THE DIVISORS OF A QUADRATIC POLYNOMIAL

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1. Let $f(n) = an^2 + bn + c$ be an irreducible quadratic polynomial with integer coefficients, and let D denote the discriminant $b^2 - 4ac$ of $f(n)$. We shall assume that $(D, k) = 1$, and that for all positive integers n, $f(n)$ is positive and coprime with k, where k is a fixed integer greater than 1.

We denote by $d(m\,; h)$ the number of positive divisors d of a positive integer m which satisfy $d \equiv h \pmod{k}$. The object of this paper is to prove the following result:

THEOREM 1. *If* $(h, k) = 1$, *then*

$$\sum_{n=1}^{x} d(f(n)\,; h) = A_1 x \log x + O(x \log\log x),$$

where x is a large positive integer.

Throughout this paper $A_1, A_2, A_3, \ldots$ denote positive constants, and they and the constants implied by the O-notation depend at most on k and the coefficients of f, and in some cases h.

The method used to prove this theorem may also be used to show that

$$\sum_{n=1}^{x} d(f(n)) = A_2 x \log x + O(x \log\log x), \tag{1}$$

where $d(m)$ denotes the number of positive divisors of a positive integer m. This latter result is mentioned in Erdös [**1**] as an unpublished result of Bellman and Shapiro.

The problem of proving a result analogous to (1), or the theorem, for irreducible polynomials $g(n)$ of degree greater than 2, appears to be very difficult. An important step in this direction is a paper due to Erdös [**1**] in which he proves that

$$B_1 x \log x < \sum_{n=1}^{x} d(g(n)) < B_2 x \log x,$$

where B_1 and B_2 are positive constants depending only on the coefficients and degree of g.

We shall give an elementary proof of Theorem 1 which will depend ultimately on estimating a multiple sum involving the Jacobi symbol $(a^2k^2D \mid t)$. It will be convenient to assume that $(a, 2k) = 1$ and that $(2, D) = 1$, although these conditions are not at all essential; they merely simplify the notation. The proof of the theorem remains valid even if $2^{2\tau} \parallel D$† for some positive integer τ, and a and k are even, provided that we replace D by $2^{-2\tau}D$ and a by $2^{-\gamma}a$, where $2^{\gamma} \parallel a$; we shall assume this remark when deducing our last theorem.

For certain polynomials $f(n)$ we may deduce from Theorem 1 a result, analogous to (1), for the function $r(f(n))$, where $r(m)$ denotes the number of representations of a positive integer m as the sum of two integer squares. We shall assume for this result that $f(n)$ is odd for all positive integers n and congruent to 1 (mod 4) for at least some positive integers n; furthermore we shall assume that $D = -\mu^2$, where μ is a positive integer. Then we have

† The notation $2^{2\tau} \parallel D$ is used when $2^{2\tau}$ is the highest power of 2 dividing D.

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THEOREM 2.

$$\sum_{n=1}^{x} r(f(n)) = A_3 x \log x + O(x \log \log x).$$

If $f(n) \equiv 3 \pmod 4$, then it is well known that $r(f(n)) = 0$; hence if $f(n) \equiv 3 \pmod 4$ for all positive integers n, then

$$\sum_{n=1}^{x} r(f(n)) = 0.$$

As an illustration of the results which may be obtained from Theorems 1 and 2, we shall consider the polynomial $f(n) = n^2+1$, and prove

THEOREM 3.

$$\sum_{n=1}^{x} r(n^2+1) = \frac{8}{\pi} x \log x + O(x \log \log x).$$

The error terms obtained in Theorems 1, 2 and 3 are certainly not the best possible. In fact, by the method of this paper one may show that the error terms are $O(xL_M(x))$, where $L_i(x)$ is defined by

$$L_1(x) = \log x, \quad L_i(x) = \log(L_{i-1}(x)) \quad \text{for} \quad i \geqq 2,$$

and where M is a positive integer independent of x; we shall indicate how this may be done in § 2.

I should like to thank Professor R. A. Rankin for his valuable advice during the preparation of this paper, and Dr H. Halberstam and Dr K. F. Roth for their helpful comments on my treatment of the related problem of obtaining the estimate (1).

2. We shall consider first the *large* divisors of $f(n)$; by a large divisor (corresponding to a given x) we mean a divisor greater than X, where X is defined to be the least positive integer such that

$$f(n) \leqq X^2 \quad \text{for} \quad 1 \leqq n \leqq x. \tag{2}$$

Clearly there exist positive constants C_1 and C_2 such that

$$C_1 x < X \leqq C_2 x. \tag{3}$$

If $d_X(m; h)$ denotes the number of positive divisors d of a positive integer m which satisfy $d \equiv h \pmod k$ and $d \leqq X$, then we may write

$$d(f(n); h) = d_X(f(n); h) + \sum_{\substack{d \mid f(n) \\ d > X \\ d \equiv h \pmod k}} 1. \tag{4}$$

The sum on the right contains what we have called the large divisors of $f(n)$. It may be empty (which is certainly the case if $f(n) \leqq X$). If it is not empty, then consider a typical large divisor d of $f(n)$ giving rise to one term of this sum. We have that $f(n) = d\delta$, where by (2)

$$\delta = f(n)/d < X^2/X = X \quad \text{and} \quad d\delta \equiv h\delta \equiv f(n) \pmod k. \tag{5}$$

We define h_1 by the congruence $hh_1 \equiv 1 \pmod k$; since $(h, k) = 1$, h_1 is unique modulo k and

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$(h_1, k) = 1$. The congruence in (5) may now be written in the form $\delta \equiv h_1 f(n) \pmod k$; we observe that, since $(f(n), k) = 1$, $(\delta, k) = 1$. We now see that to every large divisor d of $f(n)$, with $d \equiv h \pmod k$, there corresponds a unique divisor δ with $\delta < X$ and $\delta \equiv h_1 f(n) \pmod k$. The correspondence is not one-one, for clearly it is possible for both δ and $f(n)/\delta$ to be less than X. However we may rewrite (4) in the form

$$d(f(n)\,;h) = d_X(f(n)\,;h) + d_X(f(n)\,;h_1 f(n)) - \sum_{\substack{\delta \mid f(n) \\ f(n)/X \leqq \delta \leqq X \\ \delta \equiv h_1 f(n) \pmod k}} 1,$$

so that

$$\sum_{n=1}^{x} d(f(n)\,;h) = \sum_{n=1}^{x} \{d_X(f(n)\,;\ h) + d_X(f(n)\,;\ h_1 f(n))\} - \Delta, \tag{6}$$

where

$$\Delta = \sum_{n=1}^{x} \sum_{\substack{\delta \mid f(n) \\ f(n)/X \leqq \delta \leqq X \\ \delta \equiv h_1 f(n) \pmod k}} 1. \tag{7}$$

We observe that the expression on the right of (6) does not contain any large divisors.

Let $y = [x/\log x]$; then there exists a positive constant C_3 such that

$$f(n) > C_3 y^2 \quad \text{for} \quad y < n \leqq x.$$

From (7) it follows that

$$\begin{aligned} 0 \leqq \Delta &\leqq \sum_{n=1}^{y} d_X(f(n)\,;h_1 f(n)) + \sum_{n=y+1}^{x} \sum_{\substack{\delta \mid f(n) \\ Y < \delta \leqq X \\ \delta \equiv h_1 f(n) \pmod k}} 1 \\ &\leqq \sum_{n=1}^{y} d_X(f(n)\,;h_1 f(n)) + \sum_{n=1}^{x} \{d_X(f(n)\,;h_1 f(n)) - d_Y(f(n)\,;h_1 f(n))\}, \end{aligned} \tag{8}$$

where $Y = [C_3 y^2 X^{-1}]$.

[We may improve the upper bound for Δ by splitting up the sum on the right of (7) into more than two parts in the following way. Write $y_0 = 0$, $y_M = x$ and $y_m = [x/L_m(x)]$ for $1 \leqq m \leqq M-1$, where M and $L_m(x)$ are defined at the end of § 1. Then put $Y_0 = 0$, and $Y_m = [C_3^{(m)} y_m^2 X^{-1}]$ for $1 \leqq m \leqq M-1$, where $C_3^{(m)}$ is a positive constant such that $f(n) > C_3^{(m)} y_m^2$ for $n > y_m$. Then Δ satisfies

$$0 \leqq \Delta \leqq \sum_{m=1}^{M} \sum_{n=y_{m-1}+1}^{y_m} \sum_{\substack{\delta \mid f(n) \\ Y_{m-1} < \delta \leqq X \\ \delta \equiv h_1 f(n) \pmod k}} 1 \quad \leqq \sum_{m=1}^{M} \sum_{n=1}^{y_m} \{d_X(f(n);\, h_1 f(n)) - d_{Y_{m-1}}(f(n);\, h_1 f(n))\}.$$

Using this estimate for Δ instead of (8), we can obtain the improvement of the error terms of Theorems 1, 2 and 3 mentioned in § 1.]

We now put $n = mk + l$, where $0 \leqq l < k$. Then we may regard $f(n) = f(mk+l)$ as a polynomial in m with coefficients depending on l, k and the coefficients of $f(n)$, so that we may write $f(n) = F_l(m)$, say. We observe that the discriminant of $F_l(m)$ is $k^2 D$, and that $F_l(m) \equiv f(l) \pmod k$. We now have that

$$\sum_{n=1}^{x} d_X(f(n)\,;h_1 f(n)) = \sum_{l=0}^{k-1} \sum_{m} d_X(F_l(m)\,;\ h_1 f(l)), \tag{9}$$

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where in the summation over m, m runs through the integers of the interval $0 \leqq m \leqq (x-l)/k$ and where $m \neq 0$ if $l = 0$; there are corresponding expressions for the other sums of (6) and (8). Hence in order to find an estimate for the right side of (6), our main task must be to consider sums of the type

$$\sum_{m=1}^{z} d_X(F_l(m)\,;\,h_2), \tag{10}$$

where $(h_2, k) = 1$.

3. This section contains some definitions and lemmas which will be used in estimating the sum (10).

Let $\rho(q)$ denote the number of solutions in m of the congruence

$$F_l(m) \equiv 0 \pmod q, \quad 1 \leqq m \leqq q.$$

Then, if p denotes a prime, σ any positive integer and C_4 a positive constant depending only on k and D, $\rho(q)$ has the following properties:

LEMMA 1. (i) $\rho(q_1 q_2) = \rho(q_1)\rho(q_2)$ *if* $(q_1, q_2) = 1$.

(ii) $\rho(p^\sigma) = \rho(p) \leqq 2$ *if* $p \nmid k^2 D$.

(iii) $\rho(p^\sigma) = \rho(p^{2\varepsilon+1})$ *if* $p^\varepsilon \| k^2 D$ *and* $\sigma > 2\varepsilon$.

(iv) $\rho(p^\sigma) \leqq C_4$ *always*.

The proofs of (i) and (ii) are straightforward, and (iv) follows immediately from (ii) and (iii); (iii) is due to Nagell [**2**, p. 349], and from his result it also follows that (iv) is true with $C_4 = 2(k^2 D)^2$.

In several places we shall need to consider separately from other possibilities the case when $4 \mid k$ and $D = -\mu^2$, where μ is a positive integer. We shall refer to this as *condition* I: we shall use μ only in this context.

We define $\chi(t)$ by

$$\chi(1) = 1, \quad \chi(t) = \begin{cases} (a^2 k^2 D \mid t), & t \text{ odd}, \\ 0, & t \text{ even}, \end{cases}$$

where $(a^2 k^2 D \mid t)$ is the Jacobi symbol. Then we have the following result:

LEMMA 2. *If M is the lowest common multiple of* 4 *and* $|akD|$,

$$\sum_{\substack{t=1 \\ t \equiv h_3 \pmod k}}^{M} \chi(t) = 0$$

except when condition I *holds.*

Proof. Put

$$g = \begin{cases} K(a^2k^2D) = K(D) & \text{if } D \equiv 1 \pmod 4, \\ 4K(a^2k^2D) = 4K(D) & \text{if } D \not\equiv 1 \pmod 4, \end{cases}$$

where $K(m)$ denotes the squarefree kernel of m. Then g is the leader of the Jacobi symbol $(a^2k^2D \mid t)$ [**3**, p. 129]. Hence

$$\chi(t) = \chi_g(t)\chi_0(t)$$

for all t, where χ_g is a character modulo g and χ_0 is the principal character modulo M; χ is therefore a character modulo M, since $g \mid M$.

If ψ runs over all characters modulo k, we have

$$\phi(k) \sum_{\substack{t=1 \\ t \equiv h_3 (\mathrm{mod}\ k)}}^{M} \chi(t) = \sum_{t=1}^{M} \chi(t) \sum_{\psi} \psi(t) \bar{\psi}(h_3) = \sum_{\psi} \bar{\psi}(h_3) \sum_{t=1}^{M} \chi(t)\psi(t),$$

where $\phi(k)$ is Euler's function. Now $\chi\psi$ is a character modulo M and hence the inner sum on the right will equal 0, so that the required result will follow, if we show that, for all characters ψ modulo k, $\chi\psi \neq \chi_0$ except when $D = -\mu^2$ and $4 \mid k$. But $\chi\psi = \chi_0$ implies that $\chi(t) = \chi_0(t)\bar{\psi}(t)$ for all t. It follows that $g \mid k$; since $(D, k) = 1$, this means that $| K(D) | = 1$ so that $D = -\mu^2$. (Since f is irreducible, $D \neq \mu^2$). Hence $D \not\equiv 1 \pmod 4$, which implies that $4 \mid g$, so that $4 \mid k$. Thus $\chi\psi = \chi_0$ only if condition I holds.

LEMMA 3.

$$\sum_{\substack{u \leqq U \\ (u,\, 2akD) = 1 \\ u \equiv h_4 (\mathrm{mod}\ k)}} 1 = A_4 U + O(1),$$

where $(h_4, k) = 1$.

Proof. We observe that the condition $(u, k) = 1$ is automatically satisfied since $u \equiv h_4 \pmod k$ and $(h_4, k) = 1$. In particular it follows that u is odd if k is even.

Put $\theta = 0$ or 1 according as k is even or odd. Then $\phi(2^\theta a \mid D \mid)$ of the integers

$$gk + h_4 \quad (0 \leqq g < 2^\theta a \mid D \mid)$$

are coprime with $2^\theta a \mid D \mid$. Thus

$$\sum_{\substack{u < 2^\theta a | D | k \\ (u,\, 2^\theta a D) = 1 \\ u \equiv h_4 (\mathrm{mod}\ k)}} 1 = \phi(2^\theta a \mid D \mid), \tag{11}$$

whence

$$\sum_{\substack{u \leqq U \\ (u,\, 2akD) = 1 \\ u \equiv h_4 (\mathrm{mod}\ k)}} 1 = \frac{\phi(2^\theta a \mid D \mid)}{2^\theta a \mid D \mid k} U + O(1).$$

4. The sum $\sum_{m=1}^{z} d_X(F_l(m); h_2)$ is the number of solutions in m and q of the congruence

$$F_l(m) \equiv 0 \pmod q, \quad 1 \leqq m \leqq z,\ 1 \leqq q \leqq X,\ q \equiv h_2 \pmod k.$$

Let $\rho_z(q)$ denote the number of solutions in m of

$$F_l(m) \equiv 0 \pmod q \quad (1 \leqq m \leqq z);$$

then $\rho_q(q) = \rho(q)$ and $\rho_z(q)$ satisfies

$$[z/q]\rho(q) \leqq \rho_z(q) \leqq ([z/q]+1)\rho(q).$$

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It follows that

$$\sum_{m=1}^{z} d_X(F_l(m)\,;\,h_2) = \sum_{\substack{q=1 \\ q\equiv h_2 (\mathrm{mod}\ k)}}^{X} \rho_z(q)$$

$$= z \sum_{\substack{q=1 \\ q\equiv h_2 (\mathrm{mod}\ k)}}^{X} \rho(q)/q + O\Bigg(\sum_{\substack{q=1 \\ q\equiv h_2 (\mathrm{mod}\ k)}}^{X} \rho(q) \Bigg). \tag{12}$$

In order to find an estimate for the right side of (12), we shall need to consider first the sum

$$\sum_{\substack{q=1 \\ q\equiv h_2 (\mathrm{mod}\ k)}}^{X} \rho(q).$$

Each integer q may be written as a product rs where $(s, 2akD) = 1$, and where each prime dividing r also divides $2akD$. Then, by Lemma 1 (i), we have that

$$\sum_{\substack{q=1 \\ q\equiv h_2 (\mathrm{mod}\ k)}}^{X} \rho(q) = \sum_{r\leqq X} \rho(r) \sum_{\substack{s \leqq X/r \\ rs\equiv h_2 (\mathrm{mod}\ k)}} \rho(s). \tag{13}$$

Since $(rs, k) = (h_2, k) = 1$, the condition $rs \equiv h_2 \,(\mathrm{mod}\ k)$ may be rewritten in the form $s \equiv r_1 h_2 \equiv h_5 \,(\mathrm{mod}\ k)$, say, where r_1 is an integer, unique modulo k, satisfying $rr_1 \equiv 1 \,(\mathrm{mod}\ k)$, and where h_5 depends on r and $(h_5, k) = 1$.

Consider now the inner sum on the right of (13). If $s = p_1^{\sigma_1} p_2^{\sigma_2} \ldots p_l^{\sigma_l}$ where $p_1, p_2, \ldots, p_l$ are distinct primes (not dividing $2akD$), we have, from Lemma 1 (i) and (ii), that

$$\rho(s) = \rho(p_1^{\sigma_1})\rho(p_2^{\sigma_2}) \ldots \rho(p_l^{\sigma_l}) = \rho(p_1)\rho(p_2) \ldots \rho(p_l).$$

Furthermore [3, p. 140] $\qquad \rho(p) = 1+(a^2k^2D|p),$

and hence

$$\rho(s) = \prod_{p\,|\,s} \{1+(a^2k^2D \mid p)\} = \sum_{\substack{t\,|\,s \\ t \text{ squarefree}}} \chi(t). \tag{14}$$

We observe that for the special case $D = -\mu^2$, if $(p, 2ak\mu) = 1$, then

$$(a^2k^2D \mid p) = (-a^2k^2\mu^2 \mid p) = (-1)^{\frac{1}{2}(p-1)}.$$

Thus, if $p \equiv 3 \,(\mathrm{mod}\ 4)$, $\rho(p) = 0$, and it follows that $\rho(s) = 0$ if $s \equiv 3 \,(\mathrm{mod}\ 4)$.

The inner sum on the right of (13) is given by

LEMMA 4.

$$\sum\nolimits_1 = \sum_{\substack{s\leqq Z \\ (s,\, 2akD)=1 \\ s\equiv h_5 \,(\mathrm{mod}\ k)}} \rho(s) = A_5 Z + O(Z^{2/3})$$

unless condition I *holds and* $h_5 \equiv 3 \,(\mathrm{mod}\ 4)$, *in which case* $\sum_1 = 0$.

[The exponent $\frac{2}{3}$ may be replaced by any number α satisfying $\frac{1}{2} < \alpha < 1$.]

Proof. We suppose first that condition I does not hold. By (14), and since the Möbius function $\mu(t)$ satisfies

$$|\,\mu(t)\,| = \sum_{v^2\,|\,t} \mu(v),$$

$$\sum_1 = \sum_{\substack{s \leqq Z \\ (s,\, 2akD)=1 \\ s \equiv h_5 \pmod{k}}} \sum_{\substack{t \mid s \\ t \text{ squarefree}}} \chi(t) = \sum_{t \leqq Z} \chi(t) \mid \mu(t) \mid \sum_{\substack{u \leqq Z/t \\ (u,\, 2akD)=1 \\ tu \equiv h_5 \pmod{k}}} 1 = \sum_{\substack{v \leqq \sqrt{Z} \\ (v,\, 2akD)=1}} \mu(v) \sum_{w \leqq Z/v^2} \chi(w) \sum_u 1, \quad (15)$$

where $\sum_u$ stands for the summation over all positive integers u satisfying the conditions $u \leqq Z/v^2 w$, $(u, 2akD) = 1$ and $uv^2 w \equiv h_5 \pmod{k}$. We split the sum over w into two parts so that, with the above meaning for $\sum_u$,

$$\sum_{w \leqq Z/v^2} \chi(w) \sum_u 1 = \sum_{w \leqq (Z/v^2)^{1/3}} \chi(w) \sum_u 1 + \sum_{(Z/v^2)^{1/3} < w \leqq Z/v^2} \chi(w) \sum_u 1 = \sum_2 + \sum_3, \tag{16}$$

say.

In the sum $\sum_2$, we may suppose that $(w, k) = 1$, since otherwise $\chi(w) = 0$ and the congruence $uv^2 w \equiv h_5 \pmod{k}$, with $(h_5, k) = 1$, cannot be satisfied. Then the congruence $uv^2 w \equiv h_5 \pmod{k}$ is equivalent to a congruence of the form $u \equiv h_6 \pmod{k}$, where $(h_6, k) = 1$ and h_6 is unique modulo k for fixed v and w. From Lemma 3 and since

$$\left| \sum_{w=W_1}^{W_2} \chi(w) \right| = O(1)$$

for any positive integers W_1 and W_2 satisfying $W_1 < W_2$, we obtain [3, p. 240]

$$\begin{aligned}
\sum_2 &= A_4 \frac{Z}{v^2} \sum_{w \leqq (Z/v^2)^{1/3}} \frac{\chi(w)}{w} + O\Big(\sum_{w \leqq (Z/v^2)^{1/3}} \mid \chi(w) \mid \Big) \\
&= A_4 \frac{Z}{v^2} \sum_{w=1}^{\infty} \frac{\chi(w)}{w} + O\left(\frac{Z}{v^2} \left| \sum_{w > (Z/v^2)^{1/3}} \frac{\chi(w)}{w} \right| \right) + O((Z/v^2)^{1/3}) \\
&= A_4 L(\chi) Z/v^2 + O((Z/v^2)^{2/3}),
\end{aligned} \tag{17}$$

where $L(\chi) = \sum_{w=1}^{\infty} \chi(w)/w \neq 0$, the series being convergent.

In order to estimate $\sum_3$, we change the order of summation so that

$$\sum_3 = \sum_{\substack{u < (Z/v^2)^{2/3} \\ (u,\, 2akD)=1}} \sum_{\substack{(Z/v^2)^{1/3} < w \leqq Z/v^2 u \\ uv^2 w \equiv h_5 \pmod{k}}} \chi(w).$$

The congruence $uv^2 w \equiv h_5 \pmod{k}$ is equivalent to one of the form $w \equiv h_7 \pmod{k}$, where $(h_7, k) = 1$ and h_7 is unique modulo k for fixed u and v. Hence, by Lemma 2,

$$\sum_3 = O((Z/v^2)^{2/3}). \tag{18}$$

From equations (15) to (18) we obtain

$$\begin{aligned}
\sum_1 &= \sum_{\substack{v \leqq \sqrt{Z} \\ (v,\, 2akD)=1}} \mu(v) \{ A_4 L(\chi) Z/v^2 + O((Z/v^2)^{2/3}) \} \\
&= A_4 L(\chi) Z \sum_{\substack{v=1 \\ (v,\, 2akD)=1}}^{\infty} \frac{\mu(v)}{v^2} + O\left(Z \sum_{v > \sqrt{Z}} \frac{\mid \mu(v) \mid}{v^2} + Z^{2/3} \sum_{v=1}^{\infty} \frac{\mid \mu(v) \mid}{v^{4/3}} \right).
\end{aligned} \tag{19}$$

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The error term on the right is $O(Z^{2/3})$; the sum in the main term on the right is given by

$$\sum_{\substack{v=1\\(v,\,2akD)=1}}^{\infty}\frac{\mu(v)}{v^2}=\prod_{p\nmid 2akD}\{1+\mu(p)p^{-2}+\mu(p^2)p^{-4}+\dots\}$$
$$=\prod_{p\nmid 2akD}(1-p^{-2})=\{\zeta(2)\prod_{p\mid 2akD}(1-p^{-2})\}^{-1}$$
$$=\frac{6}{\pi^2}\prod_{p\mid 2akD}(1-p^{-2})^{-1}=A_6,$$

say. This, together with (19), gives the result of the Lemma, with $A_5=A_4A_6L(\chi)$, provided condition I does not hold.

In order to complete the proof of the lemma, we have to consider the case which we have so far omitted; thus we now suppose that $D=-\mu^2$ and $k=2^{\nu}k_1$, where $(k_1,2)=1$ and $\nu\geqq 2$. We recall that, in the paragraph before Lemma 4, we observed that, when $D=-\mu^2$, $\rho(s)=0$ if $s\equiv 3 \pmod 4$ and $(s,2ak\mu)=1$. Since $4\mid k$, it follows that, if $h_5\equiv 3 \pmod 4$,

$$\sum\nolimits_1=\sum_{\substack{s\leqq Z\\(s,\,2ak\mu)=1\\ s\equiv h_5 \pmod k}}\rho(s)=0. \tag{20}$$

If $h_5\equiv 1 \pmod 4$, then $h_5+2k_1\equiv 3 \pmod 4$, so that (20) holds with h_5 replaced by h_5+2k_1. Hence

$$\sum\nolimits_1=\sum_{\substack{s\leqq Z\\(s,\,2ak\mu)=1\\ s\equiv h_5 \pmod k}}\rho(s)=\sum_{\substack{s\leqq Z\\(s,\,2ak\mu)=1\\ s\equiv h_5 \pmod k}}\rho(s)+\sum_{\substack{s\leqq Z\\(s,\,2ak\mu)=1\\ s\equiv h_5+2k_1 \pmod k}}\rho(s)=\sum\nolimits_4+\sum\nolimits_5,$$

say. The method of the first part of Lemma 4 can now be applied to the sum $\sum_4+\sum_5$ provided that we use the following fact instead of Lemma 2. (Lemma 2 cannot be used in this case because condition I holds). If $(h_8,k)=1$, $h_8\equiv 1 \pmod 4$ and τ is any positive odd integer, then

$$\sum_{\substack{w<a\mu k\\ w\equiv h_8 \pmod k}}\chi(w)+\sum_{\substack{w<a\mu k\\ w\equiv h_8+2k_1\tau \pmod k}}\chi(w)=\sum_{\substack{w<a\mu k\\(w,\,a\mu)=1\\ w\equiv h_8 \pmod k}}1-\sum_{\substack{w<a\mu k\\(w,\,a\mu)=1\\ w\equiv h_8+2k_1\tau \pmod k}}1=\phi(a\mu)-\phi(a\mu)=0$$

by (11). The constant A_5 obtained for this case equals $2A_4A_6L(\chi)$.

5. We now complete the evaluation of the right side of (13). If condition I does not hold, then from (13) and Lemma 4 we obtain

$$\sum_{\substack{q=1\\ q\equiv h_2 \pmod k}}^{X}\rho(q)=A_5X\sum_{r\leqq X}\rho(r)r^{-1}+O\Big(X^{2/3}\sum_{r\leqq X}\rho(r)r^{-2/3}\Big), \tag{21}$$

where r runs through the integers divisible only by primes dividing $2aD$ and satisfying $(r,k)=1$. Similarly if $4\mid k$ and $D=-\mu^2$,

$$\sum_{\substack{q=1\\ q\equiv h_2 \pmod k}}^{X}\rho(q)=A_5X\sum_{\substack{r\leqq X\\ r\equiv h_2 \pmod 4}}\rho(r)r^{-1}+O\Big(X^{2/3}\sum_{\substack{r\leqq X\\ r\equiv h_2 \pmod 4}}\rho(r)r^{-2/3}\Big); \tag{22}$$

the extra condition $r\equiv h_2 \pmod 4$ arises from the fact that the inner sum on the right of (13) is non-zero only if $s\equiv 1 \pmod 4$.

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Let $2^{\varepsilon_0}a \mid D \mid = 2^{\varepsilon_0}p_1^{\varepsilon_1}\dots p_i^{\varepsilon_i}p_{i+1}^{\varepsilon_{i+1}}\dots p_j^{\varepsilon_j}$, where the $p_\nu(1 \leqq \nu \leqq j)$ are distinct odd primes, $p_\nu \mid a$ but $p_\nu \not\mid D$ for $1 \leqq \nu \leqq i$, $p_\nu \mid D$ for $i < \nu \leqq j$, $\varepsilon_\nu(1 \leqq \nu \leqq j)$ are positive integers and $\varepsilon_0 = 0$ or 1 according as k is even or odd. Each integer r may be written in the form $r = 2^{\varepsilon_0\sigma_0}p_1^{\sigma_1}\dots p_j^{\sigma_j}$, where the $\sigma_\nu(0 \leqq \nu \leqq j)$ are non-negative integers. Then by Lemma 1 (i) and (iv),

$$\rho(r) = \rho(2^{\varepsilon_0\sigma_0})\rho(p_1^{\sigma_1})\dots\rho(p_j^{\sigma_j}) \leqq (C_4)^{j+1}. \tag{23}$$

We put $p_0 = 2$ and define $\eta_\nu(0 \leqq \nu \leqq j)$ by

$$p_\nu^{\eta_\nu} \leqq X < p_\nu^{\eta_\nu+1}. \tag{24}$$

Then we have that

$$\sum_{\substack{r \leqq X \\ (r,k)=1}} \rho(r)r^{-2/3} \leqq (C_4)^{j+1}\prod_{\nu=0}^{j}\left(\sum_{\sigma_\nu=0}^{\eta_\nu} p_\nu^{-(2/3)\sigma_\nu}\right) = O(1),$$

so that the error terms of (21) and (22) are $O(X^{2/3})$.

In order to estimate the main terms of (21) and (22), we have

LEMMA 5.

$$\text{(i) } \sum\nolimits_6 = \sum_{\substack{r \leqq X \\ (r,k)=1}} \rho(r)r^{-1} = A_7 + O(X^{-1}(\log X)^{j+1}).$$

$$\text{(ii) } \sum\nolimits_7 = \sum_{\substack{r \leqq X \\ (r,k)=1}} \rho^*(r)r^{-1} = A_8 + O(X^{-1}(\log X)^{j}),$$

where $\rho^*(r) = \rho(r)\sin\dfrac{r\pi}{2}$, *and k is even.*

Proof. (i) Suppose first that k is odd. Then

$$\sum\nolimits_6 = \prod_{\nu=0}^{j}\left(\sum_{\sigma_\nu=0}^{\eta_\nu}\rho(p_\nu^{\sigma_\nu})p_\nu^{-\sigma_\nu}\right) + O\left(\sum_{X < r \leqq X'}\rho(r)r^{-1}\right), \tag{25}$$

where $X' = \prod_{\nu=0}^{j} p_\nu^{\eta_\nu}$. By (23) and (24) the error term on the right is

$$O\left((C_4)^{j+1}X^{-1}\prod_{\nu=0}^{j}\eta_\nu\right) = O(X^{-1}(\log X)^{j+1}).$$

Let
$$S_\nu = \sum_{\sigma_\nu=0}^{\eta_\nu}\rho(p_\nu^{\sigma_\nu})p_\nu^{-\sigma_\nu},$$

and suppose first that $0 \leqq \nu \leqq i$, so that $p_\nu \mid 2a$ but $p_\nu \not\mid D$. Then by Lemma 1 (ii),

$$S_\nu = 1 + \rho(p_\nu)\sum_{\sigma_\nu=1}^{\eta_\nu} p_\nu^{-\sigma_\nu} = 1 + \rho(p_\nu)/(p_\nu - 1) + O(X^{-1})$$

$$= E_\nu + O(X^{-1}),$$

say.

Suppose next that $i < \nu \leqq j$, so that $p_\nu \mid D$. Then by Lemma 1 (iii),

$$S_\nu = \sum_{\sigma_\nu=0}^{2\varepsilon_\nu} \rho(p_\nu^{\sigma_\nu})p_\nu^{-\sigma_\nu} + \rho(p_\nu^{2\varepsilon_\nu+1}) \sum_{\sigma_\nu=2\varepsilon_\nu+1}^{\eta_\nu} p_\nu^{-\sigma_\nu}$$

$$= \sum_{\sigma_\nu=0}^{2\varepsilon_\nu} \rho(p_\nu^{\sigma_\nu})p_\nu^{-\sigma_\nu} + \rho(p^{2\varepsilon_\nu+1})\{p_\nu^{2\varepsilon_\nu}(p_\nu-1)\}^{-1} + O(X^{-1})$$

$$= E_\nu + O(X^{-1}),$$

say. The result now follows from (25) with $A_7 = \prod_{\nu=0}^{j} E_\nu$, provided that k is odd.

If k is even, we omit the factor involving p_0 ($=2$) in the above, and we obtain the required result with $A_7 = \prod_{\nu=1}^{j} E_\nu$.

(ii) We are assuming that k is even, so that r is always odd. Then

$$\sum\nolimits_7 = \prod_{\nu=1}^{j} \left\{ \sum_{\sigma_\nu=0}^{\eta_\nu} \rho(p_\nu^{\sigma_\nu})p_\nu^{-\sigma_\nu} \sin\left(p_\nu^{\sigma_\nu}\frac{\pi}{2}\right)\right\} + O\left(\sum_{X<r\leqq X'} \rho(r)r^{-1} \left|\sin\frac{r\pi}{2}\right|\right).$$

The method used to prove (i) can now be applied to $\sum_7$, and this gives the required result with $A_8 = \prod_{\nu=1}^{j} E'_\nu$, where

$$E'_\nu = \begin{cases} E_\nu & \text{if} \quad p_\nu \equiv 1 \pmod 4, \\ 1-\rho(p_\nu)/(p_\nu+1) & \text{if} \quad p_\nu \equiv 3 \pmod 4 \quad \text{and} \quad 1 \leqq \nu \leqq i, \\ \sum_{\sigma_\nu=0}^{2\varepsilon_\nu} \rho(p_\nu^{\sigma_\nu})(-p_\nu)^{-\sigma_\nu} - \rho(p_\nu^{2\varepsilon_\nu+1})\{p_\nu^{2\varepsilon_\nu}(p_\nu+1)\}^{-1} & \text{if} \quad p_\nu \equiv 3 \pmod 4 \quad \text{and} \quad i < \nu \leqq j. \end{cases}$$

If condition I does not hold, we obtain, from (21) and Lemma 5 (i),

$$\sum_{\substack{q=1 \\ q\equiv h_2 \pmod k}}^{X} \rho(q) = A_5 A_7 X + O(X^{2/3}). \tag{26}$$

Suppose now that condition I does hold. Then, by Lemma 5 (i) and (ii),

$$\sum_{\substack{r\leqq X \\ (r,k)=1 \\ r\equiv h_2 \pmod 4}} \rho(r)r^{-1} = \tfrac{1}{2}\{\sum\nolimits_6 + (-1)^{\frac{1}{2}(h_2-1)}\sum\nolimits_7\}$$

$$= \tfrac{1}{2}\{A_7 + (-1)^{\frac{1}{2}(h_2-1)}A_8\} + O(X^{-1}(\log X)^j)$$

$$= A_9(h_2) + O(X^{-1}(\log X)^j),$$

say. Hence, if $4 \mid k$ and $D = -\mu^2$, we have by (22) that

$$\sum_{\substack{q=1 \\ q\equiv h_2 \pmod k}}^{X} \rho(q) = A_5 A_9(h_2) X + O(X^{2/3}). \tag{27}$$

B

6. In this section we shall complete the proof of Theorem 1. We suppose first that condition I does not hold and we deduce from (26) an estimate for the right side of (12). We write

$$T(q) = \sum_{\substack{u=1 \\ u \equiv h_2 \pmod k}}^{q} \rho(u), \quad T(0) = 0.$$

We have

$$\begin{aligned} \sum_{\substack{q=1 \\ q \equiv h_2 \pmod k}}^{X} \rho(q)/q &= \sum_{q=1}^{X} \{T(q) - T(q-1)\}/q = \sum_{q=1}^{X} T(q)\{q^{-1} - (q+1)^{-1}\} + T(X)/(X+1) \\ &= A_{10} \sum_{q=1}^{X} (q+1)^{-1} + O\left(\sum_{q=1}^{X} q^{-1/3}(q+1)^{-1}\right) + O(1) \\ &= A_{10} \log X + O(1), \end{aligned} \tag{28}$$

by (26), where $A_{10} = A_5 A_7$.

From (12), (26) and (28) we obtain

$$\sum_{m=1}^{z} d_X(F_l(m); h_2) = A_{10} z \log X + O(z+X), \tag{29}$$

and therefore (9) becomes

$$\begin{aligned} \sum_{n=1}^{x} d_X(f(n); h_1 f(n)) &= \sum_{l=0}^{k-1} A_{10}[(x-l)/k] \log X + O(x) \\ &= A_{10} x \log x + O(x) \end{aligned} \tag{30}$$

by (3). From (8), (9) and (29), the estimate of Δ is

$$\begin{aligned} \Delta &\leqq A_{10}\{y \log X + x \log X - x \log Y\} + O(y + Y + x) \\ &= O(x \log (X/Y)) = O(x \log \log x). \end{aligned}$$

Hence from (6) and (30) we obtain

$$\sum_{n=1}^{x} d(f(n); h) = 2A_{10} x \log x + O(x \log \log x),$$

which gives the result of the theorem with $A_1 = 2A_{10}$ provided condition I does not hold. More precisely the constant A_1 is given by

$$A_1 = \frac{12}{\pi^2 k} A_7 L(\chi) \prod_{\substack{p \mid 2aD \\ (p,k)=1}} \left(\frac{p}{p+1}\right) \prod_{p \mid k} \left(\frac{p^2}{p^2-1}\right),$$

where A_7 is the constant of Lemma 5 (i).

If condition I holds so that $4 \mid k$ and $D = -\mu^2$, we use (27) instead of (26), and the required result follows in a similar way with A_1 given by

$$A_1 = A_1(h) = A_5\left\{A_9(h) + k^{-1} \sum_{l=0}^{k-1} A_9(h_1 f(l))\right\}; \tag{31}$$

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we recall that h_1 satisfies $hh_1 \equiv 1 \pmod{k}$ and that $A_9(h_2)$ depends on the value of $h_2 \pmod{4}$. This completes the proof of Theorem 1.

7. In order to prove Theorem 2, we use a well known property of $r(m)$, and we apply Theorem 1 with $k = 4$ and $h = 1$ and 3. We assume for this theorem that $D = -\mu^2$.

It is well known [4, § 16.9] that

$$r(m) = 4\{d(m\,;1) - d(m\,;3)\};$$

hence by Theorem 1,

$$\sum_{n=1}^{x} r(f(n)) = 4\sum_{n=1}^{x}\{d(f(n)\,;1) - d(f(n)\,;3)\}$$

$$= 4\{A_1(1) - A_1(3)\}x\log x + O(x\log\log x),$$

where $A_1(1)$ and $A_1(3)$ are given by (31) with $k = 4$; this is the required result with $A_3 = 4\{A_1(1) - A_1(3)\}$. [If D is not of the form $-\mu^2$, $A_1(1) = A_1(3) = A_1$, and it follows that

$$\sum_{n=1}^{x} r(f(n)) = O(x\log\log x).]$$

We can find the value of A_3 as a product of several terms depending on a and μ. We have

$$A_1(1) - A_1(3) = A_5\{A_9(1) - A_9(3) + \tfrac{1}{4}\sum_{l=0}^{3}\{A_9(f(l)) - A_9(3f(l))\}\}.$$

Since $f(n)$ is always odd and $f(n) \equiv 1 \pmod 4$ for at least some integers n, there are two cases to consider: (i) $f(l) \equiv 1 \pmod 4$ for $l = 0, 1, 2, 3$; (ii) $f(l) \equiv 1 \pmod 4$ for exactly two of $l = 0, 1, 2, 3$ and $f(l) \equiv 3 \pmod 4$ for the remaining two of $l = 0, 1, 2, 3$. In case (i) we have

$$A_1(1) - A_1(3) = 2A_5\{A_9(1) - A_9(3)\},$$

and in case (ii)

$$A_1(1) - A_1(3) = A_5\{A_9(1) - A_9(3)\}.$$

From the definition of the constants $A_4, \ldots, A_9$ we have that

$$A_5\{A_9(1) - A_9(3)\} = \frac{6}{\pi^2}L(\chi)\prod_{p\,|\,2a\mu}\left(\frac{p}{p+1}\right)\sum_r \rho^*(r)r^{-1},$$

where the summation over r runs over all positive integers which are divisible only by odd primes dividing $a\mu$. Hence the constant A_3 of Theorem 2 is given by

$$A_3 = \begin{cases} \dfrac{48}{\pi^2}L(\chi)\displaystyle\prod_{p\,|\,2a\mu}\left(\frac{p}{p+1}\right)\sum_r \rho^*(r)r^{-1} \text{ in case (i)}, \\ \dfrac{24}{\pi^2}L(\chi)\displaystyle\prod_{p\,|\,2a\mu}\left(\frac{p}{p+1}\right)\sum_r \rho^*(r)r^{-1} \text{ in case (ii)}. \end{cases} \tag{32}$$

8. Our last result is concerned with the polynomial $f(n) = n^2+1$. As it stands this polynomial does not satisfy all the conditions of Theorem 2, for it is not always odd. However we may write

$$f(2m) = 4m^2+1 = f_1(m)$$

and
$$f(2m+1) = 2(2m^2+2m+1) = 2f_2(m);$$

then the discriminants of f_1 and f_2 are -16 and -4 respectively.

For all positive integers m, $f_1(m) \equiv 1 \pmod 4$ and $f_2(m) \equiv 1 \pmod 4$, and hence both f_1 and f_2 satisfy all the conditions of case (i) of Theorem 2. Thus, since for both these polynomials, $L(\chi) = \frac{1}{4}\pi$ and $\sum_r \rho^*(r)r^{-1} = 1$,

$$\sum_{m=1}^{y} r(f_1(m)) = \frac{8}{\pi} y \log y + O(y \log\log y)$$

and
$$\sum_{m=1}^{y} r(f_2(m)) = \frac{8}{\pi} y \log y + O(y \log\log y).$$

Since $r(2f_2(m)) = r(f_2(m))$, we obtain from above

$$\sum_{n=1}^{x} r(f(n)) = \sum_{m=1}^{[x/2]} r(f_1(m)) + \sum_{m=0}^{[(x-1)/2]} r(f_2(m))$$
$$= \frac{8}{\pi} x \log x + O(x \log\log x),$$

which is Theorem 3.

REFERENCES

1. P. Erdös, On the sum $\sum_{k=1}^{x} d(f(k))$, *J. London Math. Soc.*, **27** (1952), 7–15.

2. T. Nagell, Généralisation d'un théorème de Tchebycheff, *J. de Math.*, (8) **4** (1921), 343–356.

3. H. Hasse, *Vorlesungen über Zahlentheorie* (Springer-Verlag, Berlin/Göttingen/Heidelberg, 1950).

4. G. H. Hardy and E. M. Wright, *An introduction to the theory of numbers* (3rd Edition, Oxford, 1954).

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