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A PRIORI ESTIMATES AND EXISTENCE ON STRONGLY COUPLED COOPERATIVE ELLIPTIC SYSTEMS

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Abstract. We study the boundary value problem

$$\begin{aligned} \operatorname{div}(|\nabla u|^{m-2}\nabla u) + u^a v^b &= 0 \quad \text{in } \Omega, \\ \operatorname{div}(|\nabla v|^{m-2}\nabla v) + u^c v^d &= 0 \quad \text{in } \Omega, \\ u = v &= 0 \quad \text{on } \partial\Omega, \end{aligned}$$

where $\Omega \subset \mathbb{R}^n$ $(n \geq 2)$ is a bounded connected smooth domain, and the exponents $m > 1$ and $a, b, c, d \geq 0$ are non-negative numbers. Under appropriate conditions on the exponents m, a, b, c and d, a variety of results on a priori estimates and existence of positive solutions has been established.

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1. Introduction. In the articles [**10**, **12**], the author considered, among other matters, the following systems of elliptic differential equations

$$\begin{aligned} \Delta_m u + u^a v^b &= 0 \quad \text{in } \Omega, \\ \Delta_m v + u^c v^d &= 0 \quad \text{in } \Omega, \end{aligned} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^n$ $(n \geq 2)$ is a connected smooth domain, together with, whenever Ω has a non-empty boundary $\partial\Omega$, the boundary condition

$$u = v = 0 \quad \text{on } \partial\Omega. \tag{1.2}$$

Here the exponents m, a, b, c and d are non-negative numbers and

$$\Delta_m \cdot = \operatorname{div}(|\nabla \cdot|^{m-2}\nabla \cdot)$$

is the m–Laplace operator. Specifically, the author was concerned with the question of existence of a non-negative and non-trivial solution $\mathbf{u} = (u, v) \geq 0$[1] satisfying (1.1). This issue was raised as an open question for systems of equations such as (1.1) in the survey article [**6**] and has since been studied by a number of authors. See for example [**10**] and the references therein. Due to the presence of multiple components and multiple equations, the structure of systems is more complicated than that of scalar

[1]All relations involving vectors are understood in the component-wise sense.

equations. For instance, generically, (1.1) does not admit a variational structure and consequently variational methods typically do not apply.

Denote

$$\beta := bc - \alpha\delta; \qquad \alpha := m - 1 - a, \quad \delta := m - 1 - d,$$

and for $m \in (1, n)$,

$$m_* := \frac{n(m-1)}{n-m}, \qquad m^* := \frac{n(m-1)+m}{n-m}.$$

It is understood that $m_* = m^* = \infty$ for $m \geq n$.

We say that (1.1) is *pseudo-subcritical* provided that

$$\min(a+b, c+d) < m^*, \qquad \max(a+b, c+d) \leq m^*. \tag{1.3}$$

(When $m^* = \infty$, the convention $\max(a+b, c+d) < \infty$ is used instead.)

Also (1.1) is said to be *fully-coupled* if the exponents a, b, c, d satisfy

$$\min(b, c) > 0, \qquad \min(a, d) \geq 0,$$

and strongly-coupled if (1.1) is fully-coupled and, in addition,

$$a + d > 0.$$

See, for example, [**10**] for details.

We shall assume throughout the entire paper that $m > 1$, $\min(b, c) > 0$ and $\min(a, d) \geq 0$ (so that (1.1) is fully-coupled). Moreover, for simplicity, all solutions $\mathbf{u}$ considered throughout will be classical; i.e., in the space of $C^{2,\gamma}_{loc}(\Omega)$ for $m = 2$ and $C^{1,\gamma}_{loc}(\Omega)$ for $m \neq 2$ for some $\gamma \in (0, 1)$. When Ω is bounded, then (1.2) is prescribed and solutions will be in $C^{2,\gamma}_0(\Omega)$ and $C^{1,\gamma}_0(\Omega)$ respectively.

We say that an (**ALT**) condition holds provided that the exponents m, a, b, c and d satisfy one of the following conditions:

(A) $n \leq m$;

(B) $n > m$, $\min(\alpha, \delta) > 0$ and $\max\{b + \delta, c + \alpha\} > \dfrac{n\beta}{mm_*}$;

(C) $n > m$, $\delta \leq 0 < \alpha$ and $\max\left\{\dfrac{\beta}{c-\delta}, c + \alpha\right\} > \dfrac{n\beta}{mm_*}$;

(D) $n > m$, $\alpha \leq 0 < \delta$ and $\max\left\{b + \delta, \dfrac{\beta}{b-\alpha}\right\} > \dfrac{n\beta}{mm_*}$;

(E) $n > m$, $\max(\alpha, \delta) \leq 0$ and $\min(a + b, c + d) < m_*$.

Let $\mathcal{C}$ be a class of solutions of (1.1). Then an a priori estimate (**EST**) is said to be valid for $\mathcal{C}$, if there exists a positive constant $C > 0$ (independent of $\mathbf{u}$) such that

$$\textbf{(EST)} \qquad ||\mathbf{u}||_{L^\infty(\Omega)} \leq C$$

for all solutions $\mathbf{u} \in \mathcal{C}$ of (1.1).

For $m > 1$, the following theorem was proved in [**12**].

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THEOREM I ([**12**]). *Let $\Omega \subset \mathbb{R}^n$ be uniformly normal* [**12**, *Definition 2.2*]. *Assume that the condition* (**ALT**) *holds,*

$$\beta > 0, \qquad \max(a, d) < m^*, \tag{1.4}$$

and

$$\min(a, d) \geq m - 1. \tag{1.5}$$

Then the estimate (**EST**) *is valid for all non-negative solutions* $\mathbf{u}$ *of* (1.1) *that are monotone in* Ω [**12**, *Definition 2.3*]. *Moreover,* (1.1) *has a positive solution* $\mathbf{u}$.

REMARK. If $m \geq n$, then $(1.4)_2$ is superfluous, since $m^* = \infty$.

For the case $m = 2$ (i.e., the Laplace operator), corresponding results were obtained in [**10**, **12**] (see also the references therein). For clarity, we state the results for the spatial dimension $n = 2$ and $n > 2$ separately. The first result is for $n = 2$.

THEOREM II ([**10**]). *Let $m = 2$ and let $\Omega \subset \mathbb{R}^2$ be uniformly normal. Suppose that $\beta \neq 0$ and*

$$\min(a + b, c + d) \geq m - 1. \tag{1.6}$$

Then the estimate (**EST**) *is valid for all non-negative solutions* $\mathbf{u}$ *of* (1.1). *Moreover,* (1.1) *has a positive solution* $\mathbf{u}$ *if* $\beta > 0$.

Note particularly that there is no upper bound restriction on any of the exponents a, b, c, d in Theorem II. When the spatial dimension $n > 2$, we have the following result.

THEOREM III ([**10**, **12**]). *Let $m = 2$ and let $\Omega \subset \mathbb{R}^n$ $(n > 2)$ be uniformly normal. Then the following conclusions hold.*

(A) *Suppose that $\beta \neq 0$. Assume that* (1.3) *and* (1.6) *hold. Then the estimate* (**EST**) *is valid for all non-negative solutions* $\mathbf{u}$ *of* (1.1). *Moreover,* (1.1) *has a positive solution* $\mathbf{u}$ *if* $\beta > 0$.

(B) *Suppose that* (**ALT**), (1.4) *and* (1.5) *hold. Then the estimate* (**EST**) *is valid for all non-negative solutions* $\mathbf{u}$ *of* (1.1) *that are monotone in* Ω *and, furthermore,* (1.1) *has a positive solution* $\mathbf{u}$.

REMARKS. 1. Theorem III(B) is a special case of Theorem I.

2. The supremum a priori estimates in Theorems II and III-(A) are classical. Namely, they are valid for all non-negative solutions $\mathbf{u}$ of (1.1).

For Theorems I–III above, the essential core is to obtain a certain form of supremum a priori estimates. One typical approach in achieving this is to apply the blow-up method introduced for scalar equations in [**5**] in which establishing the desired estimates was converted to proving Liouville-type non-existence of *positive* solutions. For systems of equations with special structures (i.e., weakly-coupling), this procedure has been lately adapted directly by several authors for the very same purpose. However, in view of a possible strongly-coupling feature (i.e., $a + d > 0$), it is not yet clear whether the aforementioned straight generalization of the standard blow-up procedure remains applicable to (1.1). Nevertheless, a new blow-up procedure was then developed in **[10]** for (1.1) that recovers the original key feature of the blow-up method; non-existence implies a priori estimates. We refer the interested reader to **[10]** and the references therein for more details. To further illustrate this assertion, our first purpose is to prove the following theorem.

THEOREM 1.1. *Let* $m = 2$ *and let* $\Omega \subset \mathbb{R}^n$ $(n > 2)$ *be uniformly normal. Suppose that* (1.4) *and* (1.6) *hold. Assume that* (1.1) *has no positive solutions on* $\Omega = \mathbb{R}^n$*. Then the estimate* (**EST**) *is valid for all non-negative solutions* $\mathbf{u}$ *of* (1.1)*. Moreover,* (1.1) *has a positive solution* $\mathbf{u}$.

The proof of the existence in Theorem 1.1 is somewhat standard by the fixed point theorems, provided that a somewhat stronger version of (**EST**) is available. As mentioned earlier, via blow-up, the Liouville-type non-existence that (1.1) has no positive solutions on $\Omega = \mathbb{R}^n$ plays a crucial role in deriving such estimates. Naturally, in the light of Theorem 1.1, the second purpose of this paper is thereby to establish a corresponding Liouville-type non-existence for (1.1) and therefore, as a direct consequence of Theorem 1.1, to prove the following result.

THEOREM 1.2. *Let* $m = 2$ *and let* $\Omega \subset \mathbb{R}^n$ $(n > 2)$ *be uniformly normal. Suppose that* (**ALT**)*,* (1.4) *and* (1.6) *hold. Then the estimate* (**EST**) *is valid for all non-negative solutions* $\mathbf{u}$ *of* (1.1)*. Moreover,* (1.1) *has a positive solution* $\mathbf{u}$.

For existence alone, as observed in **[12]**, it suffices to develop a priori estimates only for monotone solutions (as in Theorem I). Based on such an observation, we are able to extend Theorems I, II and III on existence below. The first result in this direction is for $m = 2$.

THEOREM 1.3. *Let* $m = 2$ *and let* $\Omega \subset \mathbb{R}^n$ *be uniformly normal. Suppose that* (1.4) *holds. Assume that one of the following conditions holds:*

(A) (1.1) *has no positive solutions on* $\Omega = \mathbb{R}^n$*,*
(B) (1.3) *holds.*

Then the estimate (**EST**) *is valid for all non-negative solutions* $\mathbf{u}$ *of* (1.1) *that are monotone in* Ω*. Moreover,* (1.1) *has a positive solution* $\mathbf{u}$.

The next theorem is valid for arbitrary $m > 1$.

THEOREM 1.4. *Let* $\Omega \subset \mathbb{R}^n$ *be uniformly normal. Suppose that* (1.4) *holds. Assume one of the following conditions holds.*

(A) (1.1) *has no positive solutions on* $\Omega = \mathbb{R}^n$*.*
(B) (**ALT**) *holds.*

Then the estimate (**EST**) *is valid for all non-negative solutions* $\mathbf{u}$ *of* (1.1) *that are monotone in* Ω*. Moreover,* (1.1) *has a positive solution* $\mathbf{u}$.

Theorem 1.3 removes the condition (1.6) in Theorems II and III (but only for monotone solutions), while Theorem 1.4 removes (1.5) in Theorem I, and (1.6) in Theorems 1.1-1.2 (again only for monotone solutions).

When $m = 2$, (1.1) (in fact a general system of k equations) was studied in **[3]**. A priori estimates and existence of a positive solution were obtained under, among other restrictions, the upper growth bound

$$\lim_{\mathbf{u}\to\infty} \frac{|\mathbf{f}(\mathbf{u})|}{|\mathbf{u}|^{(n+1)/(n-1)}} = 0.$$

When the domain Ω is convex, then the condition above was also weakened to

$$\lim_{\mathbf{u}\to\infty} \frac{|\mathbf{f}(\mathbf{u})|}{|\mathbf{u}|^{p}} = 0, \qquad p \in (1, 2_*).$$

In [**7**], the authors further extended the studies in this direction and obtained similar estimates (and existence) for (1.1) (with $m = 2$). The results of both [**3**] and [**7**] are valid on general bounded domains. When $\Omega = B$ is a Euclidean ball, (1.1) was treated in [**2**]. Under the additional condition $a, d \leq m - 1$, the authors derived existence of radial solutions under both (1.4) and part (B) of (**ALT**). When (1.1) has a variational structure, some existence and non-existence results were obtained in [**9**].

Our main interest in studying (1.1) is to attempt to identify an optimal region of the quadruples (a, b, c, d) in which (1.1) admits a priori estimates and/or existence of positive solutions. In this regard, Theorems 1.2-1.4 are partially new, enlarging substantially such regions given in Theorems I-III. In particular, Theorems 1.2-1.4, II and III together extend earlier results in this direction for (1.1). We refer the interested reader to [**10**, **12**] and the references therein for details.

Plainly, the region defined by (1.6) is larger than that given by (1.5). On the other hand, the regions defined by (1.4) and (**ALT**), and by (1.3), respectively, overlap with each other, with neither containing the other. Indeed, the exponents $a + b$ and $c + d$ are bounded from above under (1.3), while both $a + b$ and $c + d$ may be arbitrarily large (not simultaneously though) under (1.4) and (**ALT**)). When $a + b$ and $c + d$ are close, however, (1.4) and (**ALT**) together do imply (1.3). For example, when $n > 2$, $m = 2$ and $a + b = c + d$, one readily checks that (1.4) and (**ALT**) imply $a + b = c + d < 2_* < 2^*$, but (1.3) only requires $a + b = c + d < 2^*$.

The paper is organised as follows. A new monotonicity result Lemma 2.2, which plays an important role and also is of independent interest, is proved in Section 2. Some preliminary results are given in Section 3. In Section 4, we develop the desired a priori estimates. Theorems 1.1–1.4 are then proved in Section 5.

2. An auxiliary lemma. In this section, we prove an auxiliary monotonicity result (Lemma 2.2 below) that will be used later. The result is new, particularly in view of the presence of non-Lipschitz non-linearities. See the remarks right after Lemma 2.2.

Let $k \geq 1$ be an integer. For non-negative vectors $\mathbf{p}$, $\mathbf{u} \in \mathbb{R}^k$, we write

$$|\mathbf{p}| = p_1 + p_2 + \ldots + p_k, \qquad \mathbf{u}^{\mathbf{p}} = u_1^{p_1} \cdot u_2^{p_2} \ldots u_k^{p_k},$$

where we have used the convention $0^0 = 1$.

For $\mathbb{R}^n_+ = \{(x', x_n) \in \mathbb{R}^n \mid x_n > 0\}$, consider the system of semi-linear elliptic equations

$$\begin{aligned} \Delta u_i + \kappa_i x_n^{\sigma_i} \mathbf{u}^{\mathbf{p}_i} + \delta_i = 0 &\quad \text{in } \mathbb{R}^n_+, \\ \mathbf{u} = 0 &\quad \text{on } \partial\mathbb{R}^n_+, \end{aligned} \tag{2.1}$$

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where, for $i = 1, \ldots, k$, $\mathbf{p}_i = (p_{i1}, \ldots, p_{ik}) \geq 0$ are constant vectors and $\kappa_i > 0, \sigma_i, \delta_i \geq 0$ are non-negative numbers satisfying

$$\min_i\{\sigma_i + |\mathbf{p}_i|\} \geq 1 \text{ and } \max_i |\mathbf{p}_i| \leq 2^*(< \infty \text{ if } n = 2). \tag{2.2}$$

As a direct consequence of the strong maximum principle, we have the following lemma.

LEMMA 2.1. *Let $\Gamma \subset \mathbb{R}^n_+$ be bounded and let $\mathbf{u}$ be a non-negative non-trivial solution of* (2.1). *Then $\mathbf{u}$ is strictly positive and there exists $C = C(\mathbf{u}, \Gamma) > 0$ such that*

$$C^{-1} \leq \min_{i,j} \inf_{x\in\Gamma} \frac{u_i(x)}{u_j(x)} \leq \max_{i,j} \sup_{x\in\Gamma} \frac{u_i(x)}{u_j(x)} \leq C$$

and

$$C^{-1}x_n \leq \min_i \inf_{x\in\Gamma} u_i(x) \leq \max_i \sup_{x\in\Gamma} u_i(x) \leq Cx_n.$$

Proof. The proof is essentially the same as that of Lemma 3.1 of [**10**]. □

LEMMA 2.2. *Suppose* (2.2) *holds. Then the only non-negative solutions $\mathbf{u}$ of* (2.1) *are $u = \mathbf{h}x_n$, where $\mathbf{h}$ is a non-negative constant vector. Moreover, there necessarily holds that $\delta_i = 0$ for $i = 1, \ldots, k$.*

REMARKS. 1. If $|\mathbf{p}_i| = 0$, then the condition $\sigma_i + |\mathbf{p}_i| = \sigma_i \geq 1$ is superfluous.

2. When $|\mathbf{p}_i| \geq 1$, the nonlinearity $\mathbf{u}^{\mathbf{p}_i}$ is called *locally Lipschitz in the magnitude* of $\mathbf{u}$ for $\mathbf{u} \geq 0$ in [**10**] and was treated in [**10**]. Evidently, a locally magnitude-Lipschitz $\mathbf{u}^{\mathbf{p}_i}$ need not be locally Lipschitz in $\mathbf{u}$ for $\mathbf{u} \geq 0$ since some components of $\mathbf{p}_i$ may be in the interval $(0, 1)$. In Lemma 2.2, however, it is allowed that $|\mathbf{p}_i| \in (0, 1)$ in (2.1), whence $\mathbf{u}^{\mathbf{p}_i}$ is not even locally magnitude-Lipschitz and is left untreated in [**10**]. Moreover, the appearance of $x_n^{\sigma_i}$ in (2.1) is also new.

Proof. We employ a proof used in [**10**] and refer the reader to [**10**] for further details. (See also [**8**, **11**].) We shall use some notations from [**10**].

We first show that $\mathbf{u}$ depends only on x_n and is non-decreasing. By the strong maximum principle, every component of $\mathbf{u}$ is either strictly positive or identically zero, and so is $\kappa_i x_n^{\sigma_i}\mathbf{u}^{\mathbf{p}_i}$ for $i = 1, \ldots, k$. Without loss of generality, there exists $1 \leq k' \leq k$ such that (note $k' > 0$ for otherwise there is nothing to prove)

$$u_i > 0 \quad \text{for } i = 1, \ldots, k'; \qquad u_i \equiv 0 \quad \text{for } i = k' + 1, \ldots, k.$$

It follows that for $i = 1, \ldots, k'$, either

$$\Delta u_i + \kappa_i x_n^{\sigma_i}\bar{\mathbf{u}}^{\bar{\mathbf{p}}_i} + \delta_i = 0, \qquad \text{in } \mathbb{R}^n_+,$$

where $\bar{\mathbf{u}} = (u_1, \ldots, u_{k'})$, provided that $\mathbf{p}_i = (\bar{\mathbf{p}}_i, 0, \ldots, 0) = (p_{i1}, \ldots, p_{ik'}, 0, \ldots, 0)$ (so that $\kappa_i x_n^{\sigma_i}\bar{\mathbf{u}}^{\bar{\mathbf{p}}_i} > 0$), or

$$\Delta u_i + \delta_i = 0, \qquad \text{in } \mathbb{R}^n_+, \tag{2.3}$$

provided that $p_{ij} > 0$ for some $j > k'$. If the latter occurs, by applying Theorem 4.2 of [**10**] to $(2.3)_i$, then necessarily $\delta_i = 0$ and $u_i = h_i x_n$, for some $h_i > 0$, since $u_i > 0$.

Again, without loss of generality, there exists $1 \le k'' \le k'$ such that

$$u_i \neq h_i x_n \quad \text{for } i = 1, \dots, k''; \qquad u_i = h_i x_n \quad \text{for } i = k''+1, \dots, k'.$$

(Note that $k'' > 0$ again.) It follows that, for $i = 1, \dots, k''$, we have

$$\Delta u_i + \hat{\kappa}_i x_n^{\hat{\sigma}_i} \hat{\mathbf{u}}^{\hat{\mathbf{p}}_i} + \delta_i = 0 \qquad \text{in } \mathbb{R}^n_+,$$

where $\hat{\kappa}_i > 0$, $\hat{\mathbf{u}} = (u_1, \dots, u_{k''})$, $\hat{\mathbf{p}}_i(p_{i1}, \dots, p_{ik''})$ and

$$\hat{\sigma}_i + |\hat{\mathbf{p}}_i|\sigma_i + p_{i,k''+1} + \dots + p_{ik'} + |\hat{\mathbf{p}}_i| = \sigma_i + |\bar{\mathbf{p}}_i| = \sigma_i + |\mathbf{p}_i| \ge 1, \qquad |\hat{\mathbf{p}}_i| \le |\mathbf{p}_i| \le 2^*.$$

In turn, we may consider a sub-system in $\hat{\mathbf{u}} > 0$ of $\mathbf{u}$ under the same assumptions of Lemma 2.2 (the other components of $\mathbf{u}$ are monotone and depend only on x_n as shown). Therefore we shall assume that $\mathbf{u}$ is positive in the sequel.

In spherical coordinates, with the origin at $(0', -1)$, we have

$$x_n = r\cos\phi - 1, \quad \mathbb{R}^n_+ = \left\{(r, \theta) | \theta = (\psi_1, \dots, \psi_{n-2}, \phi) \in S^{n-2} \times [0, \pi/2),\ r > \frac{1}{\cos\phi}\right\},$$

where r is the radius, and $\phi \in [0, \pi/2)$ is the (positive) angle between the positive x_n-axis and $x - (0', -1)$ for $x \in \mathbb{R}^n_+$. As in **[10]**, make the changes of variables

$$\mathbf{v}(t, \theta) = r^{(n-2)/2}\mathbf{u}(r, \theta); \qquad (t, \theta) = (\ln r, \theta),$$

(so that $t \ge 0$) and for $T \ge 0$,

$$\mathbf{v}_T(t, \theta) = \mathbf{v}((t, \theta)_T), \quad \mathbf{w}(T, t, \theta) = \mathbf{v}_T - \mathbf{v}; \qquad (t, \theta)_T = (2T - t, \theta).$$

It follows that the function $\mathbf{w}$ satisfies the equation in $\Sigma_T\{(t, \theta) \mid t \in (\ln(1/\cos\phi), T)\}$

$$\begin{aligned}
&\partial_t^2 w_i + \Delta_\theta w_i + \delta_i\big(e^{(n+2)(2T-t)/2} - e^{(n+2)t/2}\big) \\
&\quad + \kappa_i e^{\{(n+2)-(n-2)|\mathbf{p}_i|\}(2T-t)/2}(e^{2T-t}\cos\phi - 1)^{\sigma_i}\mathbf{v}_T^{\mathbf{p}_i} \\
&\quad - \kappa_i e^{\{(n+2)-(n-2)|\mathbf{p}_i|\}t/2}(e^t\cos\phi - 1)^{\sigma_i}\mathbf{v}^{\mathbf{p}_i} - \frac{1}{4}(n-2)^2 w_i = 0.
\end{aligned}$$

For $\sigma_i \ge 0$, $T \ge t \ge 0$, $|\mathbf{p}_i| \in [0, 2^*]$, $\phi \in [0, \pi/2)$ and $(t, \theta) \in \Sigma_T$, there hold

$$e^{(n+2)(2T-t)/2} \ge e^{(n+2)t/2}, \qquad e^{\{(n+2)-(n-2)|\mathbf{p}_i|\}(2T-t)/2} \ge e^{\{(n+2)-(n-2)|\mathbf{p}_i|\}t/2},$$

and

$$(e^{2T-t}\cos\phi - 1)^{\sigma_i} \ge (e^t\cos\phi - 1)^{\sigma_i} \ge 0,$$

since $e^t\cos\phi - 1 \ge 0$ for $(t, \theta) \in \Sigma_T$. Therefore, noting that $\delta_i, \kappa_i, \mathbf{v}, \mathbf{v}_T \ge 0$, $\mathbf{w}$ satisfies the inequality,

$$\partial_t^2 w_i + \Delta_\theta w_i - \frac{1}{4}(n-2)^2 w_i + \kappa_i e^{\{(n+2)-(n-2)|\mathbf{p}_i|\}t/2}(e^t\cos\phi - 1)^{\sigma_i}\big[\mathbf{v}_T^{\mathbf{p}_i} - \mathbf{v}^{\mathbf{p}_i}\big] \le 0. \qquad (2.4)$$

For $i, j = 1, \dots, k$, using the integration presentation, we rewrite

$$(e^t\cos\phi - 1)^{\sigma_i}\big[\mathbf{v}_T^{\mathbf{p}_i} - \mathbf{v}^{\mathbf{p}_i}\big]P_{ij}(t, \theta)w_j(T, t, \theta),$$

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where

$$P_{ij}(t,\theta) = (e^t\cos\phi-1)^{\sigma_i}(v_T)_1^{p_{i1}}\dots(v_T)_{j-1}^{p_{i,j-1}}\cdot v_{j+1}^{p_{i,j+1}}\dots v_k^{p_{ik}}p_{ij}\int_0^1[s(v_T)_j+(1-s)v_j]^{p_{ij}-1}ds$$
$$\geq 0.$$

For $p_{ij}\in(0,1)$ and fixed $T\geq 0$, by Lemma 2.7 of [**10**] and Lemma 2.1, we deduce that there exists $C=C(\mathbf{u},T)>0$ (independent of p_{ij}) such that

$$\begin{aligned}
&(v_T)_1^{p_{i1}}\dots(v_T)_{j-1}^{p_{i,j-1}}v_{j+1}^{p_{i,j+1}}\dots v_k^{p_{ik}}p_{ij}\int_0^1[s(v_T)_j+(1-s)v_j]^{p_{ij}-1}ds\\
&\leq (v_T)_1^{p_{i1}}\dots(v_T)_{j-1}^{p_{i,j-1}}v_{j+1}^{p_{i,j+1}}\dots v_k^{p_{ik}}\cdot\min\left\{(v_T)_j^{p_{ij}-1},v_j^{p_{ij}-1}\right\}\\
&=\left(\frac{(v_T)_1(x)}{(v_T)_j(x)}\right)^{p_{i1}}\dots\left(\frac{(v_T)_{j-1}(x)}{(v_T)_j(x)}\right)^{p_{i,j-1}}\cdot\left(\frac{v_{j+1}(x)}{v_j(x)}\right)^{p_{i,j+1}}\dots\left(\frac{v_k(x)}{v_j(x)}\right)^{p_{ik}}\\
&\quad\cdot(v_T)_j^{p_{i1}+\dots+p_{i,j-1}}(x)\cdot v_j^{p_{i,j+1}+\dots+p_{ik}}(x)\cdot\min\left\{(v_T)_j^{p_{ij}-1},v_j^{p_{ij}-1}\right\}\\
&\leq C\max\left\{(v_T)_j^{|\mathbf{p}_i|-1}(x),v_j^{|\mathbf{p}_i|-1}(x)\right\}\leq C\big(1+x_n^{|\mathbf{p}_i|-1}\big),
\end{aligned}$$

where we have used Lemma 2.1 and the fact that $(t,\theta)\in\Sigma_T$, so that we obtain

$$0<v_j(t,\theta)\leq Cx_n(t,\theta),\qquad 0<(v_T)_j(t,\theta)\leq Cx_n((t,\theta)_T),$$

and

$$0<x_n(t,\theta)=e^t\cos\phi-1\leq x_n((t,\theta)_T)=e^{2T-t}\cos\phi-1\leq e^{2T}.$$

It follows that, for all $(t,\theta)\in\Sigma_T$, we have

$$P_{ij}(t,\theta)\leq C(e^t\cos\phi-1)^{\sigma_i}\big(1+x_n^{|\mathbf{p}_i|-1}\big)=C\big(x_n^{\sigma_i}+x_n^{\sigma_i+|\mathbf{p}_i|-1}\big)\leq C,\tag{2.5}$$

since $x_n=e^t\cos\phi-1\in(0,e^T-1]$ for $(t,\theta)\in\Sigma_T$ and $\sigma_i\geq 0$, $\sigma_i+|\mathbf{p}_i|-1\geq 0$. (For $p_{ij}\geq 1$ or $p_{ij}=0$, estimate (2.5) is trivial, for the function $t^{p_{ij}}$ is locally Lipschitz for $t\geq 0$). Combining (2.4) and (2.5), we finally deduce that $\mathbf{w}$ satisfies the inequality

$$\partial_t^2 w_i+\Delta_\theta w_i-\frac{1}{4}(n-2)^2w_i+c_{ij}(t,\theta)w_i\leq 0,$$

where

$$c_{ij}(t,\theta)=\kappa_i e^{\{(n+2)-(n-2)|\mathbf{p}_i|\}t/2}P_{ij}(t,\theta)\in[0,C(\mathbf{u},T)].$$

The rest of the proof of Theorem 4.1 of [**10**], carries over, which shows that the function $r^{n-2}\mathbf{u}(r,\theta)$ is strictly increasing in the radius r.

Therefore, for all non-negative solutions $\mathbf{u}$ of (2.1), $r^{n-2}\mathbf{u}(r,\theta)$ is non-deceasing in r (with positive components being strictly increasing). Since (2.1) is independent of x', $r^{n-2}\mathbf{u}(r,\theta)$ is thus non-decreasing in every radius direction with the origin at any point $(x',-1)$. In turn, $\mathbf{u}=\mathbf{u}(x_n)$ depends only on x_n and is non-decreasing in x_n. (We refer the reader to [**8**] for a detailed proof.)

Now we claim that $\kappa_i x_n^{\sigma_i}\mathbf{u}^{\mathbf{p}_i}+\delta_i\equiv 0$, for $i=1,\dots,k$. Assume that $\kappa_i x_n^{\sigma_i}\mathbf{u}^{\mathbf{p}_i}+\delta_i=C_0>0$ for some i at some point $x_n^0>0$. Then $\kappa_i x_n^{\sigma_i}\mathbf{u}^{\mathbf{p}_i}+\delta_i\geq C_0>0$ for $x_n\geq x_n^0>0$, since $\kappa_i,\sigma_i,\delta_i\geq 0$ and $\mathbf{u}=\mathbf{u}(x_n)\geq 0$ is non-decreasing in x_n. Directly integrating the equation $u_i'' + \kappa_i x_n^{\sigma_i}\mathbf{u}^{\mathbf{p}_i} + \delta_i = 0$ yields an immediate contradiction. It follows that $\kappa_i x_n^{\sigma_i}\mathbf{u}^{\mathbf{p}_i} + \delta_i \equiv 0$ and $u_i'' = 0$ for $i = 1, \ldots, k$, whence one easily concludes that $\delta_i \equiv 0$ and $u_i = h_i x_n$, for some $h_i \geq 0$, $i = 1, \ldots, k$. The proof is complete. □

3. Preliminaries. Let $n \geq 2$ be an integer and let $\Omega \subset \mathbb{R}^n$ be a connected smooth domain. Consider

$$\begin{aligned} \Delta_m u + \kappa_1 u^a v^b + t_1 = 0 \quad &\text{in } \Omega, \\ \Delta_m v + \kappa_2 u^c v^d + t_2 = 0 \quad &\text{in } \Omega, \end{aligned} \tag{3.1}$$

together with, whenever Ω has a non-empty boundary $\partial\Omega$, the boundary condition

$$u = v = 0 \quad \text{on } \partial\Omega, \tag{3.2}$$

where $m > 1$ and $\kappa_1, \kappa_2, t_1, t_2 \geq 0$ are constants, $\mathbf{u} = (u, v) \geq 0$ is a function and a, b, c, d are non-negative numbers as given in the introduction. When Ω is bounded and $t_1 = t_2 = t$, with the help of the strong maximum principle, every non-negative solution $\mathbf{u} \geq 0$ of (3.1) must be either identically zero or strictly positive in Ω since (3.1) is fully-coupled (i.e., $b, c > 0$).

Our first lemma is a monotonicity result.

Lemma 3.1. *Let $m = 2$ and let $\Omega \subset \mathbb{R}^n$ be a uniformly normal domain. Suppose that* (1.6) *holds. Then all non-negative solutions* $\mathbf{u}$ *of* (3.1) *are necessarily monotone in* Ω.

Proof. This is essentially Theorem 3.1 of **[10]** that continues to hold for (3.1) with arbitrary $\kappa_1, \kappa_2, t_1, t_2 \geq 0$. □

For $x \in \mathbb{R}^n_+$, let $\phi > 0$ be the (positive) angle between x and the positive x_n-direction. We call $\phi = \phi(x) > 0$ the *directional angle* of x. For $\phi_0 \in (0, \pi/2]$, a non-negative function u is said to be ϕ_0-*monotone* in $\mathbb{R}^n_+$ if u is monotone in all directions $x \in \mathbb{R}^n_+$ with $\phi(x) \in [0, \phi_0)$. Our second lemma is the following Liouville-type non-existence result.

Lemma 3.2. *Suppose that* $\kappa_1, \kappa_2 > 0$.

(1) *Let* $\Omega = \mathbb{R}^n$. *Assume that* $m = 2$ *and* (1.3) *holds. Then* (3.1) *has no positive solutions* $\mathbf{u}$.
(2) *Let* $\Omega = \mathbb{R}^n$. *Assume that* (1.4) *and* (**ALT**) *hold. Then* (3.1) *has no positive solutions* $\mathbf{u}$.
(3) *Let* $\Omega = \mathbb{R}^n_+ = \{(x', x_n) \in \mathbb{R}^n \mid x_n > 0\}$ *and assume* $\beta > 0$. *Then* (3.1) *cannot have a positive solution* $\mathbf{u}$ *which is* ϕ_0*-monotone in* $\mathbb{R}^n_+$ *for some* $\phi_0 \in (0, \pi/2]$.

Proof. Part (1) follows directly from Lemmas 2.4 and 2.6 of **[10]** (no need of (1.6)!), while Part (2) follows from Lemma 2.3 and Theorem 1.1 of **[12]**.

To prove (3), suppose the contrary and let $\mathbf{u}$ be a positive solution of (3.1) that is ϕ_0-monotone in $\mathbb{R}^n_+$, for some $\phi_0 > 0$ (say $\phi_0 < \pi/4$). Then one has

$$\inf_{|x'| < \tan(\phi_0) x_n} \mathbf{u}(x', x_n) \geq \inf_{|x'| < \tan(\phi_0)} \mathbf{u}(x', 1) > 0.$$

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On the other hand, Lemma 3.2 of [**12**] applies since $\beta, \kappa_1, \kappa_2 > 0$ and $t_1, t_2 \geq 0$. In particular, one infers that there exist $\varepsilon > 0$ and $C > 0$ such that for $x_n > 0$ large[2]

$$\min\{\inf_{|x'|<\tan(\phi_0)x_n} u(x', x_n), \inf_{|x'|<\tan(\phi_0)x_n} v(x', x_n)\} \leq C x_n^{-\varepsilon}.$$

The two inequalities above clearly yield a contradiction and the proof is complete. □

The next lemma yields a bound for the ratio of the components of a solution $\mathbf{u}$ of (3.1).

LEMMA 3.3. *Let $\Gamma \subset \Omega \subset \mathbb{R}^n$ be smooth and bounded and let $\mathbf{u}$ be a positive solution of* (3.1). *Then there exists a positive constant $C = C(\mathbf{u}, t_1, t_2, \kappa_1, \kappa_2, \Gamma) > 0$ such that*

$$\frac{1}{C} \leq \inf_{x\in\Gamma} \frac{u(x)}{v(x)} \leq \sup_{x\in\Gamma} \frac{u(x)}{v(x)} \leq C.$$

Proof. When $m = 2$, this is essentially Lemma 3.1 of [**10**]. For general $m > 1$, the proof of Lemma 3.1 of [**10**] carries over by using the strong maximum principle for the m-Laplace operator. □

We conclude this section with an upper bound of the parameter $t_1 = t_2 = t$ in (3.1).

LEMMA 3.4. *Let $\Omega \subset \mathbb{R}^n$ be smooth and bounded and let $\mathbf{u}$ be a solution of* (3.1) *with $t_1 = t_2 = t \geq 0$. Suppose that $\beta, \kappa_1, \kappa_2 > 0$. Then there exists a positive constant $t_0 = t_0(a, b, c, d, m, n, \kappa_1, \kappa_2, \Omega) > 0$ such that $t \leq t_0$.*

Proof. Let $\mathbf{u}$ be a solution of (3.1). As observed earlier, $\mathbf{u}$ is either identically zero or strictly positive since $t_1 = t_2 = t$ and Ω is bounded. Hence we shall assume that $\mathbf{u} = (u, v) > 0$ ($\mathbf{u} \equiv 0 \Rightarrow t = 0$).

Let λ_1 be the first eigenvalue of $-\Delta_m$ (with homogeneous Dirichlet boundary data) and $\phi_1 > 0$ an associated first eigenfunction. For $\varepsilon > 0$ and $m > 1$, it has been observed in [**4**], as a consequence of the Young inequality, that there holds for $u_1 = u + \varepsilon$ (note that $\nabla u_1 = \nabla u$)

$$|\nabla\phi_1|^m - \nabla\left(\frac{\phi_1^m}{u_1^{m-1}}\right)|\nabla u|^{m-2}\nabla u \geq 0 \qquad (x \in \Omega).$$

The inequality above is sometimes referred to as Picone's identity and has since been used for its applications in studying equations involving the m-Laplace operator. See also, for example, [**1**] and the references therein. It follows that, by using the equations of ϕ_1 and u respectively (note that $(\phi_1^m/u_1^{m-1}) \in C_0^{1,\gamma}(\Omega)$ for some $\gamma \in (0, 1)$)

$$\lambda_1 \int_\Omega \phi_1^m - \int_\Omega (t + \kappa_1 u^a v^b)\frac{\phi_1^m}{u_1^{m-1}} = \int_\Omega \left(|\nabla\phi_1|^m - \nabla\left(\frac{\phi_1^m}{u_1^{m-1}}\right)|\nabla u|^{m-2}\nabla u\right) \geq 0.$$

[2] Lemma 3.2 in [**12**] was proved for exterior domains, but it remains valid here since for $x_n > 0$ all balls $B_r(\mathcal{O})$ centered at $\mathcal{O} = (0', x_n)$ with radius $r = \tan(\phi_0)x_n$ are contained in $\Omega = \mathbb{R}^n_+$.

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Letting $\varepsilon \to 0$ immediately yields (since $\{\phi_1^m/u_1^{m-1}\}$ are uniformly bounded in Ω for $\varepsilon > 0$)

$$\int_\Omega (t + \kappa_1 u^a v^b) u^{1-m} \phi_1^m \le \lambda_1 \int_\Omega \phi_1^m.$$

Similarly, one has

$$\int_\Omega (t + \kappa_2 u^c v^d) v^{1-m} \phi_1^m \le \lambda_1 \int_\Omega \phi_1^m.$$

Combining the two together, we have

$$\int_\Omega \{(t + \kappa_1 u^a v^b) u^{1-m} + (t + \kappa_2 u^c v^d) v^{1-m}\} \phi_1^m \le 2\lambda_1 \int_\Omega \phi_1^m. \tag{3.3}$$

Since $\beta, b, c > 0$, one readily infers that there exists $\lambda, \mu \ge 0$ ($\lambda, \mu > 0$ if $\alpha, \delta > 0$) such that $\lambda + \mu = 1$, and

$$r, s \ge 0, \quad r + s > 0, \quad \text{where } r = -\lambda\alpha + \mu c, \quad s = \lambda b - \mu\delta.$$

For example, one may take (with $\mu = 1 - \lambda$) $\lambda = 1$ if $\alpha \le 0$, $\lambda = 0$ if $\delta \le 0$ and

$$\delta/(b + \delta) < \lambda < c/(\alpha + c) \text{ if } \alpha, \delta > 0.$$

By Young's inequality, for $u, v > 0$ we have that

$$\kappa_1 (u^a v^b) u^{1-m} + \kappa_2 (u^c v^d) v^{1-m} = \kappa_1 u^{-\alpha} v^b + \kappa_2 u^c v^{-\delta} \ge C u^r v^s,$$

for some $C = C(\kappa_1, \kappa_2, r, s, \lambda, \mu) > 0$. It follows that

$$(t + \kappa_1 u^a v^b) u^{1-m} + (t + \kappa_2 u^c v^d) v^{1-m} \ge t(u^{1-m} + v^{1-m}) + C u^r v^s.$$

However, a direct computation shows that for $t \ge 0$ (since $m > 1$)

$$\inf_{u,v>0} \{t(u^{1-m} + v^{1-m}) + C u^r v^s\} = C t^{(r+s)/(r+s+m-1)},$$

for some positive constant C depending only on $\kappa_1, \kappa_2, r, s, \lambda, \mu$ and m. Now the conclusion follows from (3.2) by taking $t_0 = (2\lambda_1 C^{-1})^{(r+s+m-1)/(r+s)}$. □

4. A priori estimates. In this section, we develop supremum a priori estimates for non-negative solutions of (3.1) when $t_1 = t_2 = t$. We first consider the case $m = 2$.

THEOREM 4.1. *Let $m = 2$ and let Ω be a uniformly normal domain. Assume that $\kappa_1\kappa_2 > 0$ and $t_1 = t_2 = t \ge 0$. Suppose that* (1.4) *and* (1.6) *hold. Then there exists a positive constant $C > 0$ depending on the structural constants such that*

$$t + \max_{x\in\Omega} u(x) + \max_{x\in\Omega} v(x) \le C, \tag{4.1}$$

for all non-negative solutions $\mathbf{u}$ *of* (3.1)*, provided one of the following holds.*

(A) (1.1) *has no positive solutions on* $\Omega = \mathbb{R}^n$.

(B) (**ALT**) *holds.*

(C) (1.3) *holds.*

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Proof. We proceed similarly as in proving Theorem 5.1 of [**10**], and use the same notation. As observed earlier, one only needs to consider strictly positive solutions $\mathbf{u}$ and therefore it is assumed all solutions $\mathbf{u}$ are actually strictly positive in the sequel.

Let $\mathbf{u} = (u_1, u_2)$ be a positive solution of (3.1). Write

$$n_1(x) := u_1^{-\alpha}(x)u_2^b(x), \qquad n_2(x) := u_1^c(x)u_2^{-\delta}(x),$$

and

$$U_i := \max_{x\in\Omega} u_i(x) > 0, \quad N_i := \sup_{x\in\Omega} n_i(x) > 0 \qquad (i = 1, 2).$$

By the homogeneous Dirichlet boundary condition, there exist $\xi_i \in \Omega$ such that

$$U_i = u_i(\xi_i) < \infty \qquad (i = 1, 2).$$

Using (1.6) and Lemma 3.3, one also readily deduces that there exist $\zeta_i \in \Omega$ such that

$$n_i(\zeta_i) \geq N_i/2, \quad N_i < \infty \qquad (i = 1, 2).$$

We shall prove (4.1) by contradiction. Suppose (4.1) is false and there exist a sequence of positive solutions $\mathbf{u}_l = (u_{1,l}, u_{2,l})$ of (3.1) and a corresponding sequence of numbers $t_l \geq 0$ such that

$$0 < t_l + U_{1,l} + U_{2,l} \to \infty,$$

where $n_{i,l}(x)$, $U_{i,l}$, $N_{i,l}$, $\xi_{i,l}$ and $\zeta_{i,l}$ are the various quantities given above, corresponding to $\mathbf{u}_l$. By Lemma 3.4, the sequence $\{t_l\}$ is bounded. Hence, without loss of generality, assume that

$$\lim_{l\to\infty} U_{1,l} = \lim_{l\to\infty} \max\{U_{1,l}, U_{2,l}\} = \infty. \tag{4.2}$$

For $z^l \in \Omega$ and $Q_l \geq 1$ to be determined later, make the change of variables

$$v_{i,l}(y) = \frac{u_{i,l}(x)}{u_{i,l}(z^l)}, \quad y = (x - z^l)Q_l; \qquad i = 1, 2, \quad l = 1, 2, \ldots. \tag{4.3}$$

Put

$$\Omega_l := \{y \in \mathbb{R}^n \mid y = (x - z^l)Q_l, x \in \Omega\}; \qquad \tau_l := \mathrm{dist}(z^l, \partial\Omega)Q_l = \mathrm{dist}(0, \partial\Omega_l). \tag{4.4}$$

The proof is next divided into three cases, with

$$\bar{N}_i = \lim_{l\to\infty} N_{i,l} \in [0, \infty] \qquad (i = 1, 2),$$

where the limit can be infinite.[3]

Case (I). $\bar{N}_1 = 0$. One argues in exactly the same way as in Case (I) of Theorem 5.1 of [**10**] to derive a contradiction.

[3]It is understood all convergences here and in the sequel are up-to a subsequence.

Case (II). $\bar{N}_1 = \infty$. In (4.3), take

$$z^l := \zeta_l, \qquad Q_l^2 := \max\{N_l, t_l u_{1,l}^{-1}(\zeta_l), t_l u_{2,l}^{-1}(\zeta_l)\} \to \infty,$$

where

$$N_l := \max\{N_{1,l}, N_{2,l}\} \to \infty, \qquad \zeta_l := \zeta_{i,l} \text{ if } N_l = N_{i,l}.$$

The rest of the proof proceeds essentially the same as in **[10]**, provided one can show that all of the following hold simultaneously.[4]

(1) The system (3.1) admits no positive solutions on $\Omega = \mathbb{R}^n$.
(2) The single equation $\Delta_m u + \kappa u^\lambda = 0$, where $\kappa > 0$ and either $\lambda = a$ or $\lambda = d$, has no positive solutions on $\Omega = \mathbb{R}^n$.
(3) There holds the lower-bound estimate (4.5) below.

First, note that any one of assumptions (A), (B) and (C) of Theorem 4.1 implies statement (1). Indeed, with the aid of Lemma 2.4(B) of **[10]**, one readily verifies that (1) is equivalent to assumption (A) since $\kappa_1, \kappa_2 > 0$ and $\beta > 0$ by (1.4). By Lemma 3.2, (1) and (2), either (1.4) combined with assumption (B), or assumption (C) also implies (1). The conclusion (2) follows directly from Lemma 2.3 of **[12]** since $\max(a, d) < m^*$ by (1.4). Finally, under (1.4) and (1.6), we shall prove the lower-bound estimate (4.5); see Lemma 4.1 below. It follows that all (1)–(3) hold under the assumptions of Theorem 4.1. Now with (1)-(3) above available, with the aid of (1.4), one readily adapts the arguments of **[10]** to derive a contradiction. This finishes the proof of Case (II), pending the completion of the proof of Lemma 4.1.

Case (III). $\bar{N}_1 \in (0, \infty)$. Then, as in **[10]**, one infers that $\bar{N}_2 = \infty$ since $\beta, b, c > 0$. Take

$$z^l = \zeta_{2,l}, \qquad Q_l^2 = \max\{N_{2,l}, t_l u_{1,l}^{-1}(\zeta_{2,l}), t_l u_{2,l}^{-1}(\zeta_{2,l})\} \to \infty$$

and this becomes an analogue of Case (II) and one deduces a contradiction similarly.

To complete the proof, in view of Case (II) above, it remains to prove the next result.

LEMMA 4.1. *Let $m = 2$ and let $\Omega \subset \mathbb{R}^n$ be a uniformly normal domain. Assume that* $\mathbf{u}$ *is a positive solution of* (3.1). *Suppose that $\kappa_1 > 0$, $\kappa_2 > 0$,* (1.4) *and* (1.6) *hold, and*

$$\max(\bar{N}_1, \bar{N}_2) = \infty.$$

For $l = 1, 2, \dots,$ put

$$N_l := \max\{N_{1,l}, N_{2,l}\} \to \infty, \qquad \zeta_l := \zeta_{i,l} \textit{ if } N_l = N_{i,l},$$

and

$$Q_l^2 := \max\left\{N_l, t_l u_{1,l}^{-1}(\zeta_l), t_l u_{2,l}^{-1}(\zeta_l)\right\} \to \infty.$$

[4]The conclusions (1) and (2) below are the key ingredients Lemmas 2.6 and 5.1, respectively, used in **[10]**, that are ensured by (1.4) plus either one of the conditions (A), (B) or (C).

Then

$$\limsup_{l\to\infty} Q_l \operatorname{dist}(\zeta_l, \partial\Omega) = \infty. \tag{4.5}$$

Proof. A proof was given in [**10**, Lemma 5.1] under the pseudo-subcriticality (1.3), which can be readily carried over under assumption (1.4).

By (1.6) and Lemma 3.3, one has

$$\operatorname{dist}(\zeta_{i,l}, \partial\Omega) > 0 \qquad (i = 1, 2, \quad l = 1, 2, \ldots).$$

Suppose that (4.5) fails. Then one readily sees the arguments in Lemma 5.1 of [**10**] carry over without change to imply that there exist non-negative numbers $\sigma \in (0, 1)$, $\delta_1, \delta_2, \kappa_1, \kappa_2 \geq 0$ (abusing notation: κ_1, κ_2 here maybe different from the original κ_1, κ_2 in (3.1)) and a function $\mathbf{v} = (v_1, v_2) \in C^{1,\sigma}_{loc}(\overline{\mathbb{R}^n_+})$ such that $\mathbf{v} \geq 0$ satisfies $\mathbf{v}(0', 1) = (1, 1)$ and the limiting equations

$$\begin{aligned} \Delta v_1 + \kappa_1 v_1^a v_2^b + \delta_1 &= 0 \quad \text{in } \mathbb{R}^n_+, \\ \Delta v_2 + \kappa_2 v_1^c v_2^d + \delta_2 &= 0 \quad \text{in } \mathbb{R}^n_+, \\ \mathbf{v} &= 0 \quad \text{on } \partial\mathbb{R}^n_+ \end{aligned}$$

(which is simply (2.1) with $k = 2$). Moreover, we claim that

(1) The sequence of domains Ω_l converges to the half-space $\mathbb{R}^n_+ = \{y \in \mathbb{R}^n \mid y_n > 0\}$ and

$$\lim_{l\to\infty} \mathbf{v}_l(y) = \mathbf{v}(y) \tag{4.6}$$

on any compact set $\Gamma_1 \subset \mathbb{R}^n_+$ in the $C^{1,\sigma}$-topology, where $\{\mathbf{v}_l\}$ and $\{\Omega_l\}$ are the sequences given in the (4.3) and (4.4) respectively.

(2) There exists a positive constant $\phi_0 > 0$ such that $\mathbf{v}$ is ϕ_0-monotone in $\mathbb{R}^n_+$. See the definition in Section 2.

Claim (1) was shown in [**10**] (simply by construction).

Next we prove (2). As in [**10**], for each l and by suitable rotations and translations, the inner normal direction of $\partial\Omega$ at a boundary point $\zeta'_l \in \partial\Omega$ is chosen to be the positive x_n-direction. Moreover, under the transform (4.3)–(4.4), the point $\zeta'_l \in \partial\Omega$ is mapped to the origin $\mathcal{O} \in \partial\Omega_l$ in y-coordinates and the inner normal direction of $\partial\Omega_l$ at the origin $\mathcal{O} \in \partial\Omega_l$ becomes the positive y_n-direction (i.e., the positive x_n-direction is mapped to the positive y_n-direction).

By Lemma 3.1, all $\mathbf{u}_l$ are monotone in Ω. It follows that (thanks also to the uniform normality of Ω) there exist two positive constants $\delta_0 > 0$ and $\phi_0 > 0$ (independent of l) such that for $x \in \Omega$ and $\nu \in S^{n-1}$

$$D_\nu u_{i,l}(x) \geq 0 \qquad (i = 1, 2, \quad l = 1, 2, \ldots),$$

provided that $|x - \zeta'_l| \leq \delta_0$ and $\phi_x(\nu) < \phi_0$, where $\phi_x(\nu)$ is the directional angle of ν between ν and the positive x_n-direction and D_ν is the directional derivative in the ν-direction. In turn, by the formula (4.3), we have for $y \in \Omega_l$ and $\nu \in S^{n-1}$

$$D_\nu v_{i,l}(y) = Q_l^{-1} u_{i,l}^{-1}(z^l) D_\nu u_{i,l}(x) \geq 0 \qquad (i = 1, 2, \quad l = 1, 2, \ldots),$$

provided that $|y|(=|x-\zeta_l'|Q_l) \leq \delta_0 Q_l$ and $\phi_y(\nu)=\phi_x(\nu)<\phi_0$, where $\phi_y(\nu)$ is the directional angle of ν between ν and the positive y_n-direction. For each $r>0$, there exists $l_0=l_0(r)>0$ such that $Q_l>r\delta_0^{-1}$ for $l\geq l_0$, since $Q_l\to\infty$. It follows that, for all $l>0$ sufficiently large (i.e., $Q_l>r\delta_0^{-1}$),

$$D_\nu \mathbf{v}_l(y)\geq 0, \qquad \forall y\in(B_r(0)\cap\Omega_l) \text{ and } \nu\in S^{n-1}$$

with $\phi_y(\nu)<\phi_0$. Now invoking the convergence (4.6), we conclude that, for each $r>0$

$$D_\nu \mathbf{v}(y)\geq 0, \qquad \forall y\in(B_r(0)\cap\mathbb{R}^n_+) \text{ and } \nu\in S^{n-1}$$

with $\phi_y(\nu)<\phi_0$. It follows that $\mathbf{v}$ is ϕ_0-monotone in $\mathbb{R}^n_+$.

In summary, $\mathbf{v}$ is a (strictly positive) solution of (3.1) that is ϕ_0-monotone in $\mathbb{R}^n_+$ for some $\phi_0>0$. There are three possibilities concerning the coefficients κ_1 and κ_2 in (3.1).

(A) $\kappa_1\cdot\kappa_2>0$. Then Lemma 3.2(3) implies that (3.1) cannot have a positive solution $\mathbf{v}$ which is ϕ_0-monotone in $\mathbb{R}^n_+$ for some $\phi_0>0$, an immediate contradiction.

(B) $\kappa_1\cdot\kappa_2=0$ but $\kappa_1+\kappa_2>0$, say $\kappa_1=0$ and $\kappa_2>0$. Then $\Delta v_1+\delta_1=0$ and $v_1\geq 0$ in $\mathbb{R}^n_+$, $v_1=0$ on $\partial\mathbb{R}^n_+$ and $v_1(0',1)=1$. Clearly, Theorem 4.2 of [**10**] implies that $\delta_1=0$ and $v_1(x)=x_n$. It follows that $\Delta v_2+\kappa_2 x_n^c v_2^d+\delta_2=0$ and $v_2\geq 0$ in $\mathbb{R}^n_+$, $v_2=0$ on $\partial\mathbb{R}^n_+$ and $v_2(0',1)=1$. That is, with the aid of (1.4) and (1.6), (2.1) has a positive solution $u=v_2$ with $k=1$, $\kappa=\kappa_2>0$, $\sigma_1=c$ and $p_1=d\geq 0$ satisfying

$$c+d\geq 1, \qquad d\leq 2^*.$$

This is impossible in view of Lemma 2.2.

(C) $\kappa_1+\kappa_2=0$. Then Theorem 4.2 of [**10**] implies that $v_1(x)=v_2(x)=x_n$ with $\delta_1=\delta_2=0$. But then the proof of Lemma 5.1 in [**10**] shows that this is impossible (necessarily $\kappa_1+\kappa_2>0$). Note particularly that this part of the proof requires only (1.6).

The proof of Lemma 4.1 is complete. □

In conclusion, turning back to the proof of Theorem 4.3, we have derived a contradiction for each of the cases (I)–(III) above. In turn, (4.2) is necessarily false. Therefore the a priori estimate (4.1) holds and the proof is complete. □

For monotone solutions, it turns out that (1.6) is superfluous. We have the following corollary.

Corollary 4.1. *Let $m=2$ and let Ω be an uniformly normal domain. Suppose that $t_1=t_2=t\geq 0$, $\kappa_1,\kappa_2>0$ and* (1.4) *holds. Then there exists a positive constant $C>0$ depending on the structural constants such that*

$$t+\max_{x\in\Omega} u(x)+\max_{x\in\Omega} v(x)\leq C,$$

for all solutions $\mathbf{u}$ *of* (3.1) *that are monotone in Ω, provided one of the following holds.*

(A) (1.1) *has no positive solutions on $\Omega=\mathbb{R}^n$.*

(B) (**ALT**) *holds.*

(C) (1.3) *holds.*

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Proof. We proceed in the same way as in proving Theorem 4.1 and use the same notation. The proof here is simpler since the solutions considered are monotone. Indeed, the blown-up domain is always the entire space $\mathbb{R}^n$ and consequently Lemma 4.1 is no longer needed.

By Theorem 4.1, we only need to consider the case

$$\min(a+b, c+d) < m-1.$$

Thus, without loss of generality, we assume that

$$a = \min(a, d) < m-1 \Longrightarrow \alpha = m-1-a > 0.$$

Let $\{\mathbf{u}_l\}$ and $\{t_l\}$ be given as in Theorem 4.1 such that

$$\lim_{l\to\infty} (U_{1,l} + U_{2,l}) = \infty. \tag{4.7}$$

By assumption, Ω is uniformly normal and $\{\mathbf{u}_l\}$ are monotone in Ω. That is, there exists $\delta_0 > 0$ (cf. [**10**]) such that

$$\max_{x\in\Omega} u_{i,l}(x) = \max_{x\in\Omega_0} u_{i,l}(x) \qquad (i = 1, 2;\ \ l = 1, 2, \ldots),$$

where

$$\Omega_0 := \{x \in \Omega \mid \operatorname{dist}(x, \partial\Omega) \geq \delta_0\}$$

is non-empty. In turn, one may choose $\xi_{i,l} \in \Omega_0$ such that for $i = 1, 2$ and $l = 1, 2, \ldots$

$$U_{i,l} = u_{i,l}(\xi_{i,l}), \qquad \operatorname{dist}(\xi_{i,l}, \partial\Omega) \geq \delta_0.$$

We next consider two cases.

Case (I). We have $U_{2,l} \leq C < \infty$. Then $U_{1,l} \to \infty$ by (4.7) and, by Lemma 3.4,

$$\max\{U_{1,l}^{-1} t_l,\ U_{1,l}^{-\alpha} U_{2,l}^{b}\} \to 0$$

since $\alpha, b > 0$. Rewrite $(3.1)_1$ in the form

$$\Delta w_{1,l}(x) + \kappa_1 U_{1,l}^{-\alpha} u_{2,l}^{b}(x) w_{1,l}^{a}(x) + U_{1,l}^{-1} t_l = 0, \tag{4.8}$$

where

$$w_{1,l}(x) = u_{1,l}(x) U_{1,l}^{-1} \in (0, 1], \qquad \max_{x\in\Omega} w_{1,l}(x) = 1, \qquad w_{1,l}(x)|_{x\in\partial\Omega} = 0.$$

Letting $l \to \infty$ in (4.8) immediately yields a contradiction since

$$\max_{x\in\Omega} \left|\kappa_1 U_{1,l}^{-\alpha} u_{2,l}^{b}(x)\right| \leq \kappa_1 U_{1,l}^{-\alpha} U_{2,l}^{b} \to 0.$$

Case (II). We have $U_{2,l} \to \infty$. We claim that

$$\max\{n_{1,l}(\xi_{2,l}), n_{2,l}(\xi_{2,l})\} \to \infty.$$

To this end, suppose, say, $n_{1,l}(\xi_{2,l}) \le C$ for some $C > 0$, $l = 1, 2, \ldots$. Then $u_{1,l}(\xi_{2,l}) \ge CU_{2,l}^{b/\alpha}$ since $\alpha > 0$. In turn,

$$n_{2,l}(\xi_{2,l}) = u_{1,l}^{c}(\xi_{2,l})U_{2,l}^{-\delta} \ge CU_{2,l}^{bc/\alpha-\delta} = CU_{2,l}^{\beta/\alpha} \to \infty,$$

since $U_{2,l} \to \infty$ and $\alpha, \beta > 0$, which yields the claim.

Now, in (4.3), take

$$z^l := \xi_{2,l}, \qquad Q_l^2 := \max\left\{n_{1,l}(z^l), n_{2,l}(z^l), t_l u_{1,l}^{-1}(z^l), t_l u_{2,l}^{-1}(z^l)\right\} \to \infty.$$

The rest of the proof becomes an analogue of Case (II) of Theorem 4.1. That is, all (1)-(3) in Case (II) of Theorem 4.1 remain valid under our assumptions. Indeed, since $Q_l \mathrm{dist}(\xi_{2,l}, \partial\Omega) \ge Q_l\delta_0 \to \infty$ as $l \to \infty$, (3) holds automatically. By (1.4), (2) follows from Lemma 2.3 of [**12**] since $\max(a, d) < m^*$ by (1.4). Finally, assumption (A) is equivalent to (1) since $\kappa_1, \kappa_2 > 0$ and $\beta > 0$, by (1.4), while by Lemma 3.2, (1) and (2), either (1.4) combined with assumption (B), or assumption (C) also implies (1). It follows that the arguments there carry over and one deduces a contradiction similarly, provided that one can show that $\{v_{1,l}\}$ is uniformly bounded on any compact set $\Gamma \subset \mathbb{R}^n$ for l sufficiently large (so that $\Gamma \subset \Omega_l$). To this end, we note that, by the choice of Q_l, and the fact that $0 \le v_{2,l}$, by construction,

$$0 \le Q_l^{-2}\kappa_1 n_{1,l}(\xi_{2,l})v_{1,l}^{a}(y)v_{2,l}^{b}(y) \le \kappa_1(1 + v_{1,l}(y)), \qquad 0 \le Q_l^{-2}t_l u_{1,l}^{-1}(\xi_{2,l}) \le 1,$$

since $b > 0$ and $a \in [0, 1)$. Moreover, $v_{1,l}$ satisfies the equation

$$\Delta v_{1,l}(y) + Q_l^{-2}\kappa_1 n_{1,l}(\xi_{2,l})v_{1,l}^{a}(y)v_{2,l}^{b}(y) + Q_l^{-2}t_l u_{1,l}^{-1}(\xi_{2,l}) = 0.$$

Now the uniform boundedness of $\{v_{1,l}\}$ follows from the Harnack inequality, since $v_{1,l}(0, 0) = 1$ for $i = 1, \ldots,$ and the proof is complete. □

Finally, in this section, we prove similar a priori estimates for monotone solutions of (3.1) with arbitrary $m > 1$, without the restriction (1.6).

THEOREM 4.2. *Let Ω be uniformly normal. Suppose that $\kappa_1\kappa_2 > 0$, $t_1 = t_2 = t \ge 0$ and* (1.4) *holds. Then there exists a positive constant $C > 0$ depending on the structural constants such that*

$$t + \max_{x\in\Omega} u(x) + \max_{x\in\Omega} v(x) \le C,$$

for all solutions $\mathbf{u}$ *of* (3.1) *that are monotone in Ω, provided one of the following holds.*

(A) (1.1) *has no positive solutions on $\Omega = \mathbb{R}^n$.*
(B) (**ALT**) *holds.*

Proof. We first prove Theorem 4.2 under assumption (B). In view of Theorem I, with (1.4) and assumption (B) being valid, we may assume that

$$\min(a, d) < m - 1.$$

Now one readily verifies that the arguments of Corollary 4.1 carry over with little change (i.e., replacing 2 by m and 1 by $m - 1$ wherever applicable) and concludes the desired estimates for monotone solutions as required.

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Now note that (B) is only used to show that (3.1) with $\kappa_1\kappa_2 > 0$ has no positive solutions on $\Omega = \mathbb{R}^n$ in the entire proof. Moreover, that (3.1) with $\kappa_1\kappa_2 > 0$ has no positive solutions on $\Omega = \mathbb{R}^n$ is equivalent to (1.1) having no positive solutions on $\Omega = \mathbb{R}^n$, since $\beta > 0$ by (1.4). Therefore it suffices to prove Theorem 4.2 under the assumption (B) and the proof is complete. $\square$

We should like to point out that, under the assumption that (1.1) has no positive solutions on $\Omega = \mathbb{R}^n$, the first part $\beta > 0$ of (1.4) can be replaced by $\beta \neq 0$, provided that it is assumed that $t_1 = t_2 = t = 0$.

5. Proof of Theorems 1.1–1.4. In this section, we give proofs to Theorems 1.1–1.4 that are essentially the same as those given in [**10**, **12**]. We only sketch the proof and refer the reader to [**10**, **12**] for further details.

Proof of Theorem 1.1. One readily verifies that Theorem 4.1 applies under the assumptions of Theorem 1.1. Thus the a priori estimates (**EST**) follow directly from Theorem 4.1. The proof of the existence part is almost precisely the same as that of Theorem 1.2 in [**10**] (Theorem 1.3 in [**12**] for general $m > 1$), with the aid of Theorem 4.1. The only exception here is that one needs to prove, under the assumption $\beta > 0$, the next result.

Step 2. For $t \in [0, 1]$, there exists a positive number r such that $\mathbf{u} \neq tF(\mathbf{u})$ for $||\mathbf{u}|| = r$. Consider $\mathbf{u} = tF(\mathbf{u})$ with $t \in [0, 1]$ and $||\mathbf{u}|| = r > 0$; that is,

$$\begin{aligned} \Delta_m u + tu^a v^b &= 0, \quad \text{in } \Omega, \\ \Delta_m v + tu^c v^d &= 0, \quad \text{in } \Omega, \\ u = v &= 0, \quad \text{on } \partial\Omega. \end{aligned} \tag{5.1}$$

For $e, f \geq 1$, multiply the first equation by u^e and the second equation by v^f and integrate over Ω to obtain

$$\int_\Omega |\nabla u|^m u^{e-1} \leq C \int_\Omega u^{a+e} v^b, \qquad \int_\Omega |\nabla v|^m v^{f-1} \leq C \int_\Omega u^c v^{d+f} \tag{5.2}$$

since $t \in [0, 1]$.

We want to show that there exists $r_0 > 0$ such that the equation $\mathbf{u} = tF(\mathbf{u})$; i.e., (5.1), actually has no solution in the punctured ball $B_{r_0}(0) - \{0\}$ for all $t \in [0, 1]$. By the strong maximum principle, using (5.1), $\mathbf{u}$ is strictly positive in Ω since $||\mathbf{u}|| = r > 0$ and (5.1) is fully-coupled.

We next consider four cases. In the sequel, $\mathbf{u}$ is always taken as a positive solution of (5.1) with $r = \|\mathbf{u}\| > 0$.

Case (I). $\max\{a, d\} \geq m - 1$, say, $d \geq m - 1$. Taking $f = 1$ in $(5.2)_2$, we deduce that

$$\int_\Omega |\nabla v|^m = O(r^{c+d+1-m}) \int_\Omega v^m = O(r^{c+d+1-m}) \int_\Omega |\nabla v|^m,$$

where $r = \|\mathbf{u}\|$, $c > 0$ and $d + 1 \geq m$, and we have used the Poincaré inequality. This is impossible if $r = \|\mathbf{u}\|$ is small since $v > 0$. It follows that there exists $r_0 > 0$ such that the equation $\mathbf{u} = tF(\mathbf{u})$ has no solution in $B_{r_0}(0) - \{0\}$, for all $t \in [0, 1]$.

Case (II). $\max\{a, d\} < m - 1$, $m \in (1, n)$ and $\max\{a + b, c + d\} \le m^*$. Clearly,

$$\max\{a, d\} < m - 1 \Longrightarrow \alpha, \delta > 0. \tag{5.3}$$

We take $e = f = 1$ in (5.2), and apply the Hölder and Sobolev inequalities to deduce that

$$\begin{aligned}\int_\Omega |\nabla u|^m \le \int_\Omega u^{a+1}v^b &\le \left(\int_\Omega u^{m^*+1}\right)^{\frac{a+1}{m^*+1}} \left(\int_\Omega v^{\frac{b(m^*+1)}{m^*-a}}\right)^{\frac{m^*-a}{m^*+1}} \\ &\le C\left(\int_\Omega |\nabla u|^m\right)^{(a+1)/m} \left(\int_\Omega v^{\frac{b(m^*+1)}{m^*-a}}\right)^{\frac{m^*-a}{m^*+1}} \\ &\le C\left(\int_\Omega |\nabla u|^m\right)^{(a+1)/m} \left(\int_\Omega v^{m^*+1}\right)^{b/(m^*+1)} \\ &\le C\left(\int_\Omega |\nabla u|^m\right)^{(a+1)/m} \left(\int_\Omega |\nabla v|^m\right)^{b/m},\end{aligned}$$

where we have used the fact $b(m^* + 1)/(m^* - a) \le m^* + 1$ (by assumption $a + b \le m^*$). In turn,

$$\int_\Omega |\nabla u|^m \le C\left(\int_\Omega |\nabla v|^m\right)^{b/\alpha}.$$

Similarly, one has

$$\int_\Omega |\nabla v|^m \le C\left(\int_\Omega |\nabla u|^m\right)^{c/\delta}.$$

Combining the two inequalities together yields

$$1 \le C\left(\int_\Omega |\nabla u|^m\right)^{\beta/\alpha\delta},$$

since $\alpha, \beta, \delta > 0$ by (5.3). In turn, again, there exists $r_0 > 0$ such that the equation $\mathbf{u} = tF(\mathbf{u})$ has no solution in $B_{r_0}(0) - \{0\}$, for all $t \in [0, 1]$. (Note that $\int_\Omega |\nabla u|^m \to 0$ as $r = \|\mathbf{u}\| \to 0$.)

Case (III). $\max\{a, d\} < m - 1$, $m \in (1, n)$ and $\max\{a + b, c + d\} > m^*$. Rewrite (5.2) in the form

$$\int_\Omega |\nabla w|^m \le C\int_\Omega w^{a'+1}z^{b'}, \qquad \int_\Omega |\nabla z|^m \le C\int_\Omega w^{c'}z^{d'+1},$$

where $u = w^{m/(e+m-1)}$, $v = z^{m/(f+m-1)}$

$$a' = \frac{m(a+e)}{e+m-1} - 1 > 0, \qquad b' = \frac{mb}{f+m-1} > 0,$$

and

$$c' = \frac{mc}{e+m-1} > 0, \qquad d' = \frac{m(d+f)}{f+m-1} - 1 > 0$$

since $e, f \ge 1$.

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We claim that one can choose suitable $e, f \geq 1$ so that

$$a' = \frac{m(a+e)}{e+m-1} - 1 < m-1, \qquad d' = \frac{m(d+f)}{f+m-1} - 1 < m-1, \tag{5.4}$$

$$\beta' := b'c' - \alpha'\delta' = b'c' - (m-1-a')(m-1-d') > 0 \tag{5.5}$$

and

$$\max\{a'+b', c'+d'\} \leq m^*. \tag{5.6}$$

Indeed, direct computations show that (5.4) and (5.5) are equivalent to $\max\{a, d\} < m-1$ and to $\beta > 0$, respectively, for any $e, f > 0$. To see (5.6), first fix $f_0 \geq 1$ such that

$$m + \frac{bm}{f_0+m-1} < m^* + 1, \qquad \frac{(d+f_0)m}{f_0+m-1} < m^* + 1, \tag{5.7}$$

which is equivalent to

$$\frac{b}{f_0+m-1} < \frac{m}{n-m}, \qquad \frac{d+f_0}{f_0+m-1} < \frac{n}{n-m}.$$

This is obviously possible by taking $f_0 \geq 1$ large. Next one simply chooses $e_0 \geq 1$ so that

$$\frac{(a+e_0)m}{e_0+m-1} + \frac{bm}{f_0+m-1} < m^* + 1, \qquad \frac{(d+f_0)m}{f_0+m-1} + \frac{cm}{e_0+m-1} < m^* + 1,$$

which is (5.6) with e_0 and f_0, and is again possible by taking e_0 large since

$$\frac{(a+e)m}{e+m-1} \to m, \qquad \frac{cm}{e+m-1} \to 0$$

as $e \to \infty$, in view of (5.7). Now one readily applies the arguments of Case (II) to the pair of $(w, z) > 0$, with the positive exponents a', b', c', d' satisfying (5.4)-(5.6), to draw the same conclusion as in Case (II). (Note that $\|(w, z)\| \to 0$ as $r = \|\mathbf{u}\| \to 0$.)

Case (IV). $\max\{a, d\} < m-1$ and $m \geq n$. Plainly, the arguments of Case (II) apply (with slight modifications) since $m^* = \infty$.

Therefore, Step 2 remains valid under the assumption $\beta > 0$ and the proof of Theorem 1.1 is complete. □

The proofs of subsequent Theorems 1.2–1.4 are then essentially the same, by using various estimates developed in Section 4, and are thus left to the reader.

Proof of Theorem 1.2. The proof is essentially the same as that of Theorem 1.1, since again Theorem 4.1 applies under the assumptions of Theorem 1.2. □

Proof of Theorem 1.3. The proof is essentially the same as that of Theorem 1.1, except we use the estimates Corollary 4.1 here. □

Proof of Theorem 1.4. The proof is essentially the same as that of Theorem 1.1, except we use the estimates Theorem 4.2 this time. □

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