly finite* if any one-sided invertible element is two-sided invertible. That is, if $ab = 1$, then $ba = 1$, where $a, b \in R$. For example, left (right) Noetherian rings are Dedekind finite.

Proposition 1.6.7 *Let R be a ring with identity which is Dedekind finite and $A = R[t_+, t_-, \phi]$ a corner skew Laurent polynomial ring. Then A is crossed product if and only if $\phi(1) = 1$.*

Proof If $\phi(1) = 1$, then from relations (1.78) it follows that $t_- t_+ = t_+ t_- = 1$. Therefore all homogeneous components contain invertible elements and thus A is crossed product.

Suppose A is crossed product. Then there are $a, b \in R$ such that $(t_- a)(bt_+) = 1$ and $(bt_+)(t_- a) = 1$. Using relations (1.78), the first equality gives $ab = p$ and the second one gives $bpa = 1$, where $\phi(1) = p$. Now

$$1 = bpa = bppa = babpa = ba.$$

Since R is Dedekind finite, it follows $ab = 1$ and thus $p = \phi(1) = 1$. □

Proposition 1.6.8 *Let R be a ring with identity and $A = R[t_+, t_-, \phi]$ a corner skew Laurent polynomial ring. Then A is a graded von Neumann regular ring if and only if R is a von Neumann regular ring.*

Proof If a graded ring is graded von Neumann regular, then it is easy to see that its zero component ring is von Neumann regular. This proves one direction of the theorem. For the converse, suppose R is regular. Let $x \in A_i$, where $i > 0$. So $x = rp_i t_+^i$, for some $r \in R$, where $p_i = \phi^i(1)$. By relations (1.78) and induction, we have $t_+^i t_-^i = \phi^i(p_0) = p_i$. Since R is regular, there is an $s \in R$ such that $rp_i srp_i = rp_i$. Then, choosing $y = t_-^i p_i s$, we have

$$xyx = (rp_i t_+^i)(t_-^i p_i s)(rp_i t_+^i) = (rp_i t_+^i t_-^i p_i s)(rp_i t_+^i) = rp_i p_i p_i srp_i t_+^i = rp_i t_+^i = x.$$

A similar argument shows that for $x \in A_i$, where $i < 0$, there is a y such that $xyx = x$. This shows that A is a graded von Neumann regular ring. □

Note that in a corner skew Laurent polynomial ring $R[t_+, t_-, \phi]$, t_+ is a left invertible element with a right inverse t_- (see the relations (1.78)). In fact this property characterises such rings. Namely, a graded ring $A = \bigoplus_{i \in \mathbb{Z}} A_i$ such that A_1 has a left invertible element is a corner skew Laurent polynomial ring, as the following theorem shows. The following theorem (first established in [7]) will be used to realise Leavitt path algebras (§1.6.4) as corner skew Laurent polynomial rings (Example 1.6.14).

Theorem 1.6.9 *Let A be a $\mathbb{Z}$-graded ring which has a left invertible element $t_+ \in A_1$. Then t_+ has a right inverse $t_- \in A_{-1}$, and $A = A_0[t_+, t_-, \phi]$, where*

$$\begin{aligned} \phi : A_0 &\longrightarrow t_+t_-A_0t_+t_-, \\ a &\longmapsto t_+at_-. \end{aligned} \tag{1.80}$$

Proof Since t_+ has a right inverse, it follows easily that there is a $t_- \in A_{-1}$ with $t_-t_+ = 1$. Moreover $t_+t_- = t_+t_-t_+t_-$ is a homogeneous idempotent of degree zero. Observe that the map (1.80) is a (unital) ring isomorphism. Consider the corner skew Laurent polynomial ring $\widetilde{A} = A_0[\widetilde{t_+}, \widetilde{t_-}, \phi]$. Since $\phi(a) = t_+at_-$, it follows that $t_-\phi(a) = at_-$ and $\phi(a)t_+ = t_+a$. Thus t_+ and t_- satisfy all the relations in (1.78). Therefore there is a well-defined map $\psi : \widetilde{A} \to A$, such that $\psi(\widetilde{t_\pm}) = t_\pm$ and the restriction of ψ on A_0 is the identity and

$$\psi\Big(\sum_{k=1}^{j} \widetilde{t_-^k} a_{-k} + a_0 + \sum_{k=1}^{i} a_i \widetilde{t_+^i}\Big) = \sum_{k=1}^{j} t_-^k a_{-k} + a_0 + \sum_{k=1}^{i} a_i t_+^i.$$

This also shows that ψ is a graded homomorphism. In order to show that ψ is an isomorphism, it suffices to show that its restriction to each homogeneous component $\psi : \widetilde{A}_i \to A_i$ is a bijection. Suppose $x \in \widetilde{A}_i$, $i > 0$ such that $\psi(x) = 0$. Then $x = d\widetilde{t_+^i}$ for some $d \in A_0 p_i$ where $p_i = \phi^i(1)$ and $\psi(x) = dt_+^i$. Note that $\phi^i(1) = t_+^i t_-^i$. Thus $d\phi^i(1) = dt_+^i t_-^i = \psi(x)t_-^i = 0$. It now follows that $x = d\widetilde{t_+^i} = d\phi^i(1)\widetilde{t_+^i} = 0$ in $\widetilde{A}_i$. This shows ψ is injective. Suppose $y \in A_i$. Then $yt_-^i \in A_0$ and $yt_-^i t_+^i t_-^i = yt_-^i \phi^i(1) \in A_0\phi^i(1) = A_0 p_i$. This shows $yt_-^i t_+^i t_-^i \widetilde{t_+^i} \in \widetilde{A}_i$. But $\psi(yt_-^i t_+^i t_-^i \widetilde{t_+^i}) = yt_-^i t_+^i t_-^i t_+^i = y$. This shows that $\psi : \widetilde{A}_i \to A_i$, $i > 0$ is a bijection. A similar argument can be written for the case of $i < 0$. The case $i = 0$ is obvious. This completes the proof. □

1.6.3 Graphs

In this subsection we gather some graph-theoretic definitions which are needed for the construction of path algebras in §1.6.4.

A *directed graph* $E = (E^0, E^1, r, s)$ consists of two countable sets E^0, E^1 and maps $r, s : E^1 \to E^0$. The elements of E^0 are called *vertices* and the elements of E^1 *edges*. If $s^{-1}(v)$ is a finite set for every $v \in E^0$, then the graph is called *row-finite*. In this book we will only consider row-finite graphs. In this setting, if the number of vertices, *i.e.*, $|E^0|$, is finite, then the number of edges, *i.e.*, $|E^1|$, is finite as well and we call E a *finite* graph.

For a graph $E = (E^0, E^1, r, s)$, a vertex v for which $s^{-1}(v)$ is empty is called a *sink*, while a vertex w for which $r^{-1}(w)$ is empty is called a *source*. An edge with the same source and range is called a *loop*. A path μ in a graph E is a

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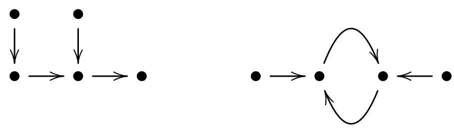
sequence of edges $\mu = \mu_1 \ldots \mu_k$, such that $r(\mu_i) = s(\mu_{i+1})$, $1 \le i \le k-1$. In this case, $s(\mu) := s(\mu_1)$ is the *source* of μ, $r(\mu) := r(\mu_k)$ is the *range* of μ, and k is the *length* of μ which is denoted by $|\mu|$. We consider a vertex $v \in E^0$ as a *trivial* path of length zero with $s(v) = r(v) = v$. If μ is a nontrivial path in E, and if $v = s(\mu) = r(\mu)$, then μ is called a *closed path based at* v. If $\mu = \mu_1 \ldots \mu_k$ is a closed path based at $v = s(\mu)$ and $s(\mu_i) \neq s(\mu_j)$ for every $i \neq j$, then μ is called a *cycle*. Throughout, we denote a cycle of length n by C_n. We call a graph without cycles a *acyclic* graph. A graph consisting of only one cycle and all the paths ending on this cycle is called a *comet* graph. A C_n-comet graph is a comet graph with a cycle of length n. Here are examples of an acyclic and a 2-comet graph.



For two vertices v and w, the existence of a path with the source v and the range w is denoted by $v \ge w$. Here we allow paths of length zero. By $v \ge_n w$, we mean there is a path of length n connecting these vertices. Therefore $v \ge_0 v$ represents the vertex v. Also, by $v > w$, we mean a path from v to w where $v \neq w$. In this book, by $v \ge w' \ge w$ it is understood that there is a path connecting v to w and going through w' (*i.e.,* w' is on the path connecting v to w). For $n \ge 2$, we define E^n to be the set of paths of length n and $E^* = \bigcup_{n\ge0} E^n$, the set of all paths.

For a graph E, let $n_{v,w}$ be the number of edges with the source v and range w. Then the *adjacency matrix* of the graph E is $A_E = (n_{v,w})$. Usually one orders the vertices and then writes A_E based on this ordering. Two different orderings of vertices give different adjacency matrices. However, if A_E and A'_E are two adjacency matrices of E, then there is a permutation matrix P such that $A'_E = PA_EP^{-1}$.

A graph E is called *essential* if E does not have sinks and sources. Moreover, a graph is called *irreducible* if for every ordered pair of vertices v and w there is a path from v to w.

1.6.4 Leavitt path algebras

A path algebra, with coefficients in the field K, is constructed as follows: consider a K-vector space with finite paths as the basis and define the multiplication by concatenation of paths. A path algebra has a natural graded structure by assigning paths as homogeneous elements of degree equal to their lengths. A formal definition of path algebras with coefficients in a ring R is given below.

Definition 1.6.10 For a graph E and a ring R with identity, we define the *path algebra of* E, denoted by $\mathcal{P}_R(E)$, to be the algebra generated by the sets $\{v \mid v \in E^0\}$, $\{\alpha \mid \alpha \in E^1\}$ with coefficients in R, subject to the relations

1 $v_i v_j = \delta_{ij} v_i$ for every $v_i, v_j \in E^0$;
2 $s(\alpha)\alpha = \alpha r(\alpha) = \alpha$ for all $\alpha \in E^1$.

Here the ring R commutes with the generators $\{v, \alpha \mid v \in E^0, \alpha \in E^1\}$. When the coefficient ring R is clear from the context, we simply write $\mathcal{P}(E)$ instead of $\mathcal{P}_R(E)$. When R is not commutative, then we consider $\mathcal{P}_R(E)$ as a left R-module. Using the above two relations, it is easy to see that when the number of vertices is finite, then $\mathcal{P}_R(E)$ is a ring with identity $\sum_{v \in E^0} v$.

When the graph has one vertex and n loops, the path algebra associated with this graph is isomorphic to $R\langle x_1, \dots, x_n\rangle$, *i.e.,* a free associative unital algebra over R with n noncommuting variables.

Setting $\deg(v) = 0$ for $v \in E^0$ and $\deg(\alpha) = 1$ for $\alpha \in E^1$, we obtain a natural $\mathbb{Z}$-grading on the free R-ring generated by $\{v, \alpha \mid v \in E^0, \alpha \in E^1\}$ (§1.6.1). Since the relations in Definition 1.6.10 are all homogeneous, the ideal generated by these relations is homogeneous and thus we have a natural $\mathbb{Z}$-grading on $\mathcal{P}_R(E)$. Note that $\mathcal{P}(E)$ is positively graded, and for any $m, n \in \mathbb{N}$,

$$\mathcal{P}(E)_m \, \mathcal{P}(E)_n = \mathcal{P}(E)_{m+n}.$$

However, by Proposition 1.1.15(2), $\mathcal{P}(E)$ is not a strongly $\mathbb{Z}$-graded ring.

The theory of Leavitt path algebras was introduced in [2, 5] which associate to directed graphs certain types of algebras. These algebras were motivated by Leavitt's construction of universal non-IBN rings [63]. Leavitt path algebras are quotients of path algebras by relations resembling those in the construction of algebras studied by Leavitt (see Example 1.3.19).

Definition 1.6.11 For a row-finite graph E and a ring R with identity, we define the *Leavitt path algebra of* E, denoted by $\mathcal{L}_R(E)$, to be the algebra generated by the sets $\{v \mid v \in E^0\}$, $\{\alpha \mid \alpha \in E^1\}$ and $\{\alpha^* \mid \alpha \in E^1\}$ with the coefficients in R, subject to the relations

1 $v_i v_j = \delta_{ij} v_i$ for every $v_i, v_j \in E^0$;
2 $s(\alpha)\alpha = \alpha r(\alpha) = \alpha$ and $r(\alpha)\alpha^* = \alpha^* s(\alpha) = \alpha^*$ for all $\alpha \in E^1$;
3 $\alpha^* \alpha' = \delta_{\alpha\alpha'} r(\alpha)$, for all $\alpha, \alpha' \in E^1$;
4 $\sum_{\{\alpha \in E^1, s(\alpha) = v\}} \alpha\alpha^* = v$ for every $v \in E^0$ for which $s^{-1}(v)$ is nonempty.

Here the ring R commutes with the generators $\{v, \alpha, \alpha^* \mid v \in E^0, \alpha \in E^1\}$. When the coefficient ring R is clear from the context, we simply write $\mathcal{L}(E)$ instead of $\mathcal{L}_R(E)$. When R is not commutative, then we consider $\mathcal{L}_R(E)$ as a

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left R-module. The elements α^* for $\alpha \in E^1$ are called *ghost edges*. One can show that $\mathcal{L}_R(E)$ is a ring with identity if and only if the graph E is finite (otherwise, $\mathcal{L}_R(E)$ is a ring with local identities, see [2, Lemma 1.6]).

Setting $\deg(v) = 0$, for $v \in E^0$, $\deg(\alpha) = 1$ and $\deg(\alpha^*) = -1$ for $\alpha \in E^1$, we obtain a natural $\mathbb{Z}$-grading on the free R-ring generated by $\{v, \alpha, \alpha^* \mid v \in E^0, \alpha \in E^1\}$. Since the relations in Definition 1.6.11 are all homogeneous, the ideal generated by these relations is homogeneous and thus we have a natural $\mathbb{Z}$-grading on $\mathcal{L}_R(E)$.

If $\mu = \mu_1 \dots \mu_k$, where $\mu_i \in E^1$, is an element of $\mathcal{L}(E)$, then we denote by μ^* the element $\mu_k^* \dots \mu_1^* \in \mathcal{L}(E)$. Further, we define $v^* = v$ for any $v \in E^0$. Since $\alpha^*\alpha' = \delta_{\alpha\alpha'} r(\alpha)$, for all $\alpha, \alpha' \in E^1$, any word in the generators $\{v, \alpha, \alpha^* \mid v \in E^0, \alpha \in E^1\}$ in $\mathcal{L}(E)$ can be written as $\mu\gamma^*$, where μ and γ are paths in E (recall that vertices were considered paths of length zero). The elements of the form $\mu\gamma^*$ are called *monomials*.

If the graph E is infinite, $\mathcal{L}_R(E)$ is a graded ring without identity (see Remark 1.1.14).

Taking the grading into account, one can write

$$\mathcal{L}_R(E) = \bigoplus_{k\in\mathbb{Z}} \mathcal{L}_R(E)_k,$$

where

$$\boxed{\mathcal{L}_R(E)_k = \Big\{ \sum_i r_i \alpha_i \beta_i^* \mid \alpha_i, \beta_i \text{ are paths}, r_i \in R, \text{ and } |\alpha_i| - |\beta_i| = k \text{ for all } i \Big\}.}$$

For simplicity we denote $\mathcal{L}_R(E)_k$, the homogeneous elements of degree k, by $\mathcal{L}_k$.

Example 1.6.12 A GRADED RING WHOSE MODULES ARE ALL GRADED

Consider the infinite line graph

$$E: \quad \cdots\cdots\to u_{-1} \xrightarrow{e_0} u_0 \xrightarrow{e_1} u_1 \cdots\cdots\to$$

Then the Leavitt path algebra $\mathcal{L}(E)$ is a $\mathbb{Z}$-graded ring. Let X be a right $\mathcal{L}(E)$-module. Set

$$X_i = X u_i, i \in \mathbb{Z},$$

and observe that $X = \bigoplus_{i\in\mathbb{Z}} X_i$. It is easy to check that X becomes a graded $\mathcal{L}(E)$-module. Moreover, any module homomorphism is a graded homomorphism. Note, however, that the module category Mod-$\mathcal{L}(E)$ is not equivalent to Gr-$\mathcal{L}(E)$. Also notice that although any ideal is a graded module over $\mathcal{L}(E)$, they are not graded ideals of $\mathcal{L}(E)$.

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The following theorem was proved in [47] and determines the finite graphs whose associated Leavitt path algebras are strongly graded.

Theorem 1.6.13 *Let E be a finite graph and K a field. Then $\mathcal{L}_K(E)$ is strongly graded if and only if E does not have sinks.*

The proof of this theorem is quite long and does not fit the purpose of this book. However, we can realise the Leavitt path algebras of finite graphs with no source in terms of corner skew Laurent polynomial rings (see §1.6.2). Using this representation, we can provide a short proof for the above theorem when the graph has no sources.

Example 1.6.14 LEAVITT PATH ALGEBRAS AS CORNER SKEW LAURENT RINGS

Let E be a finite graph with no source and $E^0 = \{v_1, \dots, v_n\}$ the set of all vertices of E. For each $1 \leq i \leq n$, we choose an edge e_i such that $r(e_i) = v_i$ and consider $t_+ = e_1 + \cdots + e_n \in \mathcal{L}(E)_1$. Then $t_- = e_1^* + \cdots + e_n^*$ is its right inverse. Thus by Theorem 1.6.9, $\mathcal{L}(E) = \mathcal{L}(E)_0[t_+, t_-, \phi]$, where

$$\begin{aligned} \phi : \mathcal{L}(E)_0 &\longrightarrow t_+t_-\,\mathcal{L}(E)_0 t_+t_- \\ a &\longmapsto t_+ a t_- \end{aligned}$$

Using this interpretation of Leavitt path algebras we are able to prove the following theorem.

Theorem 1.6.15 *Let E be a finite graph with no source and K a field. Then $\mathcal{L}_K(E)$ is strongly graded if and only if E does not have sinks.*

Proof Write $\mathcal{L}(E) = \mathcal{L}(E)_0[t_+, t_-, \phi]$, where $\phi(1) = t_+t_-$ (see Example 1.6.14). The theorem now follows from an easy to prove observation that t_+t_- is a full idempotent if and only if E does not have sinks, along with Proposition 1.6.6, that $\phi(1)$ is a full idempotent if and only if $\mathcal{L}(E)_0[t_+, t_-, \phi]$ is strongly graded. □

In the following theorem, we use the fact that $\mathcal{L}(E)_0$ is an *ultramatricial algebra*, *i.e.,* it is isomorphic to the union of an increasing countable chain of a finite product of matrix algebras over a field K (see §3.9.3).

Theorem 1.6.16 *Let E be a finite graph with no source and K a field. Then $\mathcal{L}_K(E)$ is crossed product if and only if E is a cycle.*

Proof Suppose E is a cycle with edges $\{e_1, e_2, \dots, e_n\}$. It is straightforward to check that $e_1 + e_2 + \cdots + e_n$ is an invertible element of degree 1. It then follows that each homogeneous component contains invertible elements and thus $\mathcal{L}(E)$ is crossed product.

Suppose now $\mathcal{L}(E)$ is crossed product. Write $\mathcal{L}(E) = \mathcal{L}(E)_0[t_+, t_-, \phi]$, where $\phi(1) = t_+t_-$ and $t_+ = e_1 + \cdots + e_n \in \mathcal{L}(E)_1$ (see Example 1.6.14). Since $\mathcal{L}(E)_0$ is an ultramatricial algebra, it is Dedekind finite, and thus by Proposition 1.6.7,

$$\phi(1) = e_1e_1^* + e_2e_2^* + \cdots + e_ne_n^* = v_1 + v_2 + \cdots + v_n.$$

From this it follows that (after suitable permutation), $e_ie_i^* = v_i$, for all $1 \leq i \leq n$. This in turn shows that only one edge emits from each vertex, *i.e.,* E is a cycle. □

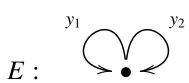
As a consequence of Theorem 1.6.13, we can show that Leavitt path algebras associated with finite graphs with no sinks are graded regular von Neumann rings (§1.1.9).

Corollary 1.6.17 *Let E be a finite graph with no sinks and K a field. Then $\mathcal{L}_K(E)$ is a graded von Neumann regular ring.*

Proof Since $\mathcal{L}(E)$ is strongly graded (Theorem 1.6.13), by Corollary 1.5.10, $\mathcal{L}(E)$ is von Neumann regular if $\mathcal{L}(E)_0$ is a von Neumann regular ring. But we know that the zero component ring $\mathcal{L}(E)_0$ is an ultramatricial algebra (§3.9.3) which is von Neumann regular (see the proof of [5, Theorem 5.3]). This finishes the proof. □

Example 1.6.18 LEAVITT PATH ALGEBRAS ARE NOT GRADED UNIT REGULAR RINGS

By analogy with the nongraded case, a graded ring is *graded von Neumann unit regular* (or *graded unit regular* for short) if for any homogeneous element x, there is an invertible homogeneous element y such that $xyx = x$. Clearly any graded unit regular ring is von Neumann regular. However, the converse is not the case. For example, Leavitt path algebras are not in general unit regular as the following example shows. Consider the graph:



Then it is easy to see that there is no homogeneous invertible element x such that $y_1xy_1 = y_1$ in $\mathcal{L}(E)$.

The following theorem determines the graded structure of Leavitt path algebras associated with acyclic graphs. It turns out that such algebras are natural examples of graded matrix rings (§1.3).

Theorem 1.6.19 *Let K be a field and E a finite acyclic graph with sinks $\{v_1, \dots, v_t\}$. For any sink v_s, let $R(v_s) = \{p_1^{v_s}, \dots, p_{n(v_s)}^{v_s}\}$ denote the set of all*

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paths ending at v_s. Then there is a $\mathbb{Z}$-graded isomorphism

$$\mathcal{L}_K(E) \cong_{\text{gr}} \bigoplus_{s=1}^{t} \mathbb{M}_{n(v_s)}(K)(|p_1^{v_s}|, \dots, |p_{n(v_s)}^{v_s}|). \tag{1.81}$$

Sketch of proof Fix a sink v_s and denote $R(v_s) = \{p_1, \dots, p_{n(v_s)}\}$. The set

$$I_{v_s} = \Big\{ \sum kp_ip_j^* \mid k \in K, p_i, p_j \in R(v_s) \Big\}$$

is an ideal of $\mathcal{L}_K(E)$, and we have an isomorphism

$$\begin{aligned} \phi : I_{v_s} &\longrightarrow \mathbb{M}_{n(v_s)}(K), \\ kp_ip_j^* &\longmapsto k(\mathbf{e}_{ij}), \end{aligned}$$

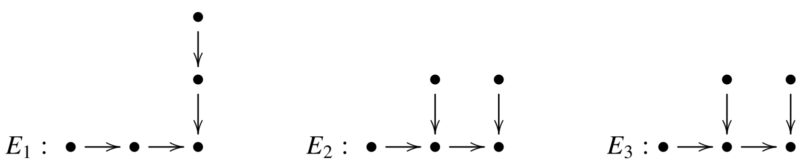
where $k \in K$, $p_i, p_j \in R(v_s)$ and $\mathbf{e}_{ij}$ is the standard matrix unit. Now, considering the grading on $\mathbb{M}_{n(v_s)}(K)(|p_1^{v_s}|, \dots, |p_{n(v_s)}^{v_s}|)$, we show that ϕ is a graded isomorphism. Let $p_ip_j^* \in I_{v_s}$. Then

$$\deg(p_ip_j^*) = |p_i| - |p_j| = \deg(\mathbf{e}_{ij}) = \deg(\phi(p_ip_j^*)).$$

So ϕ respects the grading. Hence ϕ is a graded isomorphism. One can check that

$$\mathcal{L}_K(E) = \bigoplus_{s=1}^{t} I_{v_s} \cong_{\text{gr}} \bigoplus_{s=1}^{t} \mathbb{M}_{n(v_s)}(K)(|p_1^{v_s}|, \dots, |p_{n(v_s)}^{v_s}|). \qquad \square$$

Example 1.6.20 Consider the following graphs:



Theorem 1.6.19 shows that the Leavitt path algebras of the graphs E_1 and E_2 with coefficients from the field K are graded isomorphic to $\mathbb{M}_5(K)(0, 1, 1, 2, 2)$ and thus $\mathcal{L}(E_1) \cong_{\text{gr}} \mathcal{L}(E_2)$. However

$$\mathcal{L}(E_3) \cong_{\text{gr}} \mathbb{M}_5(K)(0, 1, 2, 2, 3).$$

Similar to Theorem 1.6.19, we can characterise the graded structure of Leavitt path algebras associated with comet graphs.

Theorem 1.6.21 *Let K be a field and E a C_n-comet with the cycle C of length $n \geq 1$. Let v be a vertex on the cycle C and e be the edge in the cycle with $s(e) = v$. Eliminate the edge e and consider the set $\{p_i \mid 1 \leq i \leq m\}$ of all paths with end in v. Then*

$$\mathcal{L}_K(E) \cong_{\text{gr}} \mathbb{M}_m(K[x^n, x^{-n}])(|p_1|, \dots, |p_m|). \tag{1.82}$$

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Sketch of proof One can show that the set of monomials

$$\left\{ p_i C^k p_j^* \mid 1 \le i, j \le m, k \in \mathbb{Z} \right\}$$

is a basis of $\mathcal{L}_K(E)$ as a K-vector space. Define a map

$$\phi : \mathcal{L}_K(E) \longrightarrow \mathbb{M}_m(K[x^n, x^{-n}])(|p_1|, \ldots, |p_m|), \text{ by } \quad \phi(p_i C^k p_j^*) = \mathbf{e}_{ij}(x^{kn}),$$

where $\mathbf{e}_{ij}(x^{kn})$ is a matrix with x^{kn} in the ij-position and zero elsewhere. Extend this linearly to $\mathcal{L}_K(E)$. We have

$$\begin{aligned}\phi((p_i C^k p_j^*)(p_r C^t p_s^*)) &= \phi(\delta_{jr} p_i C^{k+t} p_s^*)\\ &= \delta_{jr} \mathbf{e}_{is}(x^{(k+t)n})\\ &= (\mathbf{e}_{ij} x^{kn})(\mathbf{e}_{rs} x^{tn})\\ &= \phi(p_i C^k p_j^*)\phi(p_r C^t p_s^*).\end{aligned}$$

Thus ϕ is a homomorphism. Also, ϕ sends the basis to the basis, so ϕ is an isomorphism.
We now need to show that ϕ is graded. We have

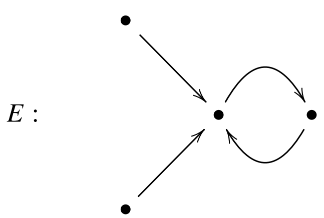
$$\deg(p_i C^k p_j^*) = |p_i C^k p_j^*| = nk + |p_i| - |p_j|$$

and

$$\deg(\phi(p_i C^k p_j^*)) = \deg(\mathbf{e}_{ij}(x^{kn})) = nk + |p_i| - |p_j|.$$

Therefore ϕ respects the grading. This finishes the proof. □

Example 1.6.22 Consider the Leavitt path algebra $\mathcal{L}_K(E)$, with coefficients in a field K, associated with the following graph:



By Theorem 1.6.13, $\mathcal{L}_K(E)$ is strongly graded. Now by Theorem 1.6.21,

$$\mathcal{L}_K(E) \cong_{\text{gr}} \mathbb{M}_4(K[x^2, x^{-2}])(0, 1, 1, 1). \tag{1.83}$$

However, this algebra is not crossed product. Set $B = K[x, x^{-1}]$ with the grading $B = \bigoplus_{n\in\mathbb{Z}} Kx^n$ and consider $A = K[x^2, x^{-2}]$ as a graded subring of B with

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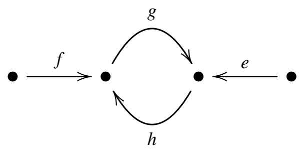
$A_n = Kx^n$ if $n \equiv 0 \pmod 2$, and $A_n = 0$ otherwise. Using the graded isomorphism of (1.83), by (1.45) a homogeneous element of degree 1 in $\mathcal{L}_K(E)$ has the form

$$\begin{pmatrix} A_1 & A_2 & A_2 & A_2 \\ A_0 & A_1 & A_1 & A_1 \\ A_0 & A_1 & A_1 & A_1 \\ A_0 & A_1 & A_1 & A_1 \end{pmatrix}.$$

Since $A_1 = 0$, the determinants of these matrices are zero, and thus no homogeneous element of degree 1 is invertible. Thus $\mathcal{L}_K(E)$ is not crossed product (see §1.1.3).

Now consider the following graph:

$E :$



By Theorem 1.6.21,

$$\mathcal{L}_K(E) \cong_{\mathrm{gr}} \mathbb{M}_4(K[x^2, x^{-2}])(0, 1, 1, 2). \tag{1.84}$$

Using the graded isomorphism of (1.84), by (1.47) homogeneous elements of degree 0 in $\mathcal{L}_K(E)$ have the form

$$\mathcal{L}_K(E)_0 = \begin{pmatrix} A_0 & A_1 & A_1 & A_2 \\ A_{-1} & A_0 & A_0 & A_1 \\ A_{-1} & A_0 & A_0 & A_1 \\ A_{-2} & A_{-1} & A_{-1} & A_0 \end{pmatrix} = \begin{pmatrix} K & 0 & 0 & Kx^2 \\ 0 & K & K & 0 \\ 0 & K & K & 0 \\ Kx^{-2} & 0 & 0 & K \end{pmatrix}.$$

In the same manner, homogeneous elements of degree 1 have the form

$$\mathcal{L}_K(E)_1 = \begin{pmatrix} 0 & Kx^2 & Kx^2 & 0 \\ K & 0 & 0 & Kx^2 \\ K & 0 & 0 & Kx^2 \\ 0 & K & K & 0 \end{pmatrix}.$$

Choose

$$u = \begin{pmatrix} 0 & 0 & x^2 & 0 \\ 0 & 0 & 0 & x^2 \\ 1 & 0 & 0 & x^2 \\ 0 & 1 & 0 & 0 \end{pmatrix} \in \mathcal{L}(E)_1$$

and observe that u is invertible; this matrix corresponds to the element

$$g + h + fge^* + ehf^* \in \mathcal{L}_K(E)_1.$$

Thus $\mathcal{L}_K(E)$ is crossed product and therefore a skew group ring as the grading is cyclic (see §1.1.4), *i.e.,*

$$\mathcal{L}_K(E) \cong_{\text{gr}} \bigoplus_{i\in\mathbb{Z}} \mathcal{L}_K(E)_0 u^i$$

and a simple calculation shows that one can describe this algebra as follows:

$$\mathcal{L}_K(E)_0 \cong \mathbb{M}_2(K) \times \mathbb{M}_2(K)$$

and

$$\mathcal{L}_K(E) \cong_{\text{gr}} \big(\mathbb{M}_2(K) \times \mathbb{M}_2(K)\big) \star_\tau \mathbb{Z}, \tag{1.85}$$

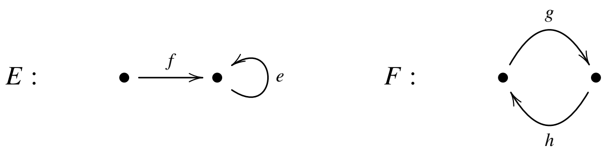
where

$$\tau\left(\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}\right) = \left(\begin{pmatrix} b_{22} & b_{21} \\ b_{12} & b_{11} \end{pmatrix}, \begin{pmatrix} a_{22} & a_{21} \\ a_{12} & a_{11} \end{pmatrix}\right).$$

Remark 1.6.23 Noncanonical gradings on Leavitt path algebras

For a graph E, the Leavitt path algebra $\mathcal{L}_K(E)$ has a canonical $\mathbb{Z}$-graded structure. This grading was obtained by assigning 0 to vertices, 1 to edges and -1 to ghost edges. However, one can equip $\mathcal{L}_K(E)$ with other graded structures as well. Let Γ be an arbitrary group with the identity element e. Let $w : E^1 \to \Gamma$ be a *weight* map and further define $w(\alpha^*) = w(\alpha)^{-1}$, for any edge $\alpha \in E^1$ and $w(v) = e$ for $v \in E^0$. The free K-algebra generated by the vertices, edges and ghost edges is a Γ-graded K-algebra (see §1.6.1). Moreover, the Leavitt path algebra is the quotient of this algebra by relations in Definition 1.6.11 which are all homogeneous. Thus $\mathcal{L}_K(E)$ is a Γ-graded K-algebra. One can write Theorems 1.6.19 and 1.6.21 with this general grading.

As an example, consider the graphs



and assign 1 for the degree of f, 2 for the degree of e in E and 1 for the degrees of g and h in F. Then the proof of Theorem 1.6.21 shows that

$$\mathcal{L}_K(E) \cong \mathbb{M}_2(K[x^2, x^{-2}])(0, 1)$$

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and

$$\mathcal{L}_K(F) \cong \mathbb{M}_2(K[x^2, x^{-2}])(0, 1)$$

are $\mathbb{Z}$-graded rings. So with these gradings, $\mathcal{L}_K(E) \cong_{\text{gr}} \mathcal{L}_K(F)$.

Example 1.6.24 LEAVITT PATH ALGEBRAS ARE STRONGLY $\mathbb{Z}_2$-GRADED

Let E be a (connected) row-finite graph with at least one edge. By Remark 1.6.23, $A = \mathcal{L}_K(E)$ has a $\mathbb{Z}_2$-grading induced by assigning 0 to vertices and $1 \in \mathbb{Z}_2$ to edges and ghost edges. Since the defining relations of Leavitt path algebras guarantee that for any $v \in E^0$, $v \in A_1 A_1$, one can easily check that $\mathcal{L}_K(E)$ is strongly $\mathbb{Z}_2$-graded for any graph (compare this with Theorem 1.6.13). In contrast to the canonical grading, in this case the 0-component ring is not necessarily an ultramatricial ring (see §3.9.3).

1.7 The graded IBN and graded type

A ring A with identity has an *invariant basis number* (IBN) or *invariant basis property* if any two bases of a free (right) A-module have the same cardinality, *i.e.,* if $A^n \cong A^m$ as A-modules, then $n = m$. When A does not have IBN, the *type* of A is defined as a pair of positive integers (n, k) such that $A^n \cong A^{n+k}$ as A-modules and these are the smallest number with this property, that is, (n, k) is the minimum under the usual lexicographic order. This means any two bases of a free A-module have the unique cardinality if one of the bases has the cardinality less than n and, further, if a free module has rank n, then a free module with the smallest cardinality (other than n) isomorphic to this module is of rank $n + k$. Another way to describe a type (n, k) is that $A^n \cong A^{n+k}$ is the first repetition in the list $A, A^2, A^3, \ldots$.

It was shown that if A has type (n, k), then $A^m \cong A^{m'}$ if and only if $m = m'$ or $m, m' \geq n$ and $m \equiv m' \pmod{k}$ (see [28, p. 225], [63, Theorem 1]).

One can show that a (right) Noetherian ring has IBN. Moreover, if there is a ring homomorphism $A \to B$, (which preserves 1), and B has IBN then A has IBN as well. Indeed, if $A^m \cong A^n$ then

$$B^m \cong A^m \otimes_A B \cong A^n \otimes_A B \cong B^n, \tag{1.86}$$

so $n = m$. One can describe the type of a ring by using the monoid of isomorphism classes of finitely generated projective modules (see Example 3.1.4). For nice discussions about these rings see [21, 28, 67].

A graded ring A has a *graded invariant basis number* (gr-IBN) if any two

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homogeneous bases of a graded free (right) A-module have the same cardinality, *i.e.,* if $A^m(\overline{\alpha}) \cong_{\mathrm{gr}} A^n(\overline{\delta})$, where $\overline{\alpha} = (\alpha_1, \dots, \alpha_m)$ and $\overline{\delta} = (\delta_1, \dots, \delta_n)$, then $m = n$. Note that, in contrast to the nongraded case, this does not imply that two graded free modules with bases of the same cardinality are graded isomorphic (see Proposition 1.3.16). A graded ring A has *IBN in* gr-A, if $A^m \cong_{\mathrm{gr}} A^n$ then $m = n$. If A has IBN in gr-A, then A_0 has IBN. Indeed, if $A_0^m \cong A_0^n$ as A_0-modules, then, similarly to (1.86),

$$A^m \cong_{\mathrm{gr}} A_0^m \otimes_{A_0} A \cong A_0^n \otimes_{A_0} A \cong_{\mathrm{gr}} A^n,$$

so $n = m$ (see [75, p. 215]).

When the graded ring A does not have gr-IBN, the *graded type* of A is defined as a pair of positive integers (n, k) such that $A^n(\overline{\delta}) \cong_{\mathrm{gr}} A^{n+k}(\overline{\alpha})$ as A-modules, for some $\overline{\delta} = (\delta_1, \dots, \delta_n)$ and $\overline{\alpha} = (\alpha_1, \dots, \alpha_{n+k})$ and these are the smallest number with this property. In Proposition 1.7.1 we show that the Leavitt algebra $\mathcal{L}(n, k+1)$ (see Example 1.6.5) has graded type (n, k).

Parallel to the nongraded setting, one can show that a graded (right) Noetherian ring has gr-IBN. Moreover, if there is a graded ring homomorphism $A \to B$, (which preserves 1), and B has gr-IBN then A has gr-IBN as well. Indeed, if $A^m(\overline{\alpha}) \cong_{\mathrm{gr}} A^n(\overline{\delta})$, where $\overline{\alpha} = (\alpha_1, \dots, \alpha_m)$ and $\overline{\delta} = (\delta_1, \dots, \delta_n)$, then

$$B^m(\overline{\alpha}) \cong_{\mathrm{gr}} A^m(\overline{\alpha}) \otimes_A B \cong A^n(\overline{\delta}) \otimes_A B \cong_{\mathrm{gr}} B^n(\overline{\delta}),$$

which implies $n = m$. Using this, one can show that any graded commutative ring has gr-IBN. For, there exists a graded maximal ideal and its quotient ring is a graded field which has gr-IBN (see §1.1.5 and Proposition 1.4.2).

Let A be a Γ-graded ring such that $A^m(\overline{\alpha}) \cong_{\mathrm{gr}} A^n(\overline{\delta})$, where $\overline{\alpha} = (\alpha_1, \dots, \alpha_m)$ and $\overline{\delta} = (\delta_1, \dots, \delta_n)$. Then there is a universal Γ-graded ring R such that

$$R^m(\overline{\alpha}) \cong_{\mathrm{gr}} R^n(\overline{\delta})$$

and a graded ring homomorphism $R \to A$ which induces the graded isomorphism

$$A^m(\overline{\alpha}) \cong_{\mathrm{gr}} R^m(\overline{\alpha}) \otimes_R A \cong_{\mathrm{gr}} R^n(\overline{\delta}) \otimes_R A \cong_{\mathrm{gr}} A^n(\overline{\delta}).$$

Indeed, by Proposition 1.3.16, there are matrices $a = (a_{ij}) \in \mathbb{M}_{n\times m}(A)[\overline{\delta}][\overline{\alpha}]$ and $b = (b_{ij}) \in \mathbb{M}_{m\times n}(A)[\overline{\alpha}][\overline{\delta}]$ such that $ab = \mathbb{I}_n$ and $ba = \mathbb{I}_m$. The free ring generated by symbols in place of a_{ij} and b_{ij} subject to relations imposed by $ab = \mathbb{I}_n$ and $ba = \mathbb{I}_m$ is the desired universal graded ring. In detail, let F be a free ring generated by x_{ij}, $1 \leq i \leq n$, $1 \leq j \leq m$ and y_{ij}, $1 \leq i \leq m$, $1 \leq j \leq n$. Assign the degrees $\deg(x_{ij}) = \delta_i - \alpha_j$ and $\deg(y_{ij}) = \alpha_i - \delta_j$ (see §1.6.1). This makes F a Γ-graded ring. Let R be a ring F modulo the relations $\sum_{s=1}^{m} x_{is} y_{sk} = \delta_{ik}$, $1 \leq i, k \leq n$ and $\sum_{t=1}^{n} y_{it} x_{tk} = \delta_{ik}$, $1 \leq i, k \leq m$, where δ_{ik} is the Kronecker delta. Since all the relations are homogeneous, R is a Γ-graded ring. Clearly the map sending x_{ij} to a_{ij} and y_{ij} to b_{ij} induces a graded ring homomorphism $R \to A$. Again Proposition 1.3.16 shows that $R^m(\overline{\alpha}) \cong_{\text{gr}} R^n(\overline{\delta})$.

Proposition 1.7.1 *Let $R = \mathcal{L}(n, k+1)$ be the Leavitt algebra of type (n, k). Then*

(1) *R is a universal $\bigoplus_n \mathbb{Z}$-graded ring which does not have gr-IBN;*
(2) *R has graded type (n, k);*
(3) *for $n = 1$, R has IBN in* gr-R.

Proof (1) Consider the algebra $\mathcal{L}(n, k+1)$ constructed in Example 1.6.5, which is a $\bigoplus_n \mathbb{Z}$-graded ring and is universal. Moreover, (1.77) combined with Proposition 1.3.16(3) shows that $R^n \cong_{\text{gr}} R^{n+k}(\overline{\alpha})$. Here $\overline{\alpha} = (\alpha_1, \dots, \alpha_{n+k})$, where $\alpha_i = (0, \dots, 0, 1, 0 \dots, 0)$ and 1 is in the ith entry. This shows that $R = \mathcal{L}(n, k+1)$ does not have gr-IBN.

(2) By [28, Theorem 6.1], R is of type (n, k). This immediately implies the graded type of R is also (n, k).

(3) Suppose $R^n \cong_{\text{gr}} R^m$ as graded R-modules. Then $R_0^n \cong R_0^m$ as R_0-modules. But R_0 is an ultramatricial algebra, *i.e.,* the direct limit of an increasing chain of a finite product of matrices over a field. Since IBN respects direct limits ([28, Theorem 2.3]), R_0 has IBN. Therefore, $n = m$. □

Remark 1.7.2 Assignment of $\deg(y_{ij}) = 1$ and $\deg(x_{ij}) = -1$, for all i, j, makes $R = \mathcal{L}(n, k+1)$ a $\mathbb{Z}$-graded algebra of graded type (n, k) with $R^n \cong_{\text{gr}} R^{n+k}(1)$.

Remark 1.7.3 Let A be a Γ-graded ring. In [77, Proposition 4.4], it was shown that if Γ is finite then A has gr-IBN if and only if A has IBN.

1.8 The graded stable rank

The notion of stable rank was defined by H. Bass [14] to study the K_1-group of rings that are finitely generated over commutative rings with finite Krull dimension. For a concise introduction to the stable rank, we refer the reader to [59, 60], and for its applications to K-theory to [13, 14]. It seems that the natural notion of graded stable rank in the context of graded ring theory has not yet been investigated in the literature. In this section we propose a definition for the graded stable rank and study the important case of graded rings with graded stable rank 1. This will be used later in Chapter 3 in relation to graded Grothendieck groups.

A row $(a_1, \dots, a_n)$ of homogeneous elements of a Γ-graded ring A is called a *graded left unimodular row* if the graded left ideal generated by a_i, $1 \leq i \leq n$, is A.

Lemma 1.8.1 *Let $(a_1, \dots, a_n)$ be a row of homogeneous elements of a Γ-graded ring A. The following are equivalent:*

(1) *$(a_1, \dots, a_n)$ is a left unimodular row;*
(2) *$(a_1, \dots, a_n)$ is a graded left unimodular row;*
(3) *the graded homomorphism*

$$\phi_{(a_1,\dots,a_n)} : A^n(\overline{\alpha}) \longrightarrow A,$$
$$(x_1, \dots, x_n) \longmapsto \sum_{i=1}^{n} x_i a_i,$$

where $\overline{\alpha} = (\alpha_1, \dots, \alpha_n)$ and $\alpha_i = -\deg(a_i)$, is surjective.

Proof The proof is straightforward. □

When $n \geq 2$, a graded left unimodular row $(a_1, \dots, a_n)$ is called *stable* if there exist homogeneous elements $b_1, \dots, b_{n-1}$ of A such that the graded left ideal generated by homogeneous elements $a_i + b_i a_n$, $1 \leq i \leq n-1$, is A.

The *graded left stable rank* of a ring A is defined to be n, denoted $\mathrm{sr}^{\mathrm{gr}}(A) = n$, if any graded unimodular row of length $n + 1$ is stable, but there exists an unstable unimodular row of length n. If such an n does not exist (*i.e.,* there are unstable unimodular rows of arbitrary length) we say that the graded stable rank of A is infinite.

In order that this definition is well-defined, one needs to show that if any graded unimodular row of fixed length n is stable, so is any unimodular row of a greater length. This can be proved similarly to the nongraded case and we omit the proof (see for example [59, Proposition 1.3]).

When the grade group Γ is a trivial group, the above definitions reduce to the standard definitions of unimodular rows and stable ranks.

The case of graded stable rank 1 is of special importance. Suppose A is a Γ-graded with $\mathrm{sr}^{\mathrm{gr}}(A) = 1$. Then from the definition it follows that, if $a, b \in A^h$ such that $Aa + Ab = A$, then there is a homogeneous element c such that the homogeneous element $a + cb$ is left invertible. When $\mathrm{sr}^{\mathrm{gr}}(A) = 1$, any left invertible homogeneous element is in fact invertible. For, suppose $c \in A^h$ is left invertible, *i.e.,* there is an $a \in A^h$ such that $ac = 1$. Then the row $(a, 1 - ca)$ is graded left unimodular. Thus, there is an $s \in A^h$ such that $u := a + s(1 - ca)$ is left invertible. But $uc = 1$. Thus u is (left and right) invertible and consequently, c is an invertible homogeneous element.

The graded stable rank 1 is quite a strong condition. In fact if $\operatorname{sr}^{\mathrm{gr}}(A) = 1$ then $\Gamma_A = \Gamma_A^*$. For, if $a \in A_\gamma$ is a nonzero element, then since $(a, 1)$ is unimodular and $\operatorname{sr}^{\mathrm{gr}}(A) = 1$, there is a $c \in A^h$, such that $a + c$ is an invertible homogeneous element, necessarily of degree γ. Thus $\gamma \in \Gamma_A^*$.

Example 1.8.2 GRADED DIVISION RINGS HAVE GRADED STABLE RANK 1

Since any nonzero homogeneous element of a graded division ring is invertible, one shows easily that its graded stable rank is 1. Thus for a field K, $\operatorname{sr}^{\mathrm{gr}}(K[x, x^{-1}]) = 1$, whereas $\operatorname{sr}(K[x, x^{-1}]) = 2$.

Example 1.8.3 FOR A STRONGLY GRADED RING A, $\operatorname{sr}^{\mathrm{gr}}(A) \neq \operatorname{sr}(A_0)$

Let $A = \mathcal{L}(1, 2)$ be the Leavitt algebra generated by x_1, x_2, y_1, y_2 (see Example 1.3.19). Then relations (1.60) show that y_1 is left invertible but it is not invertible. This shows that $\operatorname{sr}^{\mathrm{gr}}(A) \neq 1$. On the other hand, since A_0 is an ultramatricial algebra, $\operatorname{sr}(A_0) = 1$ (see §3.9.3, [59, Corollary 5.5] and [40]).

We have the following theorem which is a graded version of the cancellation theorem with a similar proof (see [60, Theorem 20.11]).

Theorem 1.8.4 (GRADED CANCELLATION THEOREM) *Let A be a Γ-graded ring and let M, N, P be graded right A-modules, with P being finitely generated. If the graded ring $\operatorname{End}_A(P)$ has graded left stable rank 1, then $P \oplus M \cong_{\mathrm{gr}} P \oplus N$ as A-modules implies $M \cong_{\mathrm{gr}} N$ as A-modules.*

Proof Set $E := \operatorname{End}_A(P)$. Let $h : P \oplus M \to P \oplus N$ be a graded A-module isomorphism. Then the composition of the maps

$$P \xrightarrow{i_1} P \oplus M \xrightarrow{h} P \oplus N \xrightarrow{\pi_1} P,$$

$$M \xrightarrow{i_2} P \oplus M \xrightarrow{h} P \oplus N \xrightarrow{\pi_1} P$$

induces a graded split epimorphism of degree zero, denoted by $(f, g) : P \oplus M \to P$. Here $(f, g)(p, m) = f(p) + g(m)$, where $f = \pi_1 h i_1$ and $g = \pi_1 h i_2$. It is clear that $\ker(f, g) \cong_{\mathrm{gr}} N$. Let $\begin{pmatrix} f' \\ g' \end{pmatrix} : P \to P \oplus M$ be the split homomorphism. Thus

$$1 = (f, g)\begin{pmatrix} f' \\ g' \end{pmatrix} = f f' + g g'.$$

This shows that the left ideal generated by f' and gg' is E. Since E has graded stable rank 1, it follows there is an $e \in E$ of degree 0 such that $u := f' + e(gg')$ is an invertible element of E. Writing $u = (1, eg)\begin{pmatrix} f' \\ g' \end{pmatrix}$ implies that both $\ker(f, g)$

and $\ker(1, eg)$ are graded isomorphic to

$$P \oplus M/\operatorname{Im}\begin{pmatrix} f' \\ g' \end{pmatrix}.$$

Thus $\ker(f, g) \cong_{\mathrm{gr}} \ker(1, eg)$. But $\ker(f, g) \cong_{\mathrm{gr}} N$ and $\ker(1, eg) \cong_{\mathrm{gr}} M$. Thus $M \cong_{\mathrm{gr}} N$. □

The following corollary will be used in Chapter 3 to show that for a graded ring with graded stable rank 1, the monoid of graded finitely generated projective modules injects into the graded Grothendieck group (see Corollary 3.1.8).

Corollary 1.8.5 *Let A be a Γ-graded ring with graded left stable rank* 1 *and M, N, P be graded right A-modules. If P is a graded finitely generated projective A-module, then $P \oplus M \cong_{\mathrm{gr}} P \oplus N$ as A-modules implies $M \cong_{\mathrm{gr}} N$ as A-modules.*

Proof Suppose $P \oplus M \cong_{\mathrm{gr}} P \oplus N$ as A-modules. Since P is a graded finitely generated A-module, there is a graded A-module Q such that $P \oplus Q \cong_{\mathrm{gr}} A^n(\overline{\alpha})$ (see (1.39)). It follows that

$$A^n(\overline{\alpha}) \oplus M \cong_{\mathrm{gr}} A^n(\overline{\alpha}) \oplus N.$$

We prove that if

$$A(\alpha) \oplus M \cong_{\mathrm{gr}} A(\alpha) \oplus N, \tag{1.87}$$

then $M \cong_{\mathrm{gr}} N$. The corollary then follows by an easy induction.

By (1.48) there is a graded ring isomorphism $\mathrm{End}_A(A(\alpha)) \cong_{\mathrm{gr}} A$. Since A has graded stable rank 1, so does $\mathrm{End}_A(A(\alpha))$. Now by Theorem 1.8.4, from (1.87) it follows that $M \cong_{\mathrm{gr}} N$. This finishes the proof. □

The stable rank imposes other finiteness properties on rings such as the IBN property (see [95, Exercise I.1.5(e)]).

Theorem 1.8.6 *Let A be a Γ-graded ring such that $A \cong_{\mathrm{gr}} A^r(\overline{\alpha})$ as left A-modules, for some $\overline{\alpha} = (\alpha_1, \dots, \alpha_r)$, $r > 1$. Then the graded stable rank of A is infinite.*

Proof Suppose the graded stable rank of A is n. Then one can find $\overline{\alpha} = (\alpha_1, \dots, \alpha_r)$, where $r > n$ and $A^r(\overline{\alpha}) \cong_{\mathrm{gr}} A$. Suppose $\phi : A^r(\overline{\alpha}) \to A$ is this given graded isomorphism. Set $a_i = \phi(e_i)$, $1 \leq i \leq r$, where $\{e_i \mid 1 \leq i \leq r\}$ are the standard (homogeneous) basis of A^r. Then for any $x \in A^r(\overline{\alpha})$,

$$\phi(x) = \phi(\sum_{i=1}^{r} x_i e_i) = \sum_{i=1}^{r} x_i a_i = \phi_{(a_1,\dots,a_r)}(x_1, \dots, x_r).$$

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Since ϕ is an isomorphism, by Lemma 1.8.1, the row $(a_1, \dots, a_r)$ is graded left unimodular. Since $r > \operatorname{sr}^{\mathrm{gr}}(A)$, there is a homogeneous row $(b_1, \dots, b_{r-1})$ such that $(a_1 + b_1 a_r, \dots, a_{r-1} + b_{r-1} a_r)$ is also left unimodular. Note that $\deg(b_i) = \alpha_r - \alpha_i$. Consider the graded left A-module homomorphism

$$\psi : A^{r-1}(\alpha_1, \dots, \alpha_{r-1}) \longrightarrow A^r(\alpha_1, \dots, \alpha_{r-1}, \alpha_r),$$
$$(x_1, \dots, x_{r-1}) \longmapsto (x_1, \dots, x_{r-1}, \sum_{i=1}^{r-1} x_i b_i)$$

and the commutative diagram

$$\begin{array}{ccc} A^{r-1}(\alpha_1, \dots, \alpha_{r-1}) & \xrightarrow{\ \psi\ } & A^r(\alpha_1, \dots, \alpha_{r-1}, \alpha_r)\,. \\ & {\scriptstyle \phi_{(a_1+b_1a_r,\dots,a_{r-1}+b_{r-1}a_r)}} \searrow \quad \swarrow {\scriptstyle \phi_{(a_1,\dots,a_r)}} & \\ & A & \end{array}$$

Since $\phi_{(a_1,\dots,a_r)}$ is an isomorphism and $\phi_{(a_1+b_1a_r,\dots,a_{r-1}+b_{r-1}a_r)}$ is an epimorphism, ψ is also an epimorphism. Thus there is $(x_1, \dots, x_{r-1})$ such that $\psi(x_1, \dots, x_{r-1}) = (0, \dots, 0, 1)$ which immediately gives a contradiction. □

Corollary 1.8.7 *The graded stable rank of the Leavitt algebra $\mathcal{L}(1, n)$ is infinite.*

Proof This follows from Proposition 1.7.1 and Theorem 1.8.6. □

Example 1.8.8 GRADED VON NEUMANN REGULAR RINGS WITH STABLE RANK 1

One can prove, similarly to the nongraded case [40, Proposition 4.12], that a graded von Neumann regular ring has a stable rank 1 if and only if it is a graded von Neumann unit regular.

1.9 Graded rings with involution

Let A be a ring with an involution denoted by *, *i.e.,* $^* : A \to A$, $a \mapsto a^*$, is an anti-automorphism of order two. Throughout this book we call A also a $*$-ring. If M is a right A-module, then M can be given a left A-module structure by defining

$$am := ma^*. \tag{1.88}$$

This gives an equivalent

$$\text{Mod-}A \approx A\text{-Mod}, \tag{1.89}$$

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where Mod-A is the category of right A-modules and A-Mod is the category of left A-modules.

Now let $A = \bigoplus_{\gamma\in\Gamma} A_\gamma$ be a Γ-graded ring. We call A a *graded $*$-ring* if there is an involution on A such that for $a \in A_\gamma$, $a^* \in A_{-\gamma}$, where $\gamma \in \Gamma$. It follows that $A_\gamma^* = A_{-\gamma}$, for any $\gamma \in \Gamma$.

Remark 1.9.1 Depending on the circumstances, one can also set another definition that $A_\gamma^* = A_\gamma$, where $\gamma \in \Gamma$.

If A is a graded $*$-ring, and M is a graded right A-module, then the multiplication in (1.88) makes M a graded left $A^{(-1)}$-module and makes $M^{(-1)}$ a graded left A-module, where $A^{(-1)}$ and $M^{(-1)}$ are Veronese rings and modules (see Examples 1.1.19 and 1.2.7). These give graded equivalences

$$\begin{aligned} \mathcal{I} : \text{Gr-}A &\longrightarrow A^{(-1)}\text{-Gr}, \\ M &\longmapsto M \end{aligned} \tag{1.90}$$

and

$$\begin{aligned} \mathcal{J} : \text{Gr-}A &\longrightarrow A\text{-Gr}, \\ M &\longmapsto M^{(-1)}. \end{aligned} \tag{1.91}$$

Here for $\alpha \in \Gamma$, $\mathcal{I}(M(\alpha)) = \mathcal{I}(M)(\alpha)$ (*i.e.,* $\mathcal{I}$ is a graded functor, see Definition 2.3.3), whereas $\mathcal{J}(M(\alpha)) = \mathcal{J}(M)(-\alpha)$.

Clearly, if the grade group Γ is trivial, the equivalences (1.90) and (1.91) both reduce to (1.89).

Let A be a graded $*$-field (*i.e.,* a graded field with $*$-involution) and R a graded A-algebra with involution denoted by * again. Then R is a graded $*$-A-algebra if $(ar)^* = a^*r^*$ (*i.e.,* the graded homomorphism $A \to R$ is a $*$-homomorphism).

Example 1.9.2 <u>GROUP RINGS</u>

For a group Γ (denoted multiplicatively here), the group ring $\mathbb{Z}[\Gamma]$ with a natural Γ-grading

$$\mathbb{Z}[\Gamma] = \bigoplus_{\gamma\in\Gamma} \mathbb{Z}[\Gamma]_\gamma, \text{ where } \mathbb{Z}[\Gamma]_\gamma = \mathbb{Z}\gamma,$$

and the natural involution $* : \mathbb{Z}[\Gamma] \to \mathbb{Z}[\Gamma], \gamma \mapsto \gamma^{-1}$ is a graded $*$-ring.

Example 1.9.3 <u>HERMITIAN TRANSPOSE</u>

If A is a graded $*$-ring, then for $a = (a_{ij}) \in \mathbb{M}_n(A)(\delta_1, \dots, \delta_n)$, the *Hermitian transpose* $a^* = (a_{ji}^*)$, makes $\mathbb{M}_n(A)(\delta_1, \dots, \delta_n)$ a graded $*$-ring (see 1.45).

Example 1.9.4 LEAVITT PATH ALGEBRAS ARE GRADED $*$-ALGEBRAS

A Leavitt path algebra has a natural $*$-involution. Let K be a $*$-field (*i.e.,* a field with $*$-involution). Define a homomorphism from the free K-algebra generated by the vertices, edges and ghost edges of the graph E to $\mathcal{L}(E)^{\text{op}}$, by $k \mapsto k^*$, $v \mapsto v$, $\alpha \mapsto \alpha^*$ and $\alpha^* \mapsto \alpha$, where $k \in K$, $v \in E^0$, $\alpha \in E^1$ and α^* is the ghost edge. The relations in the definition of a Leavitt path algebra, Definition 1.6.11, show that this homomorphism induces an isomorphism from $\mathcal{L}(E)$ to $\mathcal{L}(E)^{\text{op}}$. This makes $\mathcal{L}(E)$ a $*$-algebra. Moreover, considering the grading, it is easy to see that in fact, $\mathcal{L}(E)$ is a graded $*$-algebra.

Example 1.9.5 CORNER SKEW LAURENT RINGS AS GRADED $*$-ALGEBRAS

Recall the corner skew Laurent polynomial ring $A = R[t_+, t_-, \phi]$, where R is a ring with identity and $\phi : R \to pRp$ a corner isomorphism (see §1.6.2). Let R be a $*$-ring, p a *projection* (*i.e.,* $p = p^* = p^2$), and ϕ a $*$-isomorphism. Then A has a $*$-involution defined on generators by $(t_-^j r_{-j})^* = r_{-j}^* t_+^j$ and $(r_i t_+^i)^* = t_-^i r_i^*$. With this involution A becomes a graded $*$-ring. A $*$-ring is called *$*$-proper*, if $xx^* = 0$ implies $x = 0$. It is called *positive-definite* if $\sum_{i=1}^n x_i x_i{}^* = 0$, $n \in \mathbb{N}$, implies $x_i = 0$, $1 \leq i \leq n$. A graded $*$-ring is called *graded $*$-proper*, if $x \in A^h$ and $xx^* = 0$ then $x = 0$. The following lemma is easy to prove and we leave part of it to the reader.

Lemma 1.9.6 *Let R be a $*$-ring and $A = R[t_+, t_-, \phi]$ a $*$-corner skew Laurent polynomial ring. We have*

(1) *R is positive-definite if and only if A is positive-definite.*
(2) *R is $*$-proper if and only if A is graded $*$-proper.*

Proof (1) Since R is a $*$-subring of A, if A is positive-definite, then so is R.

For the converse, suppose R is positive-definite and

$$\sum_{k=1}^{l} x_k x_k^* = 0, \tag{1.92}$$

where $x_k \in A$. Write

$$x_k = t_-^{j_k} r^k_{-j_k} + t_-^{j_k - 1} r^k_{-j_k+1} + \cdots + t_- r^k_{-1} + r^k_0 + r^k_1 t_+ + \cdots + r^k_{i_k} t_+^{i_k}.$$

It is easy to observe that the constant term of $x_k x_k^*$ is

$$\phi^{-j_k}(r^k_{-j_k} r^{k^*}_{-j_k}) + \cdots + \phi^{-1}(r^k_{-1} r^{k^*}_{-1}) + r^k_0 r^{k^*}_0 + r^k_1 r^{k^*}_1 + \cdots + r^k_{i_k} r^{k^*}_{i_k}.$$

Now Equation 1.92 implies that the sum of these constant terms are zero. Since

https://doi.org/10.1017/CBO9781316717134.002 Published online by Cambridge University Press

R is positive-definite and ϕ is an $*$-isomorphism, it follows that all $r^k_{-j_k}$ and $r^k_{i_k}$ for $1 \leq k \leq l$ are zero and thus $x_k = 0$. This finishes the proof of (1).

(2) The proof is similar to (1) and is left to the reader. □