Can. J. Math. Vol. **46** (4), 1994 pp. 673–686

CO-H-STRUCTURES ON MOORE SPACES OF TYPE $(G, 2)$

MARTIN ARKOWITZ AND MAREK GOLASINSKI

ABSTRACT. We consider the set (of homotopy classes) of co-H-structures on a Moore space $M(G,n)$, where G is an abelian group and n is an integer ≥ 2. It is shown that for $n > 2$ the set has one element and for $n = 2$ the set is in one-one correspondence with $\mathrm{Ext}(G, G \otimes G)$. We make a detailed investigation of the co-H-structures on $M(G,2)$ in the case $G = \mathbb{Z}_m$, the integers mod m. We give a specific indexing of the co-H-structures on $M(\mathbb{Z}_m, 2)$ and of the homotopy classes of maps from $M(\mathbb{Z}_m, 2)$ to $M(\mathbb{Z}_n, 2)$ by means of certain standard homotopy elements. In terms of this indexing we completely determine the co-H-maps from $M(\mathbb{Z}_m, 2)$ to $M(\mathbb{Z}_n, 2)$ for each co-H-structure on $M(\mathbb{Z}_m, 2)$ and on $M(\mathbb{Z}_n, 2)$. This enables us to describe the action of the group of homotopy equivalences of $M(\mathbb{Z}_m, 2)$ on the set of co-H-structures of $M(\mathbb{Z}_m, 2)$. We prove that the action is transitive. From this it follows that if m is odd, all co-H-structures on $M(\mathbb{Z}_m, 2)$ are associative and commutative, and if m is even, all co-H-structures on $M(\mathbb{Z}_m, 2)$ are associative and non-commutative.

1. **Introduction.** Let G be an abelian group and n an integer ≥ 2. A *Moore space* of type (G,n) is a 1-connected, CW-complex X such that

$$\widetilde{H}_i(X) = \begin{cases} 0 & i \neq n \\ G & i = n, \end{cases}$$

where $\widetilde{H}_i$ denotes the i-th reduced homology group. It is well known that Moore spaces exist and that any two of type (G,n) have the same homotopy type. We denote a Moore space of type (G,n) by $M(G,n)$. Since the suspension $\Sigma M(G, n-1)$ of a Moore space of type $(G, n-1)$ is a Moore space of type (G,n), it follows that every Moore space $M(G,n)$, $n \geq 3$, is a co-H-group, *i.e.*, a co-H-space with associative comultiplication. This is also true for $M(G,2)$ (see §3). Thus homotopy classes of maps of $M(G,n)$ into an arbitrary space X can be given natural group structure. In this way we obtain the n-th homotopy group of X with coefficients in G, denoted $\pi_n(G; X)$. This illustrates an important use of Moore spaces, namely, to introduce coefficients into homotopy groups. Another important application of Moore spaces is in the construction of the homology decomposition of a space [Hi, Chapter 8] which is the dual of the Postnikov decomposition.

In this paper we investigate the number of homotopy classes of comultiplications of $M(G,n)$ and their properties. We see in §3 that for $n \geq 3$, $M(G,n)$ has a unique comultiplication and that the comultiplications of $M(G,2)$ are in one-one correspondence with

The first-named author is grateful to Dana Williams for TEXnical help and the second-named author is indebted to Dartmouth College for its hospitality during the time that this work was done.

Received by the editors January 8, 1993.

AMS subject classification: Primary: 55P45; secondary: 55P40.

the group $\mathrm{Ext}(G, G \otimes G)$. In §5 and §6 we specialize to the case when $G = \mathbb{Z}_m$, the integers mod m, and obtain detailed information about the comultiplcations of $M(\mathbb{Z}_m, 2)$. We give an explicit description of the comultiplications of $M(\mathbb{Z}_m, 2)$, the co-H-maps from $M(\mathbb{Z}_m, 2)$ to $M(\mathbb{Z}_n, 2)$ and the homotopy self-equivalences of $M(\mathbb{Z}_m, 2)$, all in terms of certain standard homotopy elements. Our main result (Theorem 14) is a formula which completely characterizes the co-H-maps from $M(\mathbb{Z}_m, 2)$ to $M(\mathbb{Z}_n, 2)$ with respect to each comultiplication of $M(\mathbb{Z}_m, 2)$ and each comultiplication of $M(\mathbb{Z}_n, 2)$. From this we determine the action of the group of homotopy equivalences of $M(\mathbb{Z}_m, 2)$ on the set of comultiplications of $M(\mathbb{Z}_m, 2)$. We then deduce that all comultiplications of $M(\mathbb{Z}_m, 2)$ are equivalent. It follows that they are all associative, and are either all commutative or are all non-commutative, depending on the parity of m.

We conclude this section by giving our notation and standing assumptions. All spaces are connected, based spaces of the based homotopy type of based CW-complexes. All maps and homotopies are to preserve base point. We do not distinguish notationally between a map and its homotopy class. Thus we ambiguously regard a map $f: X \longrightarrow Y$ as an element of $[X, Y]$, the homotopy classes of maps from X to Y. Maps $g: Y \longrightarrow Y'$ and $h: X' \longrightarrow X$ induce functions $g_*: [X, Y] \longrightarrow [X, Y']$ and $h^*: [X, Y] \longrightarrow [X,' Y]$ in the usual way. If ΣA is the suspension of the space A, then for every space Y, $[\Sigma A, Y]$ has a group structure, which is denoted by '+', such that $g_*: [\Sigma A, Y] \longrightarrow [\Sigma A, Y']$ is a homomorphism. The identity map of a space or group is denoted by 1 and the constant map of spaces is denoted by 0. For a map $f: X \longrightarrow Y$ the induced homomorphisms of homology and homotopy groups are written as follows:

$$f_*: H_n(X) \longrightarrow H_n(Y) \quad \text{and} \quad f_\#: \pi_n(X) \longrightarrow \pi_n(Y).$$

For an abelian group G and integer $n \geq 2$, there is the exact sequence of homotopy groups with coefficients [Hi, Chapter 5]

$$0 \longrightarrow \mathrm{Ext}\big(G, \pi_{n+1}(X)\big) \xrightarrow{\lambda} \pi_n(G; X) \xrightarrow{\eta} \mathrm{Hom}\big(G, \pi_n(X)\big) \longrightarrow 0,$$

where $\pi_n(G; X) = [M(G, n), X]$. The homomorphism η is defined by $\eta(f) = f_\# : G = \pi_n\big(M(G, n)\big) \longrightarrow \pi_n(X)$.

2. **Comultiplications.** A *comultiplication* or *co-H-structure* on a space X is a map $\varphi: X \longrightarrow X \vee X$ such that $j\varphi = \Delta$, where $j: X \vee X \longrightarrow X \times X$ is the inclusion and $\Delta: X \longrightarrow X \times X$ is the diagonal map. (Recall that this is equality of homotopy classes or homotopy of maps.) Equivalently, $\varphi: X \longrightarrow X \vee X$ is a comultiplication if and only if $q_1\varphi = 1 = q_2\varphi: X \longrightarrow X$, where $q_1, q_2: X \vee X \longrightarrow X$ are the two projections. A space X together with a comultiplication φ is called a *co-H-space*. If (X', φ') and (X, φ) are co-H-spaces and $h: X' \longrightarrow X$ is a map, we say that $h: (X', \varphi') \longrightarrow (X, \varphi)$ is a *co-H-map* if $\varphi h = (h \vee h)\varphi'$.

Now let (X, φ) be a co-H-space. Then φ is *commutative* if $T\varphi = \varphi: X \longrightarrow X \vee X$, where $T: X \vee X \longrightarrow X \vee X$ is the map which interchanges coordinates. We say that φ is *associative* if $(\varphi \vee 1)\varphi = (1 \vee \varphi)\varphi: X \longrightarrow X \vee X \vee X$. If φ is associative, we call (X, φ) an *associative co-H-space* or a *co-H-group* (see [H-M-R, Theorem 2.3]). If (X,φ) is a co-H-group, then the comultiplication φ induces group structure on $[X,Y]$ for all spaces Y such that $g_*\colon [X,Y] \longrightarrow [X,Y']$ is a homomorphism. The primary example of a co-H-group is the suspension of a space; the suspension of a map is then a co-H-map.

LEMMA 1. *If (X,φ) is an associative co-H-space, and $j_*\colon [X,Y\vee Y] \longrightarrow [X,Y\times Y]$ is the homomorphism induced by the inclusion $j\colon Y\vee Y \longrightarrow Y\times Y$, then there is a function $\rho\colon [X,Y\times Y] \longrightarrow [X,Y\vee Y]$ such that $j_*\rho = 1$. Furthermore, if $[X,Y\vee Y]$ is abelian, ρ is a homomorphism.*

PROOF. We use the following notation: $p_1,p_2\colon Y\times Y \longrightarrow Y$ for the two projections, $j_1,j_2\colon Y \longrightarrow Y\times Y$ for the two inclusions and $i_1,i_2\colon Y \longrightarrow Y\vee Y$ for the two inclusions. Clearly $\theta\colon [X,Y\times Y] \longrightarrow [X,Y]\times[X,Y]$ defined by $\theta(\alpha) = \big(p_{1*}(\alpha),p_{2*}(\alpha)\big) = (p_1\alpha,p_2\alpha)$ is an isomorphism. The inverse $\tilde\theta$ of θ is given by $\tilde\theta(\beta,\gamma) = j_{1*}(\beta)+j_{2*}(\gamma)$ for $\beta,\gamma\in[X,Y]$. Now define $\mu(\beta,\gamma)\in[X,Y\vee Y]$ by $\mu(\beta,\gamma) = i_{1*}(\beta)+i_{2*}(\gamma)$. Note that $\mu\colon [X,Y]\times[X,Y] \longrightarrow [X,Y\vee Y]$ is a homomorphism if $[X,Y\vee Y]$ is abelian, but not necessarily otherwise. We set $\rho=\mu\theta$ and need to show $j_*\rho = 1$. For this is suffices to prove $j_*\mu = \tilde\theta$, which is easily verified. ∎

For an associative co-H-space (X,φ), let $\mathcal{C}(X)\subseteq[X,X\vee X]$ be the set of comultiplications of X. We use the following notation below: If A is any space and (U,V) is a pair of spaces $(V\subseteq U)$, then $\pi_1(A;U,V)$ denotes the homotopy classes of maps of pairs $(CA,A)\longrightarrow(U,V)$, where CA is the cone on A.

PROPOSITION 2. *If (X,φ) is an associative co-H-space, then $\mathcal{C}(X)$ is in one-one correspondence with the group $\pi_1(X;X\times X,X\vee X)$.*

PROOF. Consider the exact sequence of the inclusion j [Hi, Theorem 4.1]

$$[\Sigma X,X\vee X]\xrightarrow{j_*}[\Sigma X,X\times X]\longrightarrow\pi_1(X;X\times X,X\vee X)\xrightarrow{\partial}[X,X\vee X]\xrightarrow{j_*}[X,X\times X].$$

By Lemma 1, both j_*'s are onto and so we have a short exact sequence of groups

$$1\longrightarrow\pi_1(X;X\times X,X\vee X)\xrightarrow{\partial}[X,X\vee X]\xrightarrow{j_*}[X,X\times X]\longrightarrow 1.$$

If $\Delta\in[X,X\times X]$ is the diagonal, then $\mathcal{C}(X) = j_*^{-1}(\Delta)$. But $j_*^{-1}(\Delta)$ is in one-one correspondence with $\operatorname{Ker} j_* = \operatorname{Image}\partial \approx \pi_1(X;X\times X,X\vee X)$. ∎

REMARK 3 (*cf.* [A-L, LEMMA 2.1]). If F is the homotopy fibre of $j\colon X\vee X\longrightarrow X\times X$, then in the above proof one can replace the relative group $\pi_1(X;X\times X,X\vee X)$ with $[X,F]$ and conclude that $\mathcal{C}(X)$ is in one-one correspondence with $[X,F]$.

3. **Comultiplications of Moore spaces.** As mentioned in §1, every Moore space $M(G,n)$, $n\geq 3$, is a suspension. We now show this for $M(G,2)$. We write $G = F/R$, where F is a free-abelian group. Let $\{x_i\}_{i\in I}$ be a set of free generators of F and $\{y_j\}_{j\in J}$ a set of free generators of R. We consider

$$S^1_I = \bigvee_{i\in I} S^1_i \quad\text{and}\quad S^1_J = \bigvee_{j\in J} S^1_j,$$

https://doi.org/10.4153/CJM-1994-037-0 Published online by Cambridge University Press

where $S^1_i = S^1_j = S^1$, the circle. We define a map $f\colon S^1_J \longrightarrow S^1_I$ by defining $f|S^1_j$, which is an element of $\pi_1(S^1_I)$, the free (non-abelian) group generated by the set $\{x_i\}_{i\in I}$. This element is taken to be a lifting of the element $y_j \in F$ to $\pi_1(S^1_I)$ under the canonical epimorphism $\pi_1(S^1_I) \longrightarrow F$. Now let $L(G)$ be the mapping cone of f, $L(G) = S^1_I \cup_f CS^1_J$. Clearly $\tilde{H}_i\big(L(G)\big) = 0$ for $i \neq 1$ and $\tilde{H}_1\big(L(G)\big) = G$. Thus $\Sigma L(G)$ is a Moore space of type $(G, 2)$. By uniqueness, $M(G, 2) = \Sigma L(G)$.

Thus all Moore spaces $M(G, n)$, $n \geq 2$, are co-H-groups.

PROPOSITION 4. *(i) The set of comultiplications $\mathcal{C}\big(M(G,2)\big)$ is in one-one correspondence with the group* $\mathrm{Ext}(G, G\otimes G)$.

(ii) If $n > 2$, then $\mathcal{C}\big(M(G,n)\big)$ has one element, the suspension comultiplication.

PROOF. By Proposition 2 with $X = M(G,n)$, $\mathcal{C}(X)$ and $\pi_1(X; X\times X, X\vee X)$ are in one-one correspondence, and the latter group is just the relative homotopy group with coefficients $\pi_{n+1}(G; X\times X, X\vee X)$ (see [Hi, pp. 26–27]). Thus there is a short exact universal coefficient sequence

$$0 \longrightarrow \mathrm{Ext}\big(G, \pi_{n+2}(X\times X, X\vee X)\big) \longrightarrow \pi_{n+1}(G; X\times X, X\vee X) \xrightarrow{\eta} \mathrm{Hom}\big(G, \pi_{n+1}(X\times X, X\vee X)\big) \longrightarrow 0.$$

But $H_i(X\times X, X\vee X) = 0$ for $i < 2n$ and $H_{2n}(X\times X, X\vee X) = H_n(X)\otimes H_n(X) = G\otimes G$. Hence, by the Hurewicz theorem, $\pi_i(X\times X, X\vee X) = 0$ for $i < 2n$ and $\pi_{2n}(X\times X, X\vee X) = G\otimes G$. Since $n+1 < 2n$, the above sequence yields

$$\pi_{n+1}(G; X\times X, X\vee X) = \mathrm{Ext}\big(G, \pi_{n+2}(X\times X, X\vee X)\big) = \begin{cases} 0 & n > 2 \\ \mathrm{Ext}(G, G\otimes G) & n = 2. \end{cases} \qquad \blacksquare$$

We illustrate Proposition 4 by determining the size of $\mathcal{C}\big(M(G,2)\big)$ for certain groups G. If the group $\mathrm{Ext}(G, G\otimes G)$ has order k, we shall say that $M(G,2)$ has k *comultiplications*.

EXAMPLES 5. (i) If G is free-abelian, then $\mathrm{Ext}(G, G\otimes G) = 0$.

(ii) If G is a divisible group, then $G\otimes G$ is divisible, and so $\mathrm{Ext}(G, G\otimes G) = 0$.

(iii) Let P be a set of primes and $\mathbb{Z}_{(P)} = \{\frac{a}{b} \in \mathbb{Q} \mid p$ does not divide b, for every $p \in P\}$. Then $\mathbb{Z}_{(P)} \otimes \mathbb{Z}_{(P)} = \mathbb{Z}_{(P)}$ and so $\mathrm{Ext}(\mathbb{Z}_{(P)}, \mathbb{Z}_{(P)}\otimes\mathbb{Z}_{(P)}) = \mathrm{Ext}(\mathbb{Z}_{(P)}, \mathbb{Z}_{(P)})$. But $\mathbb{Z}_{(P)}$ is P-local and so $\mathrm{Ext}(\mathbb{Z}_{(P)}, \mathbb{Z}_{(P)}) = \mathrm{Ext}(\mathbb{Z}, \mathbb{Z}_{(P)})$. However the latter group is zero.

Thus for each of the groups in G in (i)–(iii), $M(G,2)$ has one comultiplication.

(iv) Let $G = \mathbb{Z}_m$, the integers mod m. Then $\mathrm{Ext}(\mathbb{Z}_m, \mathbb{Z}_m\otimes\mathbb{Z}_m) = \mathrm{Ext}(\mathbb{Z}_m, \mathbb{Z}_m) = \mathbb{Z}_m$. Therefore $M(\mathbb{Z}_m, 2)$ has m comultiplications.

(v) Let $G = \mathbb{Z}_m \oplus \mathbb{Z}_n$. Then $G\otimes G = \mathbb{Z}_m\oplus\mathbb{Z}_n\oplus\mathbb{Z}_d\oplus\mathbb{Z}_d$, where d is the greatest common divisor (m,n). Therefore

$$\begin{aligned}\mathrm{Ext}(G, G\otimes G) &= \mathrm{Ext}(\mathbb{Z}_m, \mathbb{Z}_m\oplus\mathbb{Z}_n\oplus\mathbb{Z}_d\oplus\mathbb{Z}_d) \oplus \mathrm{Ext}(\mathbb{Z}_n, \mathbb{Z}_m\oplus\mathbb{Z}_n\oplus\mathbb{Z}_d\oplus\mathbb{Z}_d)\\ &= \mathbb{Z}_m\oplus\mathbb{Z}_n\oplus\bigoplus_{i=1}^{6}\mathbb{Z}_d.\end{aligned}$$

Thus $M(\mathbb{Z}_m \oplus \mathbb{Z}_n, 2)$ has mnd^6 comultiplications. For example, if P is the real projective plane, then $M(\mathbb{Z}_2 \oplus \mathbb{Z}_2, 2) = \Sigma P \vee \Sigma P$. Thus $\Sigma P \vee \Sigma P$ has 2^8 comultiplications.

(vi) Let $G = \mathbb{Z} \oplus \mathbb{Z}_m$. It now follows as in (v) that $\operatorname{Ext}(G, G \otimes G) = \mathbb{Z}_m \oplus \mathbb{Z}_m \oplus \mathbb{Z}_m \oplus \mathbb{Z}_m$. Therefore $M(G, 2)$ has m^4 comultiplications.

(vii) Let $G = \oplus_{i=1}^{\infty} G_i$, where $G_i = \mathbb{Z}_m$ for every i. Then $G \otimes G = G$. Hence

$$\begin{aligned} \operatorname{Ext}(G, G \otimes G) &= \operatorname{Ext}\Big(\bigoplus_{i=1}^{\infty} G_i, G\Big) \\ &= \prod_{i=1}^{\infty} \operatorname{Ext}(\mathbb{Z}_m, G) \quad \text{by [H-S, p. 97]} \\ &= \prod_{i=1}^{\infty} G/mG. \end{aligned}$$

But $mG = 0$ and so $\operatorname{Ext}(G, G \otimes G) = \prod(\oplus \mathbb{Z}_m)$. More precisely, $\operatorname{Ext}(G, G \otimes G) = \prod_{j=1}^{\infty} B_j$, where $B_j = \oplus_{i=1}^{\infty} G_i = \oplus_{i=1}^{\infty} \mathbb{Z}_m$, for each j. Note that $\prod_{j=1}^{\infty} B_j$ has the cardinality of the continuum. Therefore $M(G, 2)$ has a continuum of (homotopically distinct) comultiplications.

For the remainder of this paper we focus our attention on the Moore spaces $M(\mathbb{Z}_m, 2)$.

4. **Homotopy elements of finite order.** In this section we consider the order of certain homotopy elements with reference to the Moore space $M(\mathbb{Z}_m, 2)$. These results are needed in the next section, but may be of independent interest.

We shall adopt the following notation for the remainder of the paper: M for $M(\mathbb{Z}_m, 2)$, N for $M(\mathbb{Z}_n, 2)$, $i_1, i_2\colon M \longrightarrow M \vee M$ and $\iota_1, \iota_2\colon S^2 \longrightarrow S^2 \vee S^2$ for the inclusions, $q_1, q_2\colon M \vee M \longrightarrow M$ for the projections and $\phi\colon S^3 \longrightarrow S^2$ for the Hopf map. We also use $i_1', i_2'\colon N \longrightarrow N \vee N$ and $q_1', q_2'\colon N \vee N \longrightarrow N$ for the inclusions and projections. We let d be the greatest common divisor of m and n, $d = (m, n)$, and δ the greatest common divisor of m, $2n$ and n^2, $\delta = \big(m, (2n, n^2)\big)$. If r is an integer, then $\mathbf{r}\colon \Sigma A \longrightarrow \Sigma A$ is the element $r \cdot 1$ in the group $[\Sigma A, \Sigma A]$. For spheres, $\mathbf{r}\colon S^n \longrightarrow S^n$ is just the map of degree r. Observe that the mapping cone of $\mathbf{m}\colon S^2 \longrightarrow S^2$ is $M = M(\mathbb{Z}_m, 2) = S^2 \cup_{\mathbf{m}} e^3$. Thus there is an inclusion $i\colon S^2 \longrightarrow M$ and a projection $q\colon M \longrightarrow S^3$. The maps i and q are suspensions. We also let i' denote the inclusion $S^2 \longrightarrow N$ and q' the projection $N \longrightarrow S^3$.

We consider the composition $i'\phi q$

$$M \xrightarrow{q} S^3 \xrightarrow{\phi} S^2 \xrightarrow{i'} N.$$

PROPOSITION 6. *The element $i'\phi q \in [M, N] = \pi_2(\mathbb{Z}_m; N)$ has order δ.*

PROOF. Consider the homomorphism of universal coefficient sequences induced by i'

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \operatorname{Ext}\big(\mathbb{Z}_m, \pi_3(S^2)\big) & \xrightarrow{\lambda} & \pi_2(\mathbb{Z}_m; S^2) & \longrightarrow & \operatorname{Hom}\big(\mathbb{Z}_m, \pi_2(S^2)\big) & \longrightarrow & 0 \\ & & \Big\downarrow {\scriptstyle \operatorname{Ext}(\mathbb{Z}_m, i'_\#)} & & \Big\downarrow {\scriptstyle i'_*} & & \Big\downarrow & & \\ 0 & \longrightarrow & \operatorname{Ext}\big(\mathbb{Z}_m, \pi_3(N)\big) & \xrightarrow{\lambda'} & \pi_2(\mathbb{Z}_m; N) & \longrightarrow & \operatorname{Hom}\big(\mathbb{Z}_m, \pi_2(N)\big) & \longrightarrow & 0. \end{array}$$

We first show that i'_* carries an element of order m in $\pi_2(\mathbb{Z}_m; S^2)$ into an element of order δ in $\pi_2(\mathbb{Z}_m; N)$. Now $\mathrm{Hom}\big(\mathbb{Z}_m, \pi_2(S^2)\big) = \mathrm{Hom}(\mathbb{Z}_m, \mathbb{Z}) = 0$ and $\mathrm{Ext}\big(\mathbb{Z}_m, \pi_3(S^2)\big) = \mathrm{Ext}(\mathbb{Z}_m, \mathbb{Z}) = \mathbb{Z}_m$. Furthermore, $\pi_3(N) = \mathbb{Z}_{(2n,n^2)}$ (see *e.g.* [Si, Lemma 1]). Then

$$\mathrm{Ext}\big(\mathbb{Z}_m, \pi_3(N)\big) = \mathrm{Ext}(\mathbb{Z}_m, \mathbb{Z}_{(2n,n^2)}) = \mathbb{Z}_\delta.$$

Thus the above diagram becomes

$$\begin{array}{ccc} \mathbb{Z}_m = \mathrm{Ext}\big(\mathbb{Z}_m, \pi_3(S^2)\big) & \xrightarrow{\lambda} & \pi_2(\mathbb{Z}_m; S^2) \\ \big\downarrow \scriptstyle{\mathrm{Ext}(\mathbb{Z}_m, i'_\#)} & & \big\downarrow \scriptstyle{i'_*} \\ \mathbb{Z}_\delta = \mathrm{Ext}\big(\mathbb{Z}_m, \pi_3(N)\big) & \xrightarrow{\lambda'} & \pi_2(\mathbb{Z}_m; N) \end{array}$$

with λ an isomorphism and λ' a monomorphism. From the exact homotopy sequence of the pair (N, S^2), $i'_\#\colon \pi_3(S^2) \longrightarrow \pi_3(N)$ is onto. But $\mathrm{Ext}(\mathbb{Z}_m, G)$ is naturally isomorphic to G/mG for all G, and so the following diagram is commutative

$$\begin{array}{ccc} \mathbb{Z}_m = \mathrm{Ext}\big(\mathbb{Z}_m, \pi_3(S^2)\big) & \approx & \pi_3(S^2)/m\pi_3(S^2) = \mathbb{Z}_m \\ \big\downarrow \scriptstyle{\mathrm{Ext}(\mathbb{Z}_m, i'_\#)} & & \big\downarrow \scriptstyle{\tilde{i}'_\#} \\ \mathbb{Z}_\delta = \mathrm{Ext}\big(\mathbb{Z}_m, \pi_3(N)\big) & \approx & \pi_3(N)/m\pi_3(N) = \mathbb{Z}_\delta, \end{array}$$

where $\tilde{i}'_\#$ is induced by $i'_\#$. Since $i'_\#$ is onto, $\tilde{i}'_\#$ is onto. Hence $\mathrm{Ext}(\mathbb{Z}_m, i'_\#)\colon \mathbb{Z}_m \longrightarrow \mathbb{Z}_\delta$ is onto. Therefore $i'_*\colon \pi_2(\mathbb{Z}_m; S^2) \longrightarrow \pi_2(\mathbb{Z}_m; N) = [M, N]$ carries an element of order m onto an element of order δ. Thus to complete the proof it suffices to prove that $\phi q \in \pi_2(\mathbb{Z}_m; S^2) = [M, S^2]$ has order m.

Consider the exact sequence

$$[S^3, S^2] \xrightarrow{\mathbf{m}^*} [S^3, S^2] \xrightarrow{q^*} [M, S^2] \xrightarrow{i^*} [S^2, S^2],$$

where $\mathbf{m}^*\colon \pi_3(S^2) = \mathbb{Z} \longrightarrow \pi_3(S^2) = \mathbb{Z}$ is multiplication by m. Suppose $0 = k(\phi q) = q^*(k\phi)$ for some positive integer k. Then $k\phi = \mathbf{m}^*(\alpha) = m\alpha$ for some $\alpha \in \pi_3(S^2)$. But $\alpha = l\phi$ for some integer l. Therefore $k\phi = ml\phi$, and so m divides k. This proves that ϕq has order m, and hence $i'\phi q$ had order δ. ∎

Next consider the Whitehead product $[i'_1 i', i'_2 i'] \in \pi_3(N \vee N)$ of $i'_1 i', i'_2 i' \in \pi_2(N \vee N)$. We will show that $[i'_1 i', i'_2 i']q \in [M, N \vee N]$ has order d. To do this we first prove

LEMMA 7. *The element $[i'_1 i', i'_2 i'] \in \pi_3(N \vee N)$ has order n.*

PROOF. Let us consider the pairs $(S^2 \times S^2, S^2 \vee S^2)$ and $(N \times N, N \vee N)$. Then $H_i(S^2 \times S^2, S^2 \vee S^2) = 0$ for $i < 4$ and $H_4(S^2 \times S^2, S^2 \vee S^2) = H_2(S^2) \otimes H_2(S^2) = \mathbb{Z}$. Similarly $H_i(N \times N, N \vee N) = 0$ for $i < 4$ and $H_4(N \times N, N \vee N) = H_2(N) \otimes H_2(N) = \mathbb{Z}_n$. Let $\hat{\imath}'\colon (S^2 \times S^2, S^2 \vee S^2) \longrightarrow (N \times N, N \vee N)$ be the map of pairs obtained from $i' \times i'$. Since $i'_*\colon H_2(S^2) \longrightarrow H_2(N)$ is onto, $\hat{\imath}'_*\colon H_4(S^2 \times S^2, S^2 \vee S^2) \longrightarrow H_4(N \times N, N \vee N)$ is onto. By the Hurewicz Theorem, $\pi_4(S^2 \times S^2, S^2 \vee S^2) = \mathbb{Z}$, $\pi_4(N \times N, N \vee N) = \mathbb{Z}_n$ and

https://doi.org/10.4153/CJM-1994-037-0 Published online by Cambridge University Press

$\hat{\imath}'_\#\colon \pi_4(S^2 \times S^2, S^2 \vee S^2) \longrightarrow \pi_4(N \times N, N \vee N)$ is onto. Now the map $\hat{\imath}'$ of pairs yields a commutative diagram

$$\begin{array}{ccc} \pi_4(S^2 \times S^2, S^2 \vee S^2) & \xrightarrow{\partial} & \pi_3(S^2 \vee S^2) \\ \downarrow{\scriptstyle \hat{\imath}'_\#} & & \downarrow{\scriptstyle (i' \vee i')_\#} \\ \pi_4(N \times N, N \vee N) & \xrightarrow{\partial'} & \pi_3(N \vee N). \end{array}$$

We have that ∂ and ∂' are monomorphisms (see Proposition 2) and that $\hat{\imath}'_\#$ is onto. It follows from the definition [Hi, p. 110] that the Whitehead product $[\iota_1, \iota_2] \in \pi_3(S^2 \vee S^2)$ of the inclusions $\iota_1, \iota_2 \in \pi_2(S^2 \vee S^2)$ equals $\partial\beta$, where $\beta \in \pi_4(S^2 \times S^2, S^2 \vee S^2) = \mathbb{Z}$ is a generator. Thus $\hat{\imath}'_\#(\beta)$ is a generator of $\pi_4(N \times N, N \vee N) = \mathbb{Z}_n$ and so $\partial'\hat{\imath}'_\#(\beta)$ has order n. But

$$\begin{aligned} \partial'\hat{\imath}'_\#(\beta) &= (i' \vee i')_\#[\iota_1, \iota_2] \\ &= [i'_1 i', i'_2 i']. \end{aligned}$$

■

PROPOSITION 8. *The element* $[i'_1 i', i'_2 i']q \in [M, N \vee N] = \pi_2(\mathbb{Z}_m; N \vee N)$ *has order* d.

PROOF. We have the following commutative diagram with exact rows

$$\begin{array}{ccccc} \pi_4(N \times N, N \vee N) & \xrightarrow{\mathbf{m}^*} & \pi_4(N \times N, N \vee N) & \xrightarrow{q^*} & \pi_3(\mathbb{Z}_m; N \times N, N \vee N) \\ \downarrow{\scriptstyle \partial'} & & \downarrow{\scriptstyle \partial'} & & \downarrow{\scriptstyle \tilde{\partial}} \\ \pi_3(N \vee N) & \xrightarrow{\mathbf{m}^*} & \pi_3(N \vee N) & \xrightarrow{q^*} & \pi_2(\mathbb{Z}_m; N \vee N), \end{array}$$

where $\mathbf{m}^*$ is multiplication by m and ∂' and $\tilde{\partial}$ are boundary monomorphisms in the exact sequences of the pair $(N \times N, N \vee N)$. Clearly $j_\#[i'_1 i', i'_2 i'] = 0$, and so $[i'_1 i', i'_2 i'] = \partial'\gamma$ for some $\gamma \in \pi_4(N \times N, N \vee N)$. Now suppose $0 = k[i'_1 i', i'_2 i']q = q^*(k[i'_1 i', i'_2 i'])$ for some positive integer k. Then

$$\begin{aligned} \tilde{\partial}q^*(k\gamma) &= q^*(k[i'_1 i', i'_2 i']) \\ &= 0 \end{aligned}$$

and therefore $q^*(k\gamma) = 0$. Thus for some $\alpha \in \pi_4(N \times N, N \vee N)$, $k\gamma = m\alpha$. Hence

$$\frac{n}{d}k\gamma = \frac{n}{d}m\alpha = 0$$

since $\pi_4(N \times N, N \vee N) = \mathbb{Z}_n$. But then $\frac{n}{d}k[i'_1 i', i'_2 i'] = \partial'(\frac{n}{d}k\gamma) = 0$ and d divides k by Lemma 7. Therefore $[i'_1 i', i'_2 i']q$ has order d. ■

5. **Maps of** $M(\mathbb{Z}_m, 2)$**.** In this section we use the results of §4 to give a specific indexing of the elements of $[M(\mathbb{Z}_m, 2), M(\mathbb{Z}_n, 2)] = \pi_2\big(\mathbb{Z}_m; M(\mathbb{Z}_n, 2)\big)$ and the elements, $\mathcal{C}\big(M(\mathbb{Z}_m, 2)\big)$. We also index the elements of the group of self homotopy equivalences of $M(\mathbb{Z}_m, 2)$. This will enable us in §6 to discuss and obtain concrete results on the set of

co-H-maps, the action of the group of self homotopy equivalences on the set of comultiplications, *etc.*

For the computations of this section and the next, we collect several known results which we shall often use. But first we state a proposition which is an immediate consequence of a result of Baues. The commutator (α, β) of two group elements in an additively written group is $-\alpha - \beta + \alpha + \beta$.

PROPOSITION 9. *If X is any space and $\alpha, \beta \in \pi_2(\mathbb{Z}_m; X)$, then the commutator*

$$(\alpha, \beta) = \frac{m(m-1)}{2}[\alpha i, \beta i]q,$$

where $i\colon S^2 \longrightarrow M(\mathbb{Z}_m, 2)$ is the inclusion and $q\colon M(\mathbb{Z}_m, 2) \longrightarrow S^3$ is the projection.

See [Ba, Proposition D.22, p. 164].

Recall that a group of nilpotency < 3 is one in which all commutators of length ≥ 3 are trivial.

PROPOSITION 10. *(i) $(\alpha+\beta)\phi = \alpha\phi+\beta\phi+[\alpha, \beta]$, where $\alpha, \beta \in \pi_2(X)$ and $\phi \in \pi_3(S^2)$ is the Hopf map.*

(ii) $\mathbf{r}\phi = \phi\mathbf{r}^2$.

(iii) If $\alpha \in \pi_2(\mathbb{Z}_m; X)$ and $\beta \in \pi_3(X)$, then the commutator $(\alpha, \beta q) = 0$.

(iv) For any space X, $\pi_2(\mathbb{Z}_m; X)$ is a group of nilpotency < 3. Moreover, for m odd, $\pi_2(\mathbb{Z}_m; X)$ is abelian.

(v) If π is a group of nilpotency < 3, then the commutator of elements of π is biadditive and

$$k(\alpha+\beta) = k\alpha + k\beta - \frac{k(k-1)}{2}(\alpha, \beta)$$

for any positive integer k and $\alpha, \beta \in \pi$.

Parts (i) and (ii) are standard and well known results. Parts (iii) and (iv) follow from Proposition 9. More generally, for any abelian group G, $\pi_2(G; X)$ is of nilpotency < 3 by [Wh, p. 464]. Part (v) consists of familiar facts from group theory.

Let $M = M(\mathbb{Z}_m, 2)$ and $N = M(\mathbb{Z}_n, 2)$ with $d = (m, n)$ and $\delta = \big(m, (2n, n^2)\big)$. We consider the group $[M, N] = \pi_2(\mathbb{Z}_m; N)$. By Proposition 6, $i'\phi q \in \pi_2(\mathbb{Z}_m; N)$ has order δ, and so generates a cyclic subgroup $H \subseteq \pi_2(\mathbb{Z}_m; N)$ of order δ. We then have the sequence

$$0 \longrightarrow H \longrightarrow \pi_2(\mathbb{Z}_m; N) \xrightarrow{\eta} \mathrm{Hom}(\mathbb{Z}_m, \mathbb{Z}_n) \longrightarrow 0$$

where $H = \mathbb{Z}_\delta$, $\mathrm{Hom}(\mathbb{Z}_m, \mathbb{Z}_n) = \mathbb{Z}_d$ and $\eta(\alpha) = \alpha_*\colon H_2(M) \longrightarrow H_2(N)$. We claim that this sequence is exact. This is seen by comparing it to the exact coefficient sequence (§1)

$$0 \longrightarrow \mathrm{Ext}\big(\mathbb{Z}_m, \pi_3(N)\big) \longrightarrow \pi_2(\mathbb{Z}_m; N) \xrightarrow{\eta} \mathrm{Hom}(\mathbb{Z}_m, \mathbb{Z}_n) \longrightarrow 0,$$

and showing that $H = \mathbb{Z}_\delta = \mathrm{Ext}\big(\mathbb{Z}_m, \pi_3(N)\big)$.

Next we consider the group $\mathrm{Hom}(\mathbb{Z}_m, \mathbb{Z}_n) = \mathbb{Z}_d$. We write ι to generically denote a generator of a finite cyclic group. Then if $\phi\colon \mathbb{Z}_m \longrightarrow \mathbb{Z}_n$ is a function, $\phi\iota = s\iota$ for some integer s, $0 \le s < n$. Clearly $\phi \in \mathrm{Hom}(\mathbb{Z}_m, \mathbb{Z}_n)$ if and only if $s = 0, \frac{n}{d}, 2\frac{n}{d}, \ldots, (d-1)\frac{n}{d}$. We write the homomorphism given by $\phi(\iota) = \frac{n}{d}\iota$ as $\phi = \frac{\mathbf{n}}{\mathbf{d}}$. The elements of $\mathrm{Hom}(\mathbb{Z}_m, \mathbb{Z}_n)$ are then $r\frac{\mathbf{n}}{\mathbf{d}}$, $0 \le r < d$.

Now we turn to the group $\pi_2(\mathbb{Z}_m; N) = [M, N]$. Clearly the following diagram commutes

$$\begin{array}{ccc} S^2 & \xrightarrow{\mathbf{m}} & S^2 \\ \downarrow \frac{\mathbf{m}}{\mathbf{d}} & & \downarrow \frac{\mathbf{n}}{\mathbf{d}} \\ S^2 & \xrightarrow{\mathbf{n}} & S^2 \end{array}$$

and so we obtain a map $\lambda\colon M = S^2 \cup_{\mathbf{m}} e^3 \longrightarrow N = S^2 \cup_{\mathbf{n}} e^3$ such that $\lambda_*\colon H_2(M) = \mathbb{Z}_m \longrightarrow H_2(N) = \mathbb{Z}_n$ is $\frac{\mathbf{n}}{\mathbf{d}} \in \mathrm{Hom}(\mathbb{Z}_m, \mathbb{Z}_n)$. Clearly λ is a suspension and the following diagram commutes

$$\begin{array}{ccccc} S^2 & \xrightarrow{i} & M & \xrightarrow{q} & S^3 \\ \downarrow \frac{\mathbf{n}}{\mathbf{d}} & & \downarrow \lambda & & \downarrow \frac{\mathbf{m}}{\mathbf{d}} \\ S^2 & \xrightarrow{i'} & N & \xrightarrow{q'} & S^3. \end{array}$$

Furthermore, $(r\lambda)_* = r\frac{\mathbf{n}}{\mathbf{d}}$, $0 \le r < d$.

Thus we have

PROPOSITION 11. *Let $M = M(\mathbb{Z}_m, 2)$, $N = M(\mathbb{Z}_n, 2)$, $d = (m, n)$ and $\delta = \big(m, (2n, n^2)\big)$. Then every element $x \in \pi_2(\mathbb{Z}_m; N) = [M, N]$ can be uniquely written*

$$x = r\lambda + k(i'\phi q),$$

where r and k are integers such that $0 \le r < d$ and $0 \le k < \delta$.

We frequently write the element x as $x_{r,k}$.

Now we specialize to the case $M = N = M(\mathbb{Z}_m, 2)$ and consider the set of self maps $[M, M] = \pi_2(\mathbb{Z}_m; M)$. Note that here λ is the identity map of M so that $r\lambda = \mathbf{r}\colon M \longrightarrow M$ and that $i\phi q$ has order m. Then any $x \in \pi_2(\mathbb{Z}_m; M) = [M, M]$ can be uniquely written

$$x = x_{r,k} = \mathbf{r} + k(i\phi q),$$

where $0 \le r,\ k < m$. Now let y be another element in $\pi_2(\mathbb{Z}_m; M)$ written $y = x_{s,l} = \mathbf{s} + l(i\phi q)$, $0 \le s, l < m$. We investigate the composition $xy \in \pi_2(\mathbb{Z}_m; M)$:

$$\begin{aligned} xy &= \big(\mathbf{r} + k(i\phi q)\big)\big(\mathbf{s} + l(i\phi q)\big) \\ &= s\big(\mathbf{r} + k(i\phi q)\big) + l\big(\mathbf{r} + k(i\phi q)\big)(i\phi q) \\ &= s\mathbf{r} + sk(i\phi q) + l\big(\mathbf{r} + k(i\phi q)\big)i(\phi q), \quad \text{by Proposition 10(iii)} \\ &= \mathbf{rs} + ks(i\phi q) + l(\mathbf{r}i\phi q), \quad \text{since } i \text{ is a suspension and } qi = 0 \\ &= \mathbf{rs} + ks(i\phi q) + li\phi\mathbf{r}^2 q, \quad \text{by Proposition 10(ii)} \\ &= \mathbf{rs} + (ks + lr^2)i\phi q. \end{aligned}$$

Now let $\mathcal{E}(M) = \mathcal{E}\big(M(\mathbb{Z}_m, 2)\big) \subseteq [M, M]$ denote the group of homotopy classes of homotopy equivalences of M. The group operation, written multiplicatively, is composition and the neutral element of the group is the identity map of M. The following proposition is then an immediate consequence of the previous calculation.

https://doi.org/10.4153/CJM-1994-037-0 Published online by Cambridge University Press

PROPOSITION 12. *Every $x \in \mathcal{E}(M)$ can be uniquely written as*

$$x = x_{r,k} = \mathbf{r} + k(i\phi q),$$

where $0 \leq r, k < m$ and $(r,m) = 1$. If $y \in \mathcal{E}(M)$ with $y = \mathbf{s} + l(i\phi q)$, $0 \leq s, l < m$ and $(s,m) = 1$, then

$$xy = \mathbf{rs} + (ks + lr^2)(i\phi q).$$

Next we examine the set of comultiplications $\mathcal{C}(M)$ of $M = M(\mathbb{Z}_m, 2)$.

PROPOSITION 13. *Every comultiplication of $M = M(\mathbb{Z}_m, 2)$ can be uniquely written as*

$$\varphi_k = i_1 + i_2 + k[i_1 i, i_2 i]q,$$

where $0 \leq k < m$.

PROOF. The elements $\varphi_k \in [M, M \vee M]$ are comultiplications because

$$\begin{aligned} q_1\varphi_k &= q_1 i_1 + q_1 i_2 + k[q_1 i_1 i, q_1 i_2 i]q \\ &= 1 + 0 + k[i, 0] \\ &= 1 \end{aligned}$$

and similarly $q_2\varphi_k = 1$. Since $[i_1 i, i_2 i]q$ has order m by Proposition 8, the φ_k are m distinct comultiplications of M. But the Proposition 4, M has exactly m comultiplications. ∎

6. **Co-H-Maps and comultiplications of $M(\mathbb{Z}_m, 2)$.** The main result of this section is Theorem 14 which deals with co-H-maps between Moore spaces. We explicitly determine the co-H-maps $M(\mathbb{Z}_m, 2) \longrightarrow M(\mathbb{Z}_n, 2)$ with respect to any comultiplication of $M(\mathbb{Z}_m, 2)$ and any comultiplication of $M(\mathbb{Z}_n, 2)$. As a consequence we specify the action of the group of homotopy self-equivalences of $M(\mathbb{Z}_m, 2)$ on the set of comultiplications of $M(\mathbb{Z}_m, 2)$ and show that all comultiplications lie in a single orbit. From this we determine associativity and commutativity of the comultiplications of $M(\mathbb{Z}_m, 2)$.

Let $M = M(\mathbb{Z}_m, 2)$ and $N = M(\mathbb{Z}_n, 2)$. Propositions 11 and 13 provide an indexing for the elements of $[M, N]$, $\mathcal{C}(M)$ and $\mathcal{C}(N)$. Let $\varphi_l \in \mathcal{C}(M)$ and $\psi_j \in \mathcal{C}(N)$ be given by

$$\varphi_l = i_1 + i_2 + l[i_1 i, i_2 i]q \quad \text{and}$$
$$\psi_j = i'_1 + i'_2 + j[i'_1 i', i'_2 i']q',$$

where $0 \leq l < m$ and $0 \leq j < n$. Let $x = x_{r,k} \in [M, N]$ be given by

$$x = x_{r,k} = r\lambda + k(i'\phi q)$$

for $0 \leq r < d$ and $0 \leq k < \delta$. Recall that $d = (m,n)$ and $\delta = \big(m, (2n, n^2)\big)$.

THEOREM 14. *The map* $x_{r,k}\colon (M, \varphi_l) \longrightarrow (N, \psi_j)$ *is a co-H-map if and only if*

$$\left(\frac{n}{d}\right)^2 lr^2 \equiv rj\frac{m}{d} + k - \frac{r(r-1)}{2}\frac{m(m-1)}{2}\left(\frac{n}{d}\right)^2 \pmod{d}.$$

PROOF. First note

$$\begin{aligned}(x \vee x)i_1 &= i_1' x \\ &= i_1'\big(r\lambda + k(i'\phi q)\big) \\ &= r(i_1'\lambda) + k(i_1' i'\phi q)\end{aligned}$$

and similarly $(x \vee x)i_2 = r(i_2'\lambda) + k(i_2' i'\phi q)$.

Now

$$\begin{aligned}(x \vee x)\varphi_l &= (x \vee x)(i_1 + i_2 + l[i_1 i, i_2 i]q) \\ &= r(i_1'\lambda) + k(i_1' i'\phi q) + r(i_2'\lambda) + k(i_2' i'\phi q) \\ &\qquad + l[r(i_1'\lambda)i + k(i_1' i'\phi q)i, r(i_2'\lambda)i + k(i_2' i'\phi q)i]q \\ &= r(i_1'\lambda) + k(i_1' i'\phi q) + r(i_2'\lambda) + k(i_2' i'\phi q) + lr^2[i_1'\lambda i, i_2'\lambda i]q, \quad \text{since } qi = 0 \\ &= r(i_1'\lambda) + k(i_1' i'\phi q) + r(i_2'\lambda) + k(i_2' i'\phi q) + lr^2\left(\frac{n}{d}\right)^2[i_1' i', i_2' i']q\end{aligned}$$

and

$$\begin{aligned}\psi_j x &= \psi_j\big(r\lambda + k(i'\phi q)\big) \\ &= r(i_1' + i_2' + j[i_1' i', i_2' i']q')\lambda + k(i_1' + i_2' + j[i_1' i', i_2' i']q')(i'\phi q) \\ &= r\Big(i_1'\lambda + i_2'\lambda + j[i_1' i', i_2' i']\frac{m}{d}q\Big) + k(i_1' i' + i_2' i')\phi q, \quad \text{since } q'i' = 0 \\ &= r(i_1'\lambda + i_2'\lambda) + rj\frac{m}{d}[i_1' i', i_2' i']q \\ &\qquad + k\Big((i_1' i'\phi + i_2' i'\phi + [i_1' i', i_2' i'])q\Big), \quad \text{by Proposition 10(i), (iii) and (v)} \\ &= r(i_1'\lambda) + r(i_2'\lambda) - \frac{r(r-1)}{2}(i_1'\lambda, i_2'\lambda) + rj\frac{m}{d}[i_1' i', i_2' i']q + k(i_1' i'\phi q) \\ &\qquad + k(i_2' i'\phi q) + k[i_1' i', i_2' i']q, \quad \text{by Proposition 10(v)} \\ &= r(i_1'\lambda) + r(i_2'\lambda) - \frac{r(r-1)}{2}\frac{m(m-1)}{2}[i_1'\lambda i, i_2'\lambda i]q + rj\frac{m}{d}[i_1' i', i_2' i']q \\ &\qquad + k(i_1' i'\phi q) + k(i_2' i'\phi q) + k[i_1' i', i_2' i']q, \quad \text{by Proposition 9} \\ &= r(i_1'\lambda) + r(i_2'\lambda) - \frac{r(r-1)}{2}\frac{m(m-1)}{2}\left(\frac{n}{d}\right)^2[i_1' i', i_2' i']q + rj\frac{m}{d}[i_1' i', i_2' i']q \\ &\qquad + k(i_1' i'\phi q) + k(i_2' i'\phi q) + k[i_1' i', i_2' i']q.\end{aligned}$$

The theorem now follows from Proposition 10(iii) since $[i_1' i', i_2' i']q$ has order d by Proposition 8. ∎

The formula in Theorem 14 is cumbersome and we simplify it in special cases in Remark 18. For now we derive some consequences of the theorem.

We let $M = M(\mathbb{Z}_m, 2)$ and define an operation of $\mathcal{E}(M)$ on $\mathcal{C}(M)$ (which can be done for any co-H-space). If $x \in \mathcal{E}(M)$ and $\varphi \in \mathcal{C}(M)$, then $x * \varphi$ is the composition

$$M \xrightarrow{x^{-1}} M \xrightarrow{\varphi} M \vee M \xrightarrow{x \vee x} M \vee M.$$

Clearly $x * \varphi \in \mathcal{C}(M)$, and $\mathcal{E}(M)$ acts as a group on the set $\mathcal{C}(M)$. We call two comultiplications $\varphi, \psi \in \mathcal{C}(M)$ *equivalent* if there is an $x \in \mathcal{E}(M)$ such that $x * \varphi = \psi$. For $x \in \mathcal{E}(M)$ we note the following: $x * \varphi = \psi$ if and only if $x\colon (M, \varphi) \longrightarrow (M, \psi)$ is a co-H-map.

We next determine the action of $\mathcal{E}(M)$ on $\mathcal{C}(M)$.

PROPOSITION 15. *Let $M = M(\mathbb{Z}_m, 2)$ and let $x \in \mathcal{E}(M)$ and $\varphi \in \mathcal{C}(M)$ be given by $x = x_{r,k} = \mathbf{r} + k(i\phi q)$ and $\varphi = \varphi_l = i_1 + i_2 + l[i_1 i, i_2 i]q$, with $0 \leq k, l, r < m$ and $(r, m) = 1$. Then*

$$x_{r,k} * \varphi_l = \varphi_j,$$

where

$$j = rl + \frac{r(r-1)}{2}\frac{m(m-1)}{2}s - ks \quad \textit{reduced} \bmod m,$$

$0 \leq s < m$ and $rs \equiv 1 \pmod m$.

PROOF. We must show that $x_{r,k}\colon (M, \varphi_l) \longrightarrow (M, \varphi_j)$ is a co-H-map. But this follows from Theorem 14 with $m = n = d$. ∎

COROLLARY 16. *Any two comultiplications of $M(\mathbb{Z}_m, 2)$ are equivalent, i.e, the action of $\mathcal{E}\big(M(\mathbb{Z}_m, 2)\big)$ on $\mathcal{C}\big(M(\mathbb{Z}_m, 2)\big)$ is transitive. Consequently, all comultiplications of $M(\mathbb{Z}_m, 2)$ are associative. If m is odd, all comultiplications of $M(\mathbb{Z}, 2)$ are commutative. If m is even, no comultiplication of $M(\mathbb{Z}_m, 2)$ is commutative.*

PROOF. Since $x_{1,m-k} * \varphi_0 = \varphi_k$, the first assertion follows. Since associativity and commutativity are properties of an equivalence class of comultiplications, it suffices to verify that the standard comultiplication φ_0 of $M(\mathbb{Z}_m, 2)$ is commutative for m odd and not commutative for m even. But this follows immediately from Propositions 8 and 9. ∎

We close the paper with a number of remarks.

REMARK 17. For any abelian group G, every comultiplication of $M(G, 2)$ is associative. This is a consequence of the following result of Berstein-Hilton [B-H, Theorem A, p. 77]: If X is a $(q-1)$-connected finite CW-complex of dimension $\leq 3q - 3$, $q \geq 1$, and φ is a comultiplication of X, then φ is equivalent to a suspension comultiplication. In particular, φ is associative.

REMARK 18. The formula in Theorem 14 can be simplified in several special cases.

(i) If m is odd, $\big(m(m-1)\big)/2 \equiv 0 \pmod d$, and so we have:

$$x_{r,k}\colon (M, \varphi_l) \longrightarrow (N, \psi_j) \text{ is a co-}H\text{-map} \iff \left(\frac{n}{d}\right)^2 lr^2 \equiv rj\frac{m}{d} + k \pmod d.$$

https://doi.org/10.4153/CJM-1994-037-0 Published online by Cambridge University Press

(ii) If $j = l = 0$, we have:

$$x_{r,k}\colon (M, \varphi_0) \longrightarrow (N, \psi_0) \text{ is a co-}H\text{-map} \iff k \equiv \frac{r(r-1)}{2}\frac{m(m-1)}{2}\left(\frac{n}{d}\right)^2 \pmod d.$$

(iii) If $m = n$, then $M = N$ and $d = m = n$. Note that $x_{r,k} = \mathbf{r} + k(i\phi q)$. Then

$$x_{r,k}\colon (M, \varphi_l) \longrightarrow (M, \varphi_j) \text{ is a co-}H\text{-map}$$
$$\iff lr^2 \equiv rj + k - \frac{r(r-1)}{2}\frac{m(m-1)}{2} \pmod m.$$

(iv) Let us combine the last two cases: $m = n$ and $j = l = 0$. Then we have:

$$x_{r,k}\colon (M, \varphi_0) \longrightarrow (M, \varphi_0) \text{ is a co-}H\text{-map} \iff k \equiv \frac{r(r-1)}{2}\frac{m(m-1)}{2} \pmod m.$$

Thus if m is odd, $\frac{m(m-1)}{2} \equiv 0 \pmod m$, and so $x_{r,k}$ is a co-H-map $\Leftrightarrow k \equiv 0 \pmod m$. If m is even, we distinguish two cases: (a) $\frac{r(r-1)}{2}$ odd and (b) $\frac{r(r-1)}{2}$ even. In (a), $x_{r,k}$ is a co-H-map $\Leftrightarrow k = \frac{m}{2}$. In (b), $x_{r,k}$ is a co-H-map $\Leftrightarrow k = 0$. In particular, for m even, the power map $x_{r,0} = \mathbf{r}\colon (M, \varphi_0) \longrightarrow (M, \varphi_0)$ is a co-H-map $\Leftrightarrow \frac{r(r-1)}{2}$ is even. For example, in the case $m = 6$ and $j = l = 0$, $x_{3,3}$ and $x_{4,0}$ are co-H-maps and $x_{3,0}$ and $x_{4,3}$ are not.

(v) In addition, the formula for the action of $\mathcal{E}(M)$ on $\mathcal{C}(M)$ in Proposition 15 becomes simpler in special cases. For example, if m is odd, we have $x_{r,k} * \varphi_l = \varphi_{rl-ks}$.

REMARK 19. We briefly consider Moore spaces of type $(G, 1)$. Given an abelian group G, by [Va] a connected CW-complex X is called a *Moore space of type* $(G, 1)$ if $\pi_1(X) = G$ and $H_i(X) = 0$ for $i > 1$. Necessary and sufficient conditions are given in [Va] for the existence of Moore spaces of type $(G, 1)$. However, if X is any co-H-space, it is known that $\pi_1(X)$ is free [E-G]. Thus we see that there exists a Moore space of type $(G, 1)$ which is a co-H-space if and only if $G = \mathbb{Z}$. In particular, the circle S^1 is a Moore-co-H-space of type $(\mathbb{Z}, 1)$. It is easily seen that S^1 admits infinitely many comultiplications.

Finally, we ask if the results of this section can be generalized. More specifically, we raise the following question.

REMARK 20. For which abelian groups G are all of the comultiplications of $M(G, 2)$ equivalent?

Of course it follows from Corollary 16 that $G = \mathbb{Z}_m \oplus \mathbb{Z}_n$ with $(m, n) = 1$ is such a group.

REFERENCES

[A-L] M. Arkowitz and G. Lupton, *Equivalence Classes of Homotopy-Associative Comultiplications of Finite Complexes*, J. Pure Appl. Algebra, to appear.

[Ba] H.-J. Baues, *Combinatorial Homotopy and 4-Dimensional Complexes*, Expositions in Math. **2**, Walter de Gruyter, 1991.

[B-H] I. Berstein and P. J. Hilton, *Suspensions and Comultiplications*, Topology **2**(1963), 73–82.

[E-G] S. Eilenberg and T. Ganea, *On the Lusternik-Schnirelmann Category of Abstract Groups*, Ann. of Math. **65**(1957), 517–518.

[Hi] P. J. Hilton, *Homotopy Theory and Duality*, Notes on Math. and its Applications, Gordon and Breach, 1965.

[H-M-R] P. J. Hilton, G. Mislin and J. Roitberg, *On Co-H-Spaces*, Comment. Math. Helv. **53**(1978), 1–14.

[H-S] P. J. Hilton and U. Stammbach, *A Course in Homological Algebra*, Graduate Texts in Math. **4**, Springer-Verlag, 1971.

[Si] A. J. Sieradski, *Stabilization of Self-Equivalences of the Pseudoprojective Spaces*, Michigan Math. J. **19**(1972), 109–119.

[Va] K. Varadarajan, *Groups for which Moore Spaces* $M(\pi, 1)$ *Exist*, Ann. of Math. **84**(1966), 368–371.

[Wh] G. W. Whitehead, *Elements of Homotopy Theory*, Graduate Texts in Math. **61**, Springer-Verlag, 1978.

Department of Mathematics
Dartmouth College
Hanover, New Hampshire 03755
U.S.A.

Instytut Matematyki UMK
u. Chopina 12/18
87-100 Torun
Poland