NON-EXISTENCE OF A CERTAIN PROJECTIVE PLANE

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1

The question whether any projective plane of order ten exists or not, is an unsolved problem that has attracted some interest (see, for instance, [2]). A method, by which a plane might have been discovered, was suggested to me by a theorem in [1]: 'If the order of a plane is greater than 10, a six-arc is not complete'. Elementary arguments do not, it appears, exclude the possibility of a complete six-arc in a plane of order ten: but they do show that such a figure would be of an extreme type, and that the whole plane would fit round it in a particular way. The limitation, in fact, is so severe that it becomes feasible to consider, for a good many of the incidences in the plane, all the alternative arrangements that seem possible. With the help of the Elliott 4130 computer of the University of Leicester, I have carried out an exhaustive search, and discovered that it is impossible to build up a projective plane by this method. So I can assert:

THEOREM. *A projective plane of order ten, having one or more of its six-arcs complete, does not exist.*

2

Supposing if possible that there is a plane of order 10, and a complete 6-arc in that plane, let us work out some properties. These are in fact particular cases of theorems to be found in [1], but it will not take long to deduce them from the definitions. To say that points A, B, C, D, E, F make up a *six-arc*, means simply that no three of them are in line. The 111 lines of the plane then fall into three classes: there are 15 *secants*, joining points of the arc in pairs; 36 *tangents*, each through just one point of the arc (36, because the 11 lines through A include five secants); and 60 *exterior lines* with no point of the arc on them.

To say that the arc is *complete*, means that every point of the plane lies on at least one secant. Consider, this being so, the 11 points on any given tangent at A: since five of the 15 secants go through A, each of the other points has just one secant through it. But the intersection of BC and DE must be joined to A by some line; as this is not a tangent, it can only be AF. In this way we see that *there is no point of the plane through which pass just two secants*. The 105 points not belonging to the arc, we conclude, fall into two classes: there are 15 *interior points* at which the secants meet in threes, and 90 *exterior points*, each on just one secant. There must be one interior point at which AB, CD, EF meet; let us call this point G, and so on as follows:

$AB, CD, EF; G.$	$AB, CE, DF; H.$	$AB, CF, DE; J.$
$AC, BD, EF; K.$	$AC, BE, DF; L.$	$AC, BF, DE; M.$
$AD, BC, EF; N.$	$AD, BE, CF; O.$	$AD, BF, CE; P.$
$AE, BC, DF; Q.$	$AE, BD, CF; R.$	$AE, BF, CD; S.$
$AF, BC, DE; T.$	$AF, BD, CE; U.$	$AF, BE, CD; V.$

Now consider the 11 points on any given exterior line: each of the 15 secants meets the line somewhere, and each of the points lies either on one secant or on three. The only possibility is, that two of the points are interior and nine exterior. We find, as we should expect, that just 60 pairs of interior points are not joined by secants: and so each of the 60 exterior lines is recognisable as the join of one of these pairs.

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On the secant AB we have the interior points G, H, J, and six exterior points. Any one of these latter has through it, besides AB, four tangents joining it to C, D, E, F, and six exterior lines each joining it to two of $K, L, \ldots, V$. Tangents being harder to identify, let us concentrate on the set of six exterior lines, and call it 'a star on AB'. I began the search by listing all the possible stars on AB (that is, the ways of arranging $K, L, \ldots, V$ in six disjoint pairs, no pair joined by a secant): there are 344 of these, and I gave them reference numbers, which later appeared as items in the lists output by the computer.

Now suppose, for instance, that one of the stars on AB in the hypothetical figure is

$$\{KO, LN, MQ, PT, RV, SU\},$$

and that we interchange the symbols E and F attached to points of the arc. Then the symbols for the interior points will be subjected to the permutation

$$(LM)(OP)(QT)(RV)(SU),$$

and the star will become

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$$\{KP, LT, MN, OQ, RV, SU\}.$$

Let us say that two of the 'possible stars' are 'equivalent' when a permutation of $ABCDEF$ transforms one into the other, as (EF) does in the case just considered. This relation of equivalence (which in fact arranges all the possible stars in 15 equivalence classes) was useful in shortening the search. When I had made sure that all the possible cases in which

$$\{KO, LN, MQ, PT, RV, SU\}$$

appeared as a star were eliminated, I could disregard any later cases that included an equivalent star.

Let us say that two possible stars are 'compatible' when they have in common no exterior line (if on the same secant), or not more than one (if on different secants). Obviously, if the projective plane with the complete six-arc did exist, its 90 stars would be mutually compatible. After trying various possibilities, I thought that, although the problem of finding six compatible stars on AB had perhaps 200,000 solutions, that of finding 18 on AB, CD, EF would have fewer (about 7000, as it turned out), and might be within the reach of the computer.

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So I wrote a programme, the first part of which packed into the store information about which lines belonged to various possible stars. The second part stored information about the compatibility of possible stars on AB, CD, EF: by packing 16 or 20 bits into a word, I could fit all the information required into an array of 14,928 words.

The third part began by reading in the reference numbers of two stars on AB: so I could use as data a list of pairs of stars, which need not include any pair which was equivalent (as explained above) to a pair that remained on the list. The programme then tried all possible ways of building up a set of compatible stars, the two given stars being included. When the number of stars in the set reached 14 for the first time (after a pair of data had been read), the reference numbers of these 14 were output: I could get as far as this by hand, and actually did so for five pairs of data. These five results agreed with what afterwards came out of the machine; and this was evidence that the programme was not leaving out any possibilities that should have been included. The rest of the output consisted of all the sets of 18 compatible stars that were found.

The list of data amounted to 202 pairs of reference numbers; the total running time was about an hour and a half; and the output was 1043 sets of 18 compatible stars. Simple equivalence arguments showed that many of

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these 1043 could be disregarded: and a rather less obvious procedure of the same kind was surprisingly effective. This was to note, given a set of 18 stars, the equivalence classes of the six stars on CD, say; to choose, from one of these classes, a suitable representative on AB; and to search for a set of 18 stars that included the chosen star, and had the other stars on AB belonging to the other five equivalence classes. If the result of this search was negative, it followed that the given stars on CD were not compatible with any set of 12 on AE and BF (or, it might be, on AF and BE), and were to be disregarded.

By such arguments, the possibilities were reduced from 1043 to 40 sets of 18 stars. For each of these, I tried by hand to go one stage further, and include the set in a set of 24 compatible stars on AB, CD, EF, AF. In every case, a careful search ended in failure; and this brought the whole investigation to a negative conclusion.

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Having so far provided very few assertions that the reader can verify, I should like to end the paper with an example (almost the only one that I reached) of 23 compatible stars. Namely, on AB we have

$$\begin{array}{llllll@{\qquad}llllll}
KO & LT & MU & NS & PQ & RV, & KP & LN & MO & QV & RT & SU,\\
KQ & LS & MR & NU & OT & PV, & KS & LR & MQ & NV & OU & PT,\\
KT & LP & MV & NR & OS & QU, & KV & LU & MN & OQ & PR & ST;
\end{array}$$

on CD we have

$$\begin{array}{llllll@{\qquad}llllll}
HK & JN & LP & MU & OQ & RT, & HM & JQ & KP & LU & NR & OT,\\
HN & JL & KO & MR & PT & QU, & HO & JU & KT & LN & MQ & PR,\\
HR & JP & KQ & LT & MN & OU, & HT & JK & LR & MO & NU & PQ;
\end{array}$$

on EF we have

$$\begin{array}{llllll@{\qquad}llllll}
HM & JU & LS & OQ & PT & RV, & HO & JQ & LR & MU & PV & ST,\\
HR & JV & LP & MQ & OT & SU, & HS & JL & MV & OU & PQ & RT,\\
HT & JP & LU & MR & OS & QV, & HV & JS & LT & MO & PR & QU:
\end{array}$$

but all we can manage on AF is

$$\begin{array}{llllll@{\qquad}llllll}
GL & HS & JP & KO & MQ & NR, & GM & HR & JL & KP & NS & OQ,\\
GO & HK & JS & LN & MR & PQ, & GP & HM & JN & KQ & LR & OS,\\
GR & HN & JQ & KS & LP & MO. & & & & & &
\end{array}$$

In fact, this leaves HO and PR, which already meet on CD, to go through the sixth exterior point on AF.

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References

[1] G. E. Martin, 'On arcs in a finite projective plane', *Canadian J. Math.* **19 (1967), 376–393.**
[2] E. T. Parker, 'Orthogonal latin squares', *Proc. Nat. Acad. Sci. U.S.A.* **45 (1959), 859–862.**

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