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OMEGA RESULTS FOR THE ERROR TERM IN THE SQUARE-FREE DIVISOR PROBLEM FOR SQUARE-FULL INTEGERS

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ABSTRACT. In this paper, we investigate the distributive properties of square-free divisors over square-full integers. We first compute the mean value of the number of such divisors and obtain the error term which appears in its asymptotic formula. We then show that if one assumes the Riemann Hypothesis then the omega estimate of such error term can be drastically improved. Finally, we compute the omega estimate of the mean square of such an error term.

1. Introduction

Let $d(n)$ denote the number of positive divisors of an integer n. The distribution of $d(n)$ over integers has been widely studied. Apart from studying the asymptotic estimates of divisor sums $\sum_{n\leq x} d(n)$, one can also study different kinds of restricted divisor sums, and these restrictions can be considered either on the range of values of n or on the nature of the divisors of n or on both.

The case when the divisors of n belong to square-free integers was first studied by Mertens [13] in 1874. Let k be an integer greater than or equal to 2. Then an integer n is called k-free if p^k does not divide n for any prime p. Let $d^{(k)}(n)$ denotes the number of k-free divisor of an integer n and define the summatory function

$$D^{(k)}(x) := \sum_{n\leq x} d^{(k)}(n).$$

The asymptotic formula for $D^{(k)}(x)$ is

$$D^{(k)}(x) = \frac{1}{\zeta(k)} x\log x + \left(\frac{2\gamma-1}{\zeta(k)} - \frac{k\zeta'(k)}{\zeta^2(k)}\right) x + \Delta^{(k)}(x),$$

where $\Delta^{(k)}(x)$ is the error term. Mertens [13] first computed the trivial bound $\Delta^{(2)}(x) = O(x^{\frac{1}{2}}\log x)$. In 1932, Holder [7] considered the general case and established the estimates

$$\Delta^{(k)}(x) = \begin{cases} x^{\frac{1}{2}} & \text{if } k=2,\\ x^{\frac{1}{3}} & \text{if } k=3,\\ x^{\frac{33}{100}} & \text{if } k\geq 4.\end{cases}$$

However, one can improve the previous error terms under the assumption of the Riemann Hypothesis. In particular, Nowak and Schmeier [15] obtained the estimate $\Delta^{(2)}(x) =$

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$O(x^{\frac{15}{38}+\varepsilon})$. Baker [1] in 1994 improved this further to $O(x^{\frac{3}{8}+\varepsilon})$ by reducing the problem to the estimation of the bilinear sum

$$\sum \mu(m)d(n)e(m^{-1}\sqrt{xn}),$$

and applying a method of Heath-Brown [6].

In contrast to the above problem, naturally one can ask for the distribution of the summatory function of the divisor function where the sum is taken over the k-free integers. A similar question can be asked when the sum is taken over the k-full integers where a k-full integer n is an integer if $p^k|n$ for every prime factor p of n. It is worth mentioning that the study of distributions of square full integers was initiated by Erdös and Szekeres [5] in 1934. Batman and Grosswald studied the distribution of k-full integers [3] in 1958. Later, in 1973, Suryanarayana and Rao [17] improved the result of Batman and Grosswald [3]. Unfortunately, there are no extensive studies on the distribution of the divisor functions over k-free integers or k-full integers available in the literature. However, in 2010, Ledoan and Zaharescu [11], investigated some general real moments associated with square-full divisors of square-full numbers and computed the contribution to these moments given by the square divisors. Naturally, the distribution of square-free divisors over square-full numbers remains to be studied.

In this paper, we are interested in studying the following divisor function

$$d_2^{(2)}(n) = \begin{cases} 2^{w(n)} & \text{if } n \text{ is square-full,} \\ 0 & \text{otherwise,} \end{cases} \tag{1}$$

where $w(n)$ counts the number of prime divisors of n. Thus, $d_2^{(2)}(n)$ counts the number of square-free divisors of square-full integer n. Let us define

$$D_2^{(2)}(x) := \sum_{n\le x} d_2^{(2)}(n). \tag{2}$$

Before starting the main results, we need to introduce the following two constants given by the convergent series

$$C_1 = \frac{1}{2}\sum_{n=1}^{\infty}\frac{d(n)}{n^{\frac{3}{2}}}, \tag{3}$$

and

$$C_2 = \frac{3}{2}\sum_{n=1}^{\infty}\frac{d(n)\log n}{n^{\frac{3}{2}}}. \tag{4}$$

Define the Dirichlet series

$$F(s) = \sum_{n=1}^{\infty}\frac{d_2^{(2)}(n)}{n^s}. \tag{5}$$

Note that $F(s)$ is absolutely convergent for $\Re(s) > \frac{1}{2}$. Then, define the Dirichlet series

$$R(s) = \frac{F(s)}{\zeta^2(2s)\zeta^2(3s)}. \tag{6}$$

Similarly, $R(s)$ is absolutely convergent for $\Re(s) > \frac{1}{4}$. Our first main result is given below.

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Theorem 1.1. *Let $D_2^{(2)}(x)$ be the summatory function defined in* (2) *and $R(s)$ be the Dirichlet series defined in* (6). *Then for any $x \geq 1$, we have*

$$D_2^{(2)}(x) = \mathcal{C}_1 x^{\frac{1}{2}} \log x + \mathcal{C}_2 x^{\frac{1}{2}} + \delta_2^{(2)}(x),$$

where $\mathcal{C}_1 = C_1 R\left(\frac{1}{2}\right)$, $\mathcal{C}_2 = (2(2\gamma - 1)C_1 - C_2)\, R\left(\frac{1}{2}\right) + C_1 R'\left(\frac{1}{2}\right)$, *and $\delta_2^{(2)}(x)$ is the error term given by*

$$\delta_2^{(2)}(x) = O(x^{\frac{1}{4}} \exp(-D'(\log x)^{\frac{3}{5}}(\log\log x)^{-\frac{1}{5}}),$$

where $D' > 0$ is a constant.

Next, we investigate the error term further. More specifically, we study the Ω-type estimate for the error term and the Ω-type estimate of the mean square of the error term under the assumption of the Riemann Hypothesis.

The statements of the results are given in the following.

Theorem 1.2. *Assume the Riemann Hypothesis. Then we have*

$$\delta_2^{(2)}(x) = \Omega(x^{\frac{3}{20}}).$$

Theorem 1.3. *Assume the Riemann Hypothesis. Then we have*

$$\int_1^X (\delta_2^{(2)}(x))^2 dx = \Omega(X^{1+\frac{3}{10}}).$$

The paper is organised as follows. We gather some preliminary results in Section 2. We prove Theorem 1.1 in Section 3. We give a proof of Theorem 1.2 and Theorem 1.3 in Section 4.

2. Preliminaries

In this section, we present the necessary results which will be used to prove the Theorems.

Lemma 2.1. *Let $F(s)$ be the Dirichlet series defined in* (5). *Then for $\Re(s) > \frac{1}{2}$, we have*

$$F(s) = \frac{\zeta^2(2s)\zeta^2(3s)}{\zeta(4s)\zeta^2(5s)\zeta^3(6s)} H(s), \tag{7}$$

where $H(s)$ is a Dirichlet series absolutely and uniformly convergent in any compact set in the half-plane $\Re(s) \geq \frac{1}{7} + \delta$ for any $\delta > 0$.

Proof. By (1), for $\Re(s) > \frac{1}{2}$, the Euler product representation of $F(s)$ gives

$$F(s) = \prod_p (1 + 2p^{-2s} + 2p^{-3s} + 2p^{-4s} + \ldots). \tag{8}$$

Now, for any X with $|X| < 1$ one has

$$\begin{aligned} &1 + 2X^2 + 2X^3 + 2X^4 + \ldots \\ =&(1 - X^2)^{-2}(1 - X^3)^{-2}(1 - X^4)(1 - X^5)^2(1 - X^6)^3(1 + X^7 P(X)). \end{aligned} \tag{9}$$

where $P(X) = -2 + O(X)$. Next, we put $X = p^{-s}$ in (9) and substitute the resulting identity back in (8) to get

$$F(s) = \prod_p (1 + 2p^{-2s} + 2p^{-3s} + 2p^{-4s} + \ldots) = \frac{\zeta^2(2s)\zeta^2(3s)}{\zeta(4s)\zeta^2(5s)\zeta^3(6s)} H(s),$$

where the series $H(s)$ converges absolutely and uniformly for $\Re(s) > 1/7$. □

Let us denote

$$\zeta(4s)\zeta^2(5s)\zeta^3(6s) = G(s). \tag{10}$$

Clearly, $G(s)$ is absolutely convergent for $\Re(s) > \frac{1}{4}$. We also note that $R(s)$ defined in (6) can be expressed as

$$R(s) = \sum_{n=1}^{\infty} \frac{r(n)}{n^s} = G^{-1}(s)H(s) \quad \text{for } \Re(s) > \frac{1}{4}, \tag{11}$$

where $H(s)$ is defined in (7). Denote

$$\zeta^2(2s)\zeta^2(3s) = A(s) = \sum_{n=1}^{\infty} \frac{a(n)}{n^s} \quad \text{for } \Re(s) > \frac{1}{2}. \tag{12}$$

Let us assume $\varepsilon > 0$ to be any arbitrary small positive real number which may not be the same at each occurrence. We need the following lemma for estimating the sum $\sum_{n \le x} |r(n)|$.

Lemma 2.2. *Let $x \ge 1$ be any real number. Then for the arithmetical function $r(n)$ defined in (11) we have*

$$\sum_{n \le x} |r(n)| = \mathcal{B}x^{\frac{1}{4}} + O(x^{\frac{1}{5}+\varepsilon}), \tag{13}$$

where $\mathcal{B}$ is an explicit positive constant.

Proof. From the expression in (6) and (9), one can observe the Euler product representation for $R(s)$ is given by

$$\prod_p (1 - p^{-4s} - 2p^{-5s} - 3p^{-6s} + 2p^{-7s} + 4p^{-8s} + 2p^{-9s} - p^{-10s} - 2p^{-11s}),$$

for $\Re(s) > \frac{1}{4}$. Hence, the Euler product representation for the series $\sum_{n=1}^{\infty} \frac{|r(n)|}{n^s}$ is

$$\begin{aligned} &\prod_p (1 + p^{-4s} + 2p^{-5s} + 3p^{-6s} + 2p^{-7s} + 4p^{-8s} + 2p^{-9s} + p^{-10s} + 2p^{-11s}) \\ &= \zeta(4s) \prod_p (1 + 2p^{-5s} + 3p^{-6s} + 2p^{-7s} + 3p^{-8s} - 2p^{-10s} - 4p^{-12s} - 2p^{-13s} - p^{-14s} - 2p^{-15s}), \end{aligned} \tag{14}$$

for $\Re(s) > \frac{1}{4}$. If $T(s) = \sum_{n=1}^{\infty} \frac{t(n)}{n^s}$ denotes the Dirichlet series associated with the second Euler product in (14), then for any k-full integer n with $k \ge 5$ we have

$$|t(n)| \le 4^{\omega(n)} \le d(n)^2 \ll n^{\varepsilon},$$

where $\varepsilon > 0$ be any real number and $\omega(n)$ denotes the number of distinct prime factors of n. We will also note that $t(n)$ takes the value zero otherwise. Therefore, we can write

$$\sum_{n=1}^{\infty} \frac{|r(n)|}{n^s} = \zeta(4s)T(s), \tag{15}$$

where $T(s)$ given by the infinite product in (14) converges absolutely for $\Re(s) > \frac{1}{5}$. From (15), we can write

$$|r(n)| = \sum_{d^4 f = n} t(f) = \sum_{d^4 | n} t\left(\frac{n}{d^4}\right).$$

Consequently, we will get

$$\sum_{n \le x} |r(n)| = \sum_{f \le x} t(f) \sum_{d \le \left(\frac{x}{f}\right)^{\frac{1}{4}}} 1$$
$$= x^{\frac{1}{4}} \sum_{f=1}^{\infty} \left(\frac{t(f)}{f^{\frac{1}{4}}}\right) + O\left(\sum_{f \le x} |t(f)|\right) + O\left(x^{\frac{1}{4}} \sum_{f > x} \left(\frac{|t(f)|}{f^{\frac{1}{4}}}\right)\right), \tag{16}$$

where in the last step, we have used the fact that $T(s)$ converges absolutely for $\Re(s) > \frac{1}{5}$. It remains to estimate the error terms in (16).

The first error term can be estimated as

$$\sum_{f \le x} |t(f)| \le x^{\frac{1}{5}+\varepsilon} \sum_{f=1}^{\infty} \frac{|t(f)|}{f^{\frac{1}{5}+\varepsilon}} \ll x^{\frac{1}{5}+\varepsilon}, \tag{17}$$

using the fact $T(s)$ converges absolutely for $\Re(s) > \frac{1}{5}$. Similarly, for the second error term, one can show

$$\sum_{f > x} \left(\frac{|t(f)|}{f^{\frac{1}{4}}}\right) = \sum_{f > x} \frac{|t(f)|}{f^{\frac{1}{5}+\varepsilon} f^{\frac{1}{20}-\varepsilon}} \ll x^{-\frac{1}{20}+\varepsilon}. \tag{18}$$

Substituting (17) and (18) in (16) we get the desired result. □

The following version of Perron's formula is the key ingredient in obtaining the bound of the error term.

Lemma 2.3 (Liu and Ye [12, p. 483, Theorem 2.1]). *Let $f(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}$ be a Dirichlet series such that $B(\sigma) := \sum_{n=1}^{\infty} \left|\frac{a_n}{n^s}\right| = \sum_{n=1}^{\infty} \frac{|a_n|}{n^{\sigma}}$ is convergent for $\sigma > \sigma_a$. If $b > \sigma_a$ and $x, T, H \ge 2$, then we have*

$$\sum_{n \le x} a_n = \frac{1}{2\pi i} \int_{b-iT}^{b+iT} f(s) \frac{x^s}{s} ds + O\left(\sum_{x - x/H < n \le x + x/H} |a_n|\right) + O\left(\frac{x^b H B(b)}{T}\right). \tag{19}$$

Next, we will derive some lemmas needed to complete the proof of Theorem 1.1. Their proofs are based on the well-known result concerning the classical divisor function [8, p. 83]

$$\sum_{n \le X} d(n) = X \log X + (2\gamma - 1) X + O\left(X^{\frac{1}{3}}\right), \tag{20}$$

where γ is the Euler constant.

Lemma 2.4. *Let β and X be any positive real numbers with $X \ge 1$. Then*

$$\sum_{n \le X} \frac{d(n)}{n^{\beta}} = \frac{1}{1-\beta} X^{1-\beta} \log X + c_{\beta} X^{1-\beta} + O\left(X^{\frac{1}{3}-\beta}\right),$$

where $c_{\beta} = \frac{(2\gamma-1)}{1-\beta} - \frac{\beta}{(1-\beta)^2}$.

Proof. We get the desired result by employing the result in (20) and partial summation. □

Lemma 2.5. *Let β and X be any positive real numbers with $X \geq 1$. Then*

$$\sum_{n\leq X} \frac{d(n)\log n}{n^\beta} = \frac{1}{1-\beta}X^{1-\beta}(\log X)^2 - \frac{1}{(1-\beta)^2}X^{1-\beta}\log X + \frac{1}{(1-\beta)^3}X^{1-\beta} + O\left(X^{\frac{1}{3}-\beta}\right).$$

Proof. Employing the result in (20) and partial summation, we get the desired result. □

Next, we will recall a result due to Montgomery and Vaughan [14], concerned with the mean value theorems for the summatory functions of a class of the Dirichlet series. Let $\{a_n\}$ be a sequence of complex numbers such that for any $\varepsilon > 0$, $a_n \ll n^\varepsilon$ for any $\varepsilon > 0$. Define the Dirichlet polynomial

$$\mathbb{A}(s) = \sum_{n\leq N} a_n n^{-s}.$$

Then, by Montgomery and Vaughan's mean value theorem [14] one has

$$\int_0^T |\mathbb{A}(s)|^2 dt = \sum_{n\leq N} |a_n|^2 n^{-2\sigma} \left(T + O(n)\right). \tag{21}$$

Then, we recall a zero-free region of the Riemann zeta-function $\zeta(s)$ which can be written by

$$\sigma > 1 - \frac{A}{(\log t)^{2/3}(\log\log t)^{1/3}},\ t \geq T_0, \tag{22}$$

for some positive constants A and T_0. We provide a lower bound of $\zeta(s)$ in the next lemma.

Lemma 2.6. [8, Lemma 12.3] *We have*

$$\frac{1}{\zeta(s)} = O((\log t)^{2/3}(\log\log t)^{1/3}),$$

in the region

$$\sigma \geq 1 - \frac{A}{(\log t)^{2/3}(\log\log t)^{1/3}} \quad \text{and } t \geq T_0.$$

Upon assuming the Riemann Hypothesis, one has a better lower bound.

Lemma 2.7. [18, Lemma 1] *Assume the Riemann Hypothesis. For every sufficiently large number T and for any $\varepsilon > 0$, we have*

$$\frac{1}{\zeta(s)} = O_\varepsilon((|t|+2)^\varepsilon) \quad \text{in} \quad \Re(s) \geq \frac{1}{2} + \varepsilon,$$

provided $\Re(z) \neq 0$ for $\Re(z) \geq \frac{1}{2} + \varepsilon$, $|\Im(z) - \Im(s)| \leq (\log T)^5$, and $|\Im(s)| \geq 1$.

Next, we will state a zero density estimate of $\zeta(s)$ due to Ingham [18].

Lemma 2.8. [4, Lemma 13]. *The number of zeros of $\zeta(s)$ in $\Re(s) \geq \frac{1}{2} + \varepsilon$ and $|\Im(s)| \leq T$ is $O(T^{1-\frac{11\varepsilon}{10}})$.*

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3. Proof of Theorem 1.1

This section is devoted to the proof of Theorem 1.1. Before proceeding further, let us first state and prove a lemma required to prove the theorem.

Lemma 3.1. *For any $x \geq 1$ and $r(n)$ defined in* (11)*, we have*

$$\sum_{n \leq x} r(n) \ll x^{\frac{1}{4}} \exp(-D(\log x)^{\frac{3}{5}}(\log_2 x)^{-\frac{1}{5}}),$$

where $D > 0$ is a constant.

Proof. First we employ $f(s) = R(s)$ which is defined in (11) and

$$B(\sigma) = \sum_{n=1}^{\infty} \frac{|r_n|}{n^{\sigma}},$$

where $R(s)$ and $r(n)$ are defined in (11). Next, we will take $x \geq 4$, $T \geq 4$, $H \geq 2$, and $b = 1/4 + 1/\log x$. Note that, by the Laurent series expansion of $\zeta(s)$ at $s = 1$ one finds $B(b) \ll \log x$. Hence,

$$\sum_{n \leq x} r(n) = \frac{1}{2\pi i} \int_{b-iT/4}^{b+iT/4} R(s) \frac{x^s}{s} ds + O\left(\sum_{x-x/H<n\leq x+x/H} |r(n)| \right) + O\left(\frac{x^{1/4} H \log x}{T} \right). \tag{23}$$

Now, we focus on the integral in (23). Consider a positively oriented rectangular contour $\mathcal{C}$ consisting of the line segments $[b - iT/4, b + iT/4]$, $[b + iT/4, d + iT/4]$, $[d + iT/4, d - iT/4]$, and $[d - iT/4, b - iT/4]$, where

$$d = \frac{1}{4} - \frac{A}{(\log T)^{2/3}(\log\log T)^{1/3}},$$

and $A > 0$ is the constant given in (22). From Lemma 2.6 we have $G(s) \neq 0$ in the region

$$\sigma > \frac{1}{4} - \frac{A}{(\log 4t)^{2/3}(\log\log 4t)^{1/3}}, \quad T_0 < t \leq T/4.$$

In this zero free region of $G(s)$, we have

$$\frac{1}{G(s)} \ll (\log T)^{2/3}(\log\log T)^{1/3}. \tag{24}$$

The above bound can be readily obtained from Lemma 2.6. Now, appealing to Cauchy's residue theorem, we get

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$$\begin{aligned} I := & \frac{1}{2\pi i}\int_{b-iT/4}^{b+iT/4} G^{-1}(s)H(s)\frac{x^s}{s}ds \\ = & \underbrace{\frac{1}{2\pi i}\int_{d-iT/4}^{d+iT/4} G^{-1}(s)H(s)\frac{x^s}{s}ds}_{=I_1} + \underbrace{\frac{1}{2\pi i}\int_{d+iT/4}^{b+iT/4} G^{-1}(s)H(s)\frac{x^s}{s}ds}_{=I_2} \\ & + \underbrace{\frac{1}{2\pi i}\int_{b-iT/4}^{d-iT/4} G^{-1}(s)H(s)\frac{x^s}{s}ds}_{=I_3} \end{aligned} \tag{25}$$

Employing the bound given in (24) we find that

$$\begin{aligned} I_1 &\ll x^d \int_{T_0}^{T/4} |G^{-1}(d+it)|\frac{dt}{|d+it|} + x^d \\ &\ll x^d(\log T)^{\frac{5}{3}}(\log\log T)^{\frac{1}{3}} + x^d. \end{aligned} \tag{26}$$

Similarly,

$$\begin{aligned} I_2, I_3 &\ll \frac{1}{2\pi i}\int_d^b |G^{-1}(\sigma \pm iT/4)|H(\sigma \pm iT/4)|\frac{x^\sigma}{|\sigma \pm iT/4|}ds \\ &\ll (\log x)^{-1}T^{-1}(\log T)^{2/3}(\log\log T)^{1/3}\max\left(x^b, x^d\right) \\ &\ll x^{\frac{1}{4}}T^{-1}(\log x)^{-1}(\log T)^{\frac{2}{3}}(\log\log T)^{\frac{1}{3}}. \end{aligned} \tag{27}$$

At this point, we choose $T = \exp(C_0(\log x)^{\frac{3}{5}}(\log_2 x)^{-\frac{1}{5}})$ with $C_0 > 0$ a constant and substitute the estimates (26) and (27) in (25). Therefore

$$I = O\left(x^{\frac{1}{4}}\exp(-C_0(\log x)^{\frac{3}{5}}(\log_2 x)^{-\frac{1}{5}})\right). \tag{28}$$

By Lemma 2.2, we can estimate the sum in the error term in (23),

$$\sum_{x-x/H<n\le x+x/H} |r(n)| \ll x^{\frac{1}{4}}H^{-1} + x^{\frac{1}{5}+\varepsilon}.$$

Taking $H = \sqrt{T} = \exp(\frac{C_0}{2}(\log x)^{\frac{3}{5}}(\log_2 x)^{-\frac{1}{5}})$ we obtain

$$\sum_{x-x/H<n\le x+x/H} |r(n)| \ll x^{\frac{1}{4}}\exp(-C_1'(\log x)^{\frac{3}{5}}(\log_2 x)^{-\frac{1}{5}}). \tag{29}$$

With the above choice of H, the second error term in (23), is estimated as

$$\ll x^{\frac{1}{4}}\exp(-C_2'(\log x)^{\frac{3}{5}}(\log_2 x)^{-\frac{1}{5}}). \tag{30}$$

Substituting (28), (29) and (30) in (23), we get the desired result. □

Now employing Lemma 3.1 and partial summation, we obtain following results.

Lemma 3.2. *Let $x \ge 1$ be any real number and $r(n)$ be the arithmetical function defined in* (11). *Then we have*

$$\sum_{n\le x}\frac{r(n)}{n^{\frac{1}{2}}} = R\left(\frac{1}{2}\right) + O(x^{-\frac{1}{4}}\exp(-D_1(\log x)^{\frac{3}{5}}(\log_2 x)^{-\frac{1}{5}}), \tag{31}$$

and

$$\sum_{n\le x}\frac{r(n)\log n}{n^{\frac{1}{2}}}=-R'\left(\frac{1}{2}\right)+O(x^{-\frac{1}{4}}\exp(-D_2(\log x)^{\frac{3}{5}}(\log_2 x)^{-\frac{1}{5}}),\tag{32}$$

where $D_1, D_2>0$ *are the constants.*

Proof. The proof follows from Lemma 3.1 and partial summation. □

The next result is also instrumental to our proof of the theorem.

Lemma 3.3. *Let* $x\ge 1$ *be any real number and let* $a(n)$ *be the arithmetical function defined in* (12) *with* $\mathcal{A}(x)=\sum_{n\le x}a(n)$. *Then*

$$\mathcal{A}(x)=C_1x^{\frac{1}{2}}\log x+(2(2\gamma-1)C_1-C_2)\,x^{\frac{1}{2}}+O(x^{\frac{1}{3}}\log x),\tag{33}$$

where C_1 *and* C_2 *are defined in* (3) *and* (4) *respectively.*

Proof. We evaluate $\mathcal{A}(x)$ using the Dirichlet hyperbola method. From (12), one can find that the arithmetic function $a(n)$ supported over the integers of the form m^2n^3. Then

$$\mathcal{A}(x)=\sum_{n\le x}a(n)=\sum_{m^2n^3\le x}d(n)d(m)=S_1+S_2-S_3\tag{34}$$

where

$$S_1=\sum_{m\le M}d(m)\sum_{n\le\sqrt[3]{\frac{x}{m^2}}}d(n),\ S_2=\sum_{n\le N}d(n)\sum_{m\le\sqrt{\frac{x}{n^3}}}d(m)\ \text{and}\ S_3=\sum_{n\le N}d(n)\sum_{m\le M}d(m).\tag{35}$$

The values of M and N will be chosen later. Now, we compute the sum S_1. We have

$$\begin{aligned}S_1&=\sum_{m\le M}d(m)\left(\frac{1}{3}\left(\frac{x}{m^2}\right)^{\frac{1}{3}}(\log x-2\log m)+(2\gamma-1)\left(\frac{x}{m^2}\right)^{\frac{1}{3}}+O\left(\left(\frac{x}{m^2}\right)^{\frac{1}{9}}\right)\right)\\&=\left(\frac{x^{\frac{1}{3}}\log x}{3}+(2\gamma-1)x^{\frac{1}{3}}\right)\sum_{m\le M}\frac{d(m)}{m^{\frac{2}{3}}}-\frac{2x^{\frac{1}{3}}}{3}\sum_{m\le M}\frac{d(m)\log m}{m^{\frac{2}{3}}}++O\left(x^{\frac{1}{9}}\sum_{m\le M}\frac{d(m)}{m^{\frac{2}{9}}}\right)\\&=x^{\frac{1}{3}}M^{\frac{1}{3}}\log x\log M-2x^{\frac{1}{3}}M^{\frac{1}{3}}(\log M)^2+\frac{c_{\frac{2}{3}}}{3}x^{\frac{1}{3}}M^{\frac{1}{3}}\log x+(\frac{c_{\frac{2}{3}}}{3}+6)x^{\frac{1}{3}}M^{\frac{1}{3}}\log M\\&\qquad-18x^{\frac{1}{3}}M^{\frac{1}{3}}+O\left(x^{\frac{1}{3}}M^{-\frac{1}{3}}\log x\right)+O\left(x^{\frac{1}{9}}M^{\frac{1}{3}}\log M\right).\end{aligned}\tag{36}$$

where $c_{\frac{2}{3}}=3(2\gamma-3)$. In a similar fashion

$$\begin{aligned}S_2&=\frac{x^{\frac{1}{2}}\log x}{2}\sum_{n\le N}\frac{d(n)}{n^{\frac{3}{2}}}-\frac{3x^{\frac{1}{2}}}{2}\sum_{n\le N}\frac{d(n)\log n}{n^{\frac{3}{2}}}+(2\gamma-1)x^{\frac{1}{2}}\sum_{n\le N}\frac{d(n)}{n^{\frac{3}{2}}}+O\left(x^{\frac{1}{6}}\sum_{n\le N}\frac{d(n)}{n^{\frac{1}{2}}}\right)\\&=C_1x^{\frac{1}{2}}\log x+(2(2\gamma-1)C_1-C_2)\,x^{\frac{1}{2}}+O(x^{\frac{1}{2}}N^{-\frac{1}{2}}\log x\log N)+O(x^{\frac{1}{6}}N^{\frac{1}{2}}\log N),\end{aligned}\tag{37}$$

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and

$$\begin{aligned} S_3 &= \sum_{n\le N} d(n) \sum_{m\le M} d(m) \\ &= MN\log M\log N + (2\gamma-1)MN\log N + (2\gamma-1)MN\log M + (2\gamma-1)^2MN \\ &\qquad + O(M^{\frac{1}{3}}N\log N) + O(N^{\frac{1}{3}}M\log M). \end{aligned} \tag{38}$$

Then we replace $\log M$ and $\log N$ by $\log x$ and M by $\frac{x^{\frac{1}{2}}}{N^{\frac{3}{2}}}$ and substitute (36), (37) and (38) in (34), to get

$$|\mathcal{A}(x) - C_1x^{\frac{1}{2}}\log x - (2(2\gamma-1)C_1 - C_2)\,x^{\frac{1}{2}}| \ll x^{\frac{1}{2}}N^{-\frac{1}{2}}\log^2 x + x^{\frac{1}{6}}N^{\frac{1}{2}}\log x. \tag{39}$$

Next, we choose $N = x^{\frac{1}{3}}$ and thus obtain the error term of order $x^{\frac{1}{3}}\log^2 x$. This completes the proof of the lemma. □

We are ready to prove Theorem 1.1.

Proof of Theorem 1.1. We will use the Lemmas 3.2 and 3.3 in the proof. Let ρ be a real number such that $0<\rho<1$. Then we split the sum as follows

$$D_2^{(2)}(x) = \sum_{n\le x} d_2^{(2)}(n) = \sum_{de\le x} r(d)a(e), = \mathcal{S}_1 + \mathcal{S}_2, \tag{40}$$

where

$$\mathcal{S}_1 = \sum_{d\le \rho x}\sum_{\frac{1}{\rho}<e\le\frac{x}{d}} r(d)a(e) \quad\text{and}\quad \mathcal{S}_2 = \sum_{e\le\frac{1}{\rho}}\sum_{d\le\frac{x}{e}} r(d)a(e).$$

We will first evaluate $\mathcal{S}_1$. Employing (31), (32) and (33) in the sum $\mathcal{S}_1$ we have

$$\begin{aligned} \mathcal{S}_1 &= \sum_{d\le\rho x} r(d)\left(C_1\left(\frac{x}{d}\right)^{\frac{1}{2}}\log\frac{x}{d} + (2(2\gamma-1)C_1 - C_2)\left(\frac{x}{d}\right)^{\frac{1}{2}} + O\left(\left(\frac{x}{d}\right)^{\frac{1}{3}}\log^2\frac{x}{d}\right)\right) \\ &= C_1x^{\frac{1}{2}}\log x\sum_{d\le\rho x}\frac{r(d)}{d^{\frac{1}{2}}} - C_1x^{\frac{1}{2}}\sum_{d\le\rho x}\frac{r(d)\log d}{d^{\frac{1}{2}}} + (2(2\gamma-1)C_1 - C_2)\sum_{d\le\rho x}\frac{r(d)}{d^{\frac{1}{2}}} + O(x^{\frac{1}{4}}\rho^{-\frac{1}{12}}\log^2 x) \\ &= \mathcal{C}_1x^{\frac{1}{2}}\log x + \mathcal{C}_2x^{\frac{1}{2}} + O(x^{\frac{1}{4}}\rho^{-\frac{1}{4}}\exp(-D_3(\log\rho x)^{\frac{3}{5}}(\log\log\rho x)^{-\frac{1}{5}}). \end{aligned} \tag{41}$$

where $\mathcal{C}_1 = C_1R\left(\frac{1}{2}\right)$ and $\mathcal{C}_2 = (2(2\gamma-1)C_1 - C_2)\,R\left(\frac{1}{2}\right) + C_1R'\left(\frac{1}{2}\right)$. To estimate $\mathcal{S}_2$, we will use Lemma 3.1, and the asymptotic (33). Then we have

$$\begin{aligned} \mathcal{S}_2 &\ll x^{\frac{1}{4}}\exp(-D_4(\log\rho x)^{\frac{3}{5}}(\log\log\rho x)^{-\frac{1}{5}})\sum_{e\le\rho^{-1}}\frac{a(e)}{e^{\frac{1}{4}}} \\ &\ll x^{\frac{1}{4}}\exp(-D_4(\log\rho x)^{\frac{3}{5}}(\log\log\rho x)^{-\frac{1}{5}})\rho^{-\frac{1}{4}}\log\rho^{-1}. \end{aligned} \tag{42}$$

Substituting (41) and (42) in (40) and then taking $\rho = \exp(-D_5(\log x)^{\frac{3}{5}}(\log\log x)^{-\frac{1}{5}})$ where $D_5>0$ is a constant, we complete the proof of the theorem.

□

Remark 3.1. It is useful to note that our technique can be used to generalize Theorem 1.1 to understand the distribution of l-free divisors over any k-full numbers for any integers $k, l \ge 2$.

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4. Proof of Theorem 1.2 and 1.3

The proof of Theorem 1.2 directly follows from the result of Nowak [10, Theorem 2] by substituting

$$\alpha = \frac{4-1}{2(3+3+2+2)} = \frac{3}{20}.$$

The estimate in Theorem 1.2 directly follows from the result in [10]. To get the mean square estimate in Theorem 1.3 our method relies on techniques of Ivic [9] and Balasubramanian, Ramachandra and Subbarao in [4]. let us introduce some notations. Consider a constant $c \leq \alpha$ where α is given above. Let T be a large positive number. We define the sets

$$J = \left\{ T^{1-c} \leq t \leq 2T : |t - \Im(z)| \leq (\log T)^{20} \text{ when } G(z) \neq 0 \text{ for } \Re(z) > \frac{1}{7} \right\},$$

and

$$K = \left\{ T^{1-c} - (\log T)^4 \leq t \leq 2T + (\log T)^4 : |t - \Im(z)| \leq (\log T)^{15} \text{ when } G(z) \neq 0 \text{ for } \Re(z) > \frac{1}{7} \right\}.$$

Now, we state a few lemmas which give some required bounds.

Lemma 4.1. *Assume the Riemann Hypothesis. If $\Re(s) \geq \frac{1}{7}$, $l \geq 4$ and $t \in K$, then $\frac{1}{\zeta(ls)} = O((|t|+2)^{\varepsilon})$ for any $\varepsilon > 0$.*

Proof. The proof is a direct consequence of Lemma 2.7. □

As an application of the above Lemma, we have

Corollary 4.2. *Assume the Riemann Hypothesis. Let $F(s)$ be the Dirichlet series defined in (7). For $\Re(s) > \frac{1}{7}$ and $t \in K$, and a suitable constant $A > 0$, one has*

$$F(s) = O((|t|+2)^A),$$

for a suitable constant $A > 0$.

Proof. The proof follows from lemma 4.1. □

Before proceeding further, we need a few results related to J. Define

$$J(x) = J \cap [x, 2x] \quad \text{for any } x \text{ with } \quad T^{1-c} \leq x \leq 2T,$$

and let $N(x)$ be the number of zeros of $G(s)$ with $t \in [x, 2x]$. Next, we need the following form of zero-density result which is an immediate consequence of Lemma 2.8.

Lemma 4.3. *We have*

$$N(x) \ll x^{1-\frac{11\varepsilon}{10}}.$$

Proof. The proof follows from Lemma 2.8. □

Next consider the interval $[x, 2x]$. Corresponding to every zero $\rho = \beta + i\gamma$ with $\beta \geq \frac{1}{7} + \varepsilon$ and $x - (\log T)^{20} \leq \gamma \leq 2x + (\log T)^{20}$ of $G(s)$, if we remove the interval $[\gamma - (\log T)^{20}, \gamma + (\log T)^{20}]$, then the remaining portion gives us $J(x)$. As there are $N(x)$ zeros in $[x, 2x]$, so there are $N(x)$ disjoint intervals and the total length of $J(x)$ is $\ll x$. Now we can delete from $J(x)$ the connected components, each of length $\leq x^{\varepsilon}$. The total length of the deleted portion $O(N(x)x^{\varepsilon}) = O(x^{1-\frac{\varepsilon}{10}})$. Hence if $J^{(2)}(x)$ is the remaining portion, the total length of $J^{(2)}(x)$ is $\gg x$.

To complete our proof of Theorem 1.3, we need the following estimate related to the error term $\delta_2^{(2)}(x)$.

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Lemma 4.4. *For any $\varepsilon > 0$ and $T^\varepsilon \ll H \le T$,*

$$H^{-1}\int_0^H \delta_2^{(2)}(T+u)du = \delta_2^{(2)}(T) + O\left(HT^\varepsilon\right).$$

Proof. Employing the trivial bound $d_2^{(2)}(n) \ll n^\varepsilon$, we see that that $D_2^{(2)}(u+T) - D_2^{(2)}(T) \ll uT^{\varepsilon-1}$ and using Theorem 1.1, we have

$$\delta_2^{(2)}(u+T) - \delta_2^{(2)}(T) \ll uT^\varepsilon.$$

Now using the above expression we can infer that

$$H^{-1}\int_0^H (\delta_2^{(2)}(T+u) - \delta_2^{(2)}(T))du \ll HT^\varepsilon,$$

and hence the result. □

Lemma 4.5. *Let $s_\alpha = \alpha + it$ with $t \in J$. If*

$$\int_{J^{(2)}(x)} \frac{|F(s_\alpha)|^2}{|s_\alpha|^2} dt \gg \log T, \tag{43}$$

then

$$\int_1^X (\delta_2^{(2)}(x))^2 dx = \Omega(X^{1+2\alpha}\log X). \tag{44}$$

Proof. Consider the Mellin integral

$$e^{-U^h} = \frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty} U^{-w}\Gamma(1+wh^{-1})w^{-1}dw, \tag{45}$$

where $h, U > 0$. We shall take $h = (\log T)^2$. Then setting $U = \frac{n}{Y}$ in (45) with $Y = T^B$ and B is a sufficiently large constant, we have

$$\sum_{n=1}^\infty \frac{d_2^{(2)}(n)}{n^{s_\alpha}} e^{-(\frac{n}{Y})^h} = \frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty} F(s_\alpha+w)Y^w\Gamma(1+wh^{-1})w^{-1}dw.$$

Now we break off the portion of the integral into two parts

$$\begin{aligned}\sum_{n=1}^\infty \frac{d_2^{(2)}(n)}{n^{s_\alpha}} e^{-(\frac{n}{Y})^h} =& \frac{1}{2\pi i}\int_{2-i(\log T)^4}^{2+i(\log T)^4} F(s_\alpha+w)\Gamma(1+wh^{-1})Y^w w^{-1}dw \\ &+ \frac{1}{2\pi i}\int_{|\tau|\ge(\log T)^4} \frac{F(s_\alpha+2+i\tau)\Gamma(1+(2+i\tau)h^{-1})Y^{2+i\tau}}{(2+i\tau)}d\tau,\end{aligned} \tag{46}$$

where $\tau = \Im w$. Next, we recall the asymptotic behaviour of $\Gamma(s)$ [16, p. 38] in a vertical strip, for s=$\sigma + it$ with $a \le \sigma \le b$ and $|t| \ge 1$,

$$|\Gamma(s)| = (2\pi)^{\frac{1}{2}}|t|^{\sigma-\frac{1}{2}}e^{-\frac{1}{2}\pi|t|}\left(1 + O\left(\frac{1}{|t|}\right)\right), \tag{47}$$

Upon using the above formula, the second integral in (46) becomes

$$\ll T^{2B}\int_{|\tau|\ge(\log T)^4} |\tau|^{-1/2+2h^{-1}}e^{-\frac{1}{2}\pi|\tau|h^{-1}}d\tau \ll 1,$$

as $h = (\log T)^2$. Therefore, employing the above estimate in (46), we arrive at

$$\sum_{n=1}^{\infty} \frac{d_2^{(2)}(n)}{n^{s_\alpha}} e^{-(\frac{n}{Y})^h} = \frac{1}{2\pi i} \int_{2-i(\log T)^4}^{2+i(\log T)^4} \frac{F(s_\alpha + w)\Gamma(1+wh^{-1})Y^w}{w} dw + O(1). \tag{48}$$

We shift the line of integration to $\Re(w) = \frac{1}{7} + \varepsilon - s_\alpha$ and encounter a pole at $w = 0$. Now, using the estimate given in the Corollary 4.2 and using the fact $t \pm \Im(w) \in K$ (as $t \in J$, we see that the value of the integral on the horizontals as well as vertical is small), we obtain

$$F(s_\alpha) = \sum_{n=1}^{\infty} \frac{d_2^{(2)}(n)}{n^{s_\alpha}} e^{-(\frac{n}{Y})^h} + O(1). \tag{49}$$

Then we split the sum (49) in two parts

$$F(s_\alpha) = \sum_{n \le T} \frac{d_2^{(2)}(n)}{n^{s_\alpha}} e^{-(\frac{n}{Y})^h} + \sum_{T<n\le 2Y} \frac{d_2^{(2)}(n)}{n^{s_\alpha}} e^{-(\frac{n}{Y})^h} + O(1). \tag{50}$$

Note that the infinite sum $\sum_{n>2Y} \frac{d_2^{(2)}(n)}{n^{s_\alpha}} e^{-(\frac{n}{Y})^h}$ can be estimated as $O(1)$. The sum $D_2^{(2)}(x) = \sum_{n\le x} d_2^{(2)}(n)$ is already estimated in Theorem 1.1. We may write

$$\begin{aligned} \sum_{T<n\le 2Y} \frac{d_2^{(2)}(n)}{n^{s_\alpha}} e^{-(\frac{n}{Y})^h} &= \int_T^{2Y} x^{-s_\alpha} e^{-(\frac{x}{Y})^h} dD_2^{(2)}(x) \\ &= \underbrace{\int_T^{2Y} x^{-s_\alpha} e^{-(\frac{x}{Y})^h} d\delta_2^{(2)}(x)}_{=\mathcal{I}_T} + \underbrace{\sum_{j=0}^{1} \mathcal{C}_j \int_T^{2Y} x^{-\frac{1}{2}-s_\alpha} e^{-(\frac{x}{Y})^h} (\log x)^j dx}_{=\mathcal{J}_T}. \end{aligned} \tag{51}$$

For the integral $\mathcal{J}_T$ we use integration by parts and see that

$$\begin{aligned} &\int_T^{2Y} x^{-\frac{1}{2}-s_\alpha} e^{-(\frac{x}{Y})^h} (\log x)^{j-1} dx \\ =& x^{\frac{1}{2}-s_\alpha} e^{-(\frac{x}{Y})^h} (\log x)^{j-1} \Big|_T^{2Y} + \int_T^{2Y} x^{\frac{1}{2}-s_\alpha} e^{-(\frac{x}{Y})^h} \left(-h\frac{x^{h-1}}{Y^h} + j(\log x)^j x^{-1} \right) dx \\ =& O\left(T^{\frac{1}{2}-s_\alpha+\varepsilon} \right). \end{aligned}$$

Hence using the above estimate, one can obtain

$$\mathcal{J}_T \ll T^{\frac{1}{2}-s_\alpha+\varepsilon}. \tag{52}$$

For the integral $\mathcal{I}_T$ we have

$$\begin{aligned} \mathcal{I}_T = x^{-s_\alpha} e^{-(\frac{x}{Y})^h} \delta_2^{(2)}(x) \Big|_T^{2Y} + s_\alpha \int_T^{2Y} x^{-s_\alpha-1} e^{-(\frac{x}{Y})^h} \delta_2^{(2)}(x) dx \\ + hY^{-h} \int_T^{2Y} x^{h-s_\alpha-1} e^{-(\frac{x}{Y})^h} \delta_2^{(2)}(x) dx. \end{aligned} \tag{53}$$

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Therefore using (52) and (53) in (51), we obtain

$$\begin{aligned}\frac{F(s_\alpha)}{s_\alpha} =&\frac{1}{s_\alpha}\sum_{n\le T}\frac{d_2^{(2)}(n)}{n^{s_\alpha}}e^{-(\frac{n}{Y})^h}+O\left(\frac{1}{t}\right)+O\left(\frac{\delta_2^{(2)}(T)}{T^\alpha t}\right)+O\left(\frac{T^{\frac12-\alpha+\varepsilon}}{t}\right)\\ &+\underbrace{\int_T^{2Y}x^{-s_\alpha-1}e^{-(\frac{x}{Y})^h}\delta_2^{(2)}(x)dx}_{=\mathcal{K}_T}+\underbrace{\frac{h}{s_\alpha}Y^{-h}\int_T^{2Y}x^{h-s_\alpha-1}e^{-(\frac{x}{Y})^h}\delta_2^{(2)}(x)dx}_{=\mathcal{L}_T}.\end{aligned}$$

Now upon squaring and integrating the above expression over $t\in J$, we have

$$\begin{aligned}\int_J\frac{|F(s_\alpha)|^2}{|s_\alpha|^2}dt\ll&\int_{T^{1-c}}^{2T}\left|\sum_{n\le T}\frac{d_2^{(2)}(n)}{n^{s_\alpha}}e^{-(\frac{n}{Y})^h}\right|^2|s_\alpha|^{-2}dt+(\delta_2^{(2)}(T))^2T^{c-2\alpha-1}+T^{c-1}\\ &+T^{2c-2\alpha-1+\varepsilon}+\int_{T^{1-c}}^{2T}(|\mathcal{K}_T|^2+|\mathcal{L}_T|^2)dt. \qquad (54)\end{aligned}$$

We observe that the fact $d_2^{(2)}(n)\ll n^\varepsilon$ for any $\varepsilon>0$. Applying the mean value theorem of Dirichlet polynomials (21), the first integral in (54) can be estimated as the following

$$\begin{aligned}&\int_{T^{1-c}}^{2T}\left|\sum_{n\le T}\frac{d_2^{(2)}(n)}{n^{s_\alpha}}e^{-(\frac{n}{Y})^h}\right|^2t^{-2}dt\\ &\ll T^{2c-2}\sum_{j\ge1}2^{-2j}\int_{2^{j-1}T^{1-c}}^{2^jT^{1-c}}\left|\sum_{n\le T}\frac{d_2^{(2)}(n)}{n^{s_\alpha}}e^{-(\frac{n}{Y})^h}\right|^2dt\\ &\ll T^{2c-2}\sum_{j\ge1}2^{-2j}\sum_{n\le T}\frac{(d_2^{(2)}(n))^2}{n^{2\alpha}}(n+2^jT^{1-c})\ll T^{2c-2\alpha+\varepsilon}, \qquad (55)\end{aligned}$$

whenever $c\le\frac12(\alpha-\varepsilon)$ and any $0<\varepsilon<\alpha$. For the second expression, we use Lemma 4.4 to get

$$\delta_2^{(2)}(T)=H^{-1}\int_0^H\delta_2^{(2)}(T+u)du+O\left(HT^\varepsilon\right),$$

and then by Cauchy-Schwartz inequality

$$\begin{aligned}(\delta_2^{(2)}(T))^2&\ll H^{-1}\int_0^H(\delta_2^{(2)}(T+u))^2du+H^2T^\varepsilon\\ &\ll H^{-1}Y^h\int_0^H(\delta_2^{(2)}(T+u))^2u^{-h}du+H^2T^\varepsilon\\ &\ll H^{-1}\int_T^{T+H}(\delta_2^{(2)}(x))^2e^{-2(\frac{x}{Y})^h}dx+H^2T^\varepsilon.\end{aligned}$$

Next putting $H=T^{\varepsilon+c}$ in the above expression, we have

$$\begin{aligned}(\delta_2^{(2)}(T))^2T^{c-2\alpha-1}&\ll T^{-2\varepsilon}+T^{-\varepsilon}\int_T^{T^{c+\varepsilon}}(\delta_2^{(2)}(x))^2x^{-2\alpha-1}e^{-2(\frac{x}{Y})^h}dt\\ &\ll T^{-\varepsilon}\left(\int_T^{2Y}(\delta_2^{(2)}(x))^2x^{-2\alpha-1}e^{-2(\frac{x}{Y})^h}dx+1\right), \qquad (56)\end{aligned}$$

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whenever $c \leq \frac{1}{3}(1+2\alpha-5\varepsilon)$. Thus our choice for $c = \min(\frac{1}{2}(\alpha-\varepsilon), \frac{1}{3}(1+2\alpha-5\varepsilon))$. Now again by (21),

$$\begin{aligned}\int_{T^{1-c}}^{2T} |\mathcal{K}_T|^2 dt &\leq \int_0^1 \int_{T^{1-c}}^{2T} \left| \sum_{T\leq n\leq 2Y-1} \delta_2^{(2)}(n+v)(n+v)^{-s_\alpha-1} e^{-(\frac{(n+v)}{Y})^h} \right|^2 dtdv \\ &\ll \int_0^1 \sum_{T\leq n\leq 2Y-1} (\delta_2^{(2)}(n+v))^2 (n+v)^{-2\alpha-1} e^{-2(\frac{(n+v)}{Y})^h} dv \\ &= \int_T^{2Y} (\delta_2^{(2)}(x))^2 x^{-2\alpha-1} e^{-2(\frac{x}{Y})^h} dx, \end{aligned} \tag{57}$$

if T and Y are non integers. Then it remains to estimate the integral $\int_{T^{1-c}}^{2T} |\mathcal{L}_T|^2 dt$. If T and Y are not integers, then by Cauchy-Schwartz inequality, we have

$$\int_{T^{1-c}}^{2T} |\mathcal{L}_T|^2 dt \ll \int_T^{2Y} (\delta_2^{(2)}(x))^2 x^{-2\alpha-1} e^{-2(\frac{x}{Y})^h} dx. \tag{58}$$

Now collecting (55), (56), (57) and (58) and substituting back them in (54) we get

$$\int_J \frac{|F(s_\alpha)|^2}{|s_\alpha|^2} dt \ll \int_T^{2Y} \delta_2^{(2)}(x))^2 x^{-2\alpha-1} e^{-2(\frac{x}{Y})^h} dx. \tag{59}$$

Covering the interval J into dyadic interval of the form $[2^{j-1}T^{1-c}, 2^j T^{1-c}] \cap J$, and then by (43), we have

$$\int_J \frac{|F(s_\alpha)|^2}{|s_\alpha|^2} dt \gg \sum_{j=1}^{c(T)} \int_{J^{(2)}(2^{j-1}T^{1-c})} \frac{|F(s_\alpha)|^2}{|s_\alpha|^2} dt \gg \sum_{j=1}^{c(T)} \log T \gg (\log T)^2, \tag{60}$$

where $c(T) = c\frac{\log T}{\log 2}$. Now from (59) and (60) and assuming (44) false, we infer from that

$$\begin{aligned}(\log T)^2 &\ll \int_J \frac{|F(s_\alpha)|^2}{|s_\alpha|^2} dt \ll \int_T^{2Y} (\delta_2^{(2)}(x))^2 x^{-2\alpha-1} e^{-2(\frac{x}{Y})^h} dx \\ &= \left(\int_1^x (\delta_2^{(2)}(y))^2 dy \right) x^{-2\alpha-1} e^{-2(\frac{x}{Y})^h} \Big|_T^{2Y} + \int_T^{2Y} \left(\int_1^x (\delta_2^{(2)}(y))^2 dy \right) \\ &\quad \times ((2\alpha+1)x^{-2\alpha-2} + 2x^{h-2\alpha-2}Y^{-h}) e^{-2(\frac{x}{Y})^h} dx \\ &= o(\log T) + o\left(\int_T^{2Y} x^{-1} \log x \; e^{-2(\frac{x}{Y})^h} dx \right) = o(\log^2 T), \end{aligned}$$

as $T \to \infty$ giving a contradiction and proving the lemma. □

Now if we can show $\int_{J^{(2)}(x)} \frac{|F(s_\alpha)|^2}{|s_\alpha|^2} dt \gg \log T$, then we will be done with our proof of Theorem 1.3.

Lemma 4.6. *We have for any $A > 0$,*

$$\int_A^{A+H} |F(s_\alpha)|^2 dt \gg HA \quad \textit{for } H \geq A^\varepsilon.$$

Proof. Let $s_\alpha = \alpha + it$. Now, employing the functional equation of $\zeta(s)$

$$\zeta(s) = \chi(s)\zeta(1-s),$$

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where $\chi(s) = (2\pi)^{s-1} 2 \sin \frac{\pi s}{2} \Gamma(1-s)$, and (47), one finds

$$\begin{aligned}|F(s_\alpha)| &= |\zeta^2(2s_\alpha)\zeta^2(3s_\alpha)G^{-1}(s_\alpha)H(s_\alpha)|\\ &= |\chi^2(2s_\alpha)\chi^2(3s_\alpha)\zeta^2(1-2s_\alpha)\zeta^2(1-3s_\alpha)G^{-1}(s_\alpha)H(s_\alpha)|\\ &\gg |t^{2(\frac{1}{2}-2\alpha)}t^{2(\frac{1}{2}-3\alpha)}\zeta^2(1-2s_\alpha)\zeta^2(1-3s_\alpha)G^{-1}(s_\alpha)H(s_\alpha)|\\ &= |t|^{\frac{1}{2}}|\zeta^2(1-2s_\alpha)\zeta^2(1-3s_\alpha)G^{-1}(s_\alpha)H(s_\alpha)|.\end{aligned}$$

Hence we obtain that

$$|F(s_\alpha)|^2 \gg |t||\zeta^4(1-2s_\alpha)\zeta^4(1-3s_\alpha)G^{-2}(s_\alpha)H^2(s_\alpha)|.$$

Next, we consider $U(s) := \zeta^4(1-2s)\zeta^4(1-3s)G^{-2}(s)H^2(s)$ and employ [2, Theorem 3] to get that

$$\int_A^{H+A} |U(s_\alpha)|^2 dt_0 \gg H \quad \text{for } H \geq A^\varepsilon, \tag{61}$$

and hence the result. □

Proof of Theorem 1.3. Now applying Lemma 4.6 to each connected component of $J^{(2)}(x)$ and summing them we get

$$\int_{J^{(2)}(x)} \frac{|F(s_\alpha)|^2}{|s_\alpha|^2} dt \gg \log T.$$

This result together with Lemma 4.5 would imply statement Theorem 1.3.

□

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References

[1] R. C. Baker, The square-free divisor problem, *Quart. J. Math. Oxford Ser.*, (2) **45**, no 179, (1994), 269-277.
[2] R. Balasubramanian, An improvement ona a theorem of Titchmarsh on the mean square of $\zeta(\frac{1}{2}+it)$, *Proc. London Math. Soc.*, **36**, (1978), 540-576.
[3] P. Batman and E. Grosswald, On a theorem of Erdös and Szekeres, *Illinois J. Math.*, **2**, (1958), 88-98.
[4] R. Balasubramanian, K. Ramachandra and M. V. Subbarao, On the error function in the asymptotic formula for the counting function of k-full numbers, *Acta Arith.*, **50**, (1988), 107-118.
[5] P. Erdős and G. Szekeres, Uber die Anzahl der Abelschen Gruppen gegebener Ordnung und tuber ein verwandtes zahlentheoretisches Problem, *Acta Univ Szeged*, **7**, (1934), 95-102.
[6] D. R. Heath-Brown, The Pjatecki-Sapiro prime number theorem, *J. Number Theory*, **16**, (1983), no. 2, 242-266.
[7] O. Hölder, Uber einen asymptotischen Ausdruck, *Acta Math.*, **59**, (1932), 89-97.
[8] A. Ivic, The Riemann Zeta-function. Wiley, London (1985).
[9] A. Ivic, The general divisor problem, *J. Number Theory*, **21**, (1987), 73-91.
[10] M. Kühleitner, E. G. Nowak, An Omega Theorem for a Class of Arithmetic Functions, *Math. Nachr.*, **165**, (1994), 79-98.
[11] A. H. Ledoan and A. Zaharescu, Square-Full Divisors of Square-Full Integers, *Integers*, **10**, (2010), 243-256.
[12] J. Liu and Y. Ye, Perron's formula and the prime number theorem for automorphic L-functions, *Pure Appl. Math. Quart.*, **3**, (2007), 481–497.
[13] F. Mertens, Uber einige asymptotische Gesetze der Zahlentheorie, *J. Reine. Angew. Math.*, **77**, (1874), 289-338.
[14] H. L. Montgomery and R. C. Vaughan, The large sieve, *Mathematika*, **20**, (1973), 119–134.

[15] W. G. Nowak and M. Schmeier, Conditional asymptotic formulae for a class of arithmetic functions, *Proc. Amer. Math. Soc.*, **103**, no.3, (1988), 713-717.
[16] H. Rademacher, Topics in analytic number theory, ser. Die Grundlehren der mathematischen Wissenschaften, Band 169. Springer-Verlag, New York-Heidelberg, 1973, edited by E. Grosswald, J. Lehner and M. Newman.
[17] D. Suryanarayana and S. R. C. Rao, The distribution of square-full integers, *Arkiv főr Matematik*, **11**, (1973), 195-201.
[18] E. C. Titchmarsh, The Theory of the Riemann Zeta-Function, 2nd ed. rev. by D. R. Heath-Brown, Oxford Univ. Press, (1986).

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