J. Aust. Math. Soc. **104** (2018), 338–357
doi:10.1017/S1446788717000210

PARABOLIC CLASSICAL CURVATURE FLOWS

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(Received 8 August 2016; accepted 28 June 2017; first published online 30 October 2017)

Communicated by A. Hassell

Abstract

We consider classical curvature flows: 1-parameter families of convex embeddings of the 2-sphere into Euclidean 3-space, which evolve by an arbitrary (nonhomogeneous) function of the radii of curvature (RoC). We determine conditions for parabolic flows that ensure the boundedness of various geometric quantities and investigate some examples. As a new tool, we introduce the RoC diagram of a surface and its hyperbolic or anti-de Sitter metric. The relationship between the RoC diagram and the properties of Weingarten surfaces is also discussed.

2010 *Mathematics subject classification*: primary 35K40; secondary 53A05.

Keywords and phrases: oriented lines, radii of curvature, parabolic flow, Weingarten, curvature flow.

1. Introduction and results

The evolution of a submanifold embedded in Euclidean space by a specified function of the eigenvalues of its second fundamental form has been studied for decades. In particular, there are many longtime existence and convergence results for flows by homogeneous symmetric functions of the principal curvatures, such as mean curvature, inverse mean curvature and Gauss curvature flows [1, 5, 9, 13, 17, 20–23].

The main concern of this paper is the qualitative study of general nonhomogeneous curvature flows for closed convex surfaces in 3-dimensional Euclidean space—the domain of classical surface theory. Such equations arise in many physical situations, for example, when considering the erosion of a pebble under various types of abrasion [2, 6, 7].

Define a *classical curvature flow* to be a 1-parameter family of smooth convex embeddings $\vec{X} : S^2 \times [0, t_1) \to \mathbb{R}^3$ such that

$$\frac{\partial \vec{X}}{\partial t}^{\perp} = -\mathcal{K}(\psi, |\sigma|)\, \vec{n},$$
$$\vec{X}(S^2, 0) = S_0,$$

where $\mathcal{K}$ is a given function of $\psi = \frac{1}{2}(r_1 + r_2)$ and $|\sigma| = \frac{1}{2}(r_1 - r_2)$, $r_1 \geq r_2$ being the radii of curvature (RoC) of $S = \vec{X}(S^2)$, $\vec{n}$ is the unit normal to S and S_0 is an initial convex surface.

The reasons for this particular combination of RoC will become apparent later. A flow is said to be *contracting* if $\mathcal{K} \geq 0$ and *expanding* if $\mathcal{K} \leq 0$.

A stationary solution of a classical curvature flow is a *Weingarten surface*, satisfying the equation $\mathcal{K}(\psi, |\sigma|) = 0$. While nonround Weingarten spheres exist (for example Hopf spheres [15]), many results state that the only Weingarten spheres satisfying some particular relationship are round [3, 4, 14, 16, 24].

In general, a Weingarten relation is a second-order fully-nonlinear partial differential equation for the support function of the surface. Thus ellipticity can be defined, as can the definition of parabolicity for flows [19].

This paper considers how the RoC, or rather their sum and difference ψ and $|\sigma|$, evolve under a parabolic classical curvature flow. The pair $(\psi, |\sigma|)$ take values in the upper half-plane, which we call the *RoC space*, and one can visualize the dynamics of the flow by considering the image of this map, denoted $f_t : S^2 \to \mathbb{R}^2_+$ and referred to as the *RoC diagram* of the surface.

The flow on RoC space is a second-order system of partial differential equations with many beautiful properties. Firstly, as a consequence of the derived Codazzi–Mainardi equations, the system decouples to highest order. Secondly, while the second-order differential operator depends on the principal foliation of the surface, the ellipticity of the operator is independent of it.

Furthermore, the associated *classical curvature flow ordinary differential equation (ODE)*, which neglects the spatial derivative terms, is a Hamiltonian system with canonical coordinates ψ and $|\sigma|$. With the aid of these properties, we prove the following results.

Let $\{,\}$ be the Poisson bracket and denote differentiation with respect to the canonical coordinates by ordered subscripts:

$$\mathcal{K}_{10} = \frac{\partial \mathcal{K}}{\partial \psi}, \quad \mathcal{K}_{01} = \frac{\partial \mathcal{K}}{\partial |\sigma|}, \quad \mathcal{K}_{11} = \frac{\partial^2 \mathcal{K}}{\partial \psi \partial |\sigma|} \quad \text{and so on.}$$

THEOREM 1.1. *Consider a classical curvature flow with induced flow of RoC: $f_t : S^2 \to \mathbb{R}^2_+$, $f_t = (\psi, |\sigma|)$.*

If the flow is contracting $\mathcal{K} \geq 0$ and the function satisfies:

(i) *parabolicity:* $-\mathcal{K}_{10} > |\mathcal{K}_{01}|$*;*
(ii) *convexity:* $[\mathrm{Hess}(\mathcal{K})] \geq 0$*;*

then for any function $\mathcal{H} : \mathbb{R}^2_+ \to \mathbb{R}$ satisfying:

(a) *ellipticity:* $\mathcal{H}_{10} \geq |\mathcal{H}_{01}|$*;*
(b) *convexity:* $[\mathrm{Hess}(\mathcal{H})] \geq 0$*;*
(c) *Poisson* : $\{\mathcal{H}, \mathcal{K}\} \geq 0$*;*

https://doi.org/10.1017/S1446788717000210 Published online by Cambridge University Press

the following a-priori estimate holds for $\mathcal{H} \circ f_t : S^2 \to \mathbb{R}$*:*

$$\mathcal{H} \circ f_t \leq \max_{S^2} \mathcal{H} \circ f_0.$$

If, conversely, the flow is expanding $\mathcal{K} \leq 0$ *and the function satisfies:*

(i) *parabolicity:* $-\mathcal{K}_{10} > |\mathcal{K}_{01}|$*;*
(ii) *concavity:* $[\mathrm{Hess}(\mathcal{K})] \leq 0$*;*

then for any function $\mathcal{H} : \mathbb{R}^2_+ \to \mathbb{R}$ *satisfying:*

(a) *ellipticity:* $\mathcal{H}_{10} \geq |\mathcal{H}_{01}|$*;*
(b) *concavity:* $[\mathrm{Hess}(\mathcal{H})] \leq 0$*;*
(c) $Poisson : \{\mathcal{H}, \mathcal{K}\} \leq 0$*;*

the following a-priori estimate holds:

$$\mathcal{H} \circ f_t \geq \min_{S^2} \mathcal{H} \circ f_0.$$

This follows from a more general technical Main Theorem proven in Section 4.1, where conditions (ii), (a) and (b) are replaced by a single condition. The proof involves computing the flow of an arbitrary function $\mathcal{H}$, studying the sign of lower-order terms of the flow and applying the parabolic maximum principle.

There are a variety of applications of this result. For example, clearly for $\mathcal{H} = -\mathcal{K}$, $\mathcal{H}_{10} = -\mathcal{K}_{10}$ and $[\mathrm{Hess}(\mathcal{H})] = [\mathrm{Hess}(\mathcal{K})]$, so conditions (i) and (ii) imply conditions (a) and (b), while condition (c) is automatically satisfied. In fact, we can drop the convexity condition on $\mathcal{K}$ and prove the following result.

THEOREM 1.2. *For a parabolic classical curvature flow on* $[0, t_1) \times S^2$*, the following estimate holds:*

$$|\mathcal{K}(t)| \geq \min_{S^2} |\mathcal{K}(0)|.$$

By appealing to the more general result, one can also relax the convexity assumption on $\mathcal{K}$ in Theorem 1.1 and prove the following bound on the mean radius of curvature.

THEOREM 1.3. *For a parabolic classical curvature flow with* $\mathcal{K} + |\sigma|\mathcal{K}_{01} \geq 0$ *and* $\mathcal{K}_{01} + |\sigma|\mathcal{K}_{02} \geq 0$ *on* $[0, t_1) \times S^2$*,*

$$\psi(t) \leq \max_{S^2} \psi(0).$$

For a parabolic classical curvature flow with $\mathcal{K} + |\sigma|\mathcal{K}_{01} \leq 0$ *and* $\mathcal{K}_{01} + |\sigma|\mathcal{K}_{02} \leq 0$ *on* $[0, t_1) \times S^2$*,*

$$\psi(t) \geq \min_{S^2} \psi(0).$$

Similarly, the ellipticity assumption on $\mathcal{H}$ can be relaxed and we can still obtain such estimates as the following result.

Theorem 1.4. *Consider a parabolic classical curvature flow such that* $-\mathcal{K}_{10} > \epsilon \geq 0$. *If the flow satisfies* $-\mathcal{K}_{10} \geq |\sigma||\mathcal{K}_{20}|$ *on* $S^2 \times [0, t_1)$, *then*

$$|\sigma(t)| \leq \max_{S^2} |\sigma(0)|e^{-\epsilon t}.$$

This indicates curvature flows that, if they last forever, tend to round spheres (for which $|\sigma| = 0$), so that the RoC diagram shrinks to a point on the boundary of the upper half-plane.

As a further application, we prove the nonexistence of homothetic solitons other than round spheres, for a wide class of flows. A homothetic soliton of a classical flow is a surface that evolves by dilation under the flow.

Theorem 1.5. *The only homothetic soliton for a contracting parabolic classical curvature flow with* $\mathcal{K} + |\sigma|\mathcal{K}_{01} \geq 0$ *and* $\mathcal{K}_{01} + |\sigma|\mathcal{K}_{02} \geq 0$ *is the evolving round sphere.*

Similarly, the only homothetic soliton for an expanding parabolic classical curvature flow with $\mathcal{K} + |\sigma|\mathcal{K}_{01} \leq 0$ *and* $\mathcal{K}_{01} + |\sigma|\mathcal{K}_{02} \leq 0$ *is the evolving round sphere.*

Turning to specific classes of flows, we establish the following result.

Theorem 1.6. *Consider the following flows:*

(i) *positive powers of mean curvature* $\mathcal{K} = H^n$ *for* $n \neq 0$;
(ii) *positive powers of Gauss curvature* $\mathcal{K} = K^n$ *for* $n \neq 0$;
(iii) *powers of mean radius of curvature* $\mathcal{K} = \pm K^n H^{-n}$ *for* $n \neq 0$;
(iv) *Bloore flow* $\mathcal{K} = a + 2bH + cK$ *for* a, b, c *positive;*

and the linear Weingarten flow. These flows are all parabolic.

Linear Weingarten flow is convex, as are powers of mean curvature for $n \geq -1$, *powers of Gauss curvature for* $n \geq \frac{1}{2}$ *and powers of mean radius of curvature for* $n \geq -1$. *Powers of mean radius of curvature are concave for* $n \leq -1$.

For each of these flows, we have the estimate

$$|\mathcal{K}(t)| \geq \min_{S^2} |\mathcal{K}(0)|,$$

while for flows (i) and (iii) with $n > 0$, *flow (ii) with* $n > 1/2$ *and flow (iv) for all positive* a, b *and* c,

$$\psi(t) \leq \max_{S^2} \psi(0).$$

For these values, the flows do not admit homothetic solitons, other than round spheres.

For negative powers of mean radius of curvature, we have $\psi(t) \geq \min_{S^2} \psi(0)$ *and there are no homothetic solitons, other than round spheres.*

The rest of this paper is organized as follows: the next section contains background material on convex surfaces and introduces the RoC diagram. The connection of this work to results on Weingarten surfaces is explored and the significance of the hyperbolic or anti-de Sitter metric on the RoC diagram is also discussed.

Section 3 derives the evolution equations for the RoC and investigates their properties, while Section 4 contains the proofs of Theorems 1.1 to 1.5. Finally, Section 5 looks at examples of curvature flows and contains the proof of Theorem 1.6.

2. Convex surfaces and RoC diagram

2.1. Classical surface theory redux. Consider a smooth closed convex surface $S \subset \mathbb{R}^3$ given by a map $X : S^2 \to \mathbb{R}^3$. We now outline our approach to classical surface theory—further details can be found in Guilfoyle and Klingenberg [10] and references therein.

Let ξ be the standard complex coordinate on S^2; since S is convex, we can use the inverse of the Gauss map $S \to S^2$ to choose ξ as a local coordinate on S. We refer to these as Gauss coordinates and use them exclusively for all computations that follow.

Let $r : S^2 \to \mathbb{R}$ be the support function of S and define the complex derivative

$$F = \tfrac{1}{2}(1 + \xi\bar{\xi})^2 \bar{\partial} r. \tag{2.1}$$

This is a Lagrangian section of the space of oriented lines in $\mathbb{R}^3$, which can be identified with $T\mathbb{S}^2$ endowed with its canonical neutral Kaehler structure.

The surface S can be reconstructed from the support function and its derivatives by $\vec{X}(\xi, \bar{\xi}) = (x^1(\xi, \bar{\xi}), x^2(\xi, \bar{\xi}), x^3(\xi, \bar{\xi}))$, where

$$x^1 + ix^2 = \frac{2(F - \bar{F}\xi^2) + 2\xi(1 + \xi\bar{\xi})r}{(1 + \xi\bar{\xi})^2}, \quad x^3 = \frac{-2(F\bar{\xi} + \bar{F}\xi) + (1 - \xi^2\bar{\xi}^2)r}{(1 + \xi\bar{\xi})^2}. \tag{2.2}$$

Moving up a derivative, label the complex slopes of F as

$$\psi = r + (1 + \xi\bar{\xi})^2 \partial\Big(\frac{F}{(1 + \xi\bar{\xi})^2}\Big), \quad \sigma = -\partial\bar{F}. \tag{2.3}$$

By its definition and Equation (2.1), ψ is clearly real. The average and difference of the RoC $r_1 \geq r_2$ of S can be expressed as

$$\psi = \tfrac{1}{2}(r_1 + r_2), \quad |\sigma| = \tfrac{1}{2}(r_1 - r_2). \tag{2.4}$$

The argument of $\sigma = |\sigma| e^{i\phi}$ gives the principal directions of S and, as we will see, is the parameter ϕ that appears in the differential second-order partial differential operator $\triangle_\phi$.

These quantities satisfy the derived Codazzi–Mainardi equations, which in Gauss coordinates are

$$\partial\psi = -(1 + \xi\bar{\xi})^2 \bar{\partial}\Big(\frac{\sigma}{(1 + \xi\bar{\xi})^2}\Big). \tag{2.5}$$

This can also be written

$$i|\sigma|\partial\phi = \partial|\sigma| + e^{i\phi}\bar{\partial}\psi - \frac{2\bar{\xi}|\sigma|}{1 + \xi\bar{\xi}}. \tag{2.6}$$

For future use, note the following proposition.

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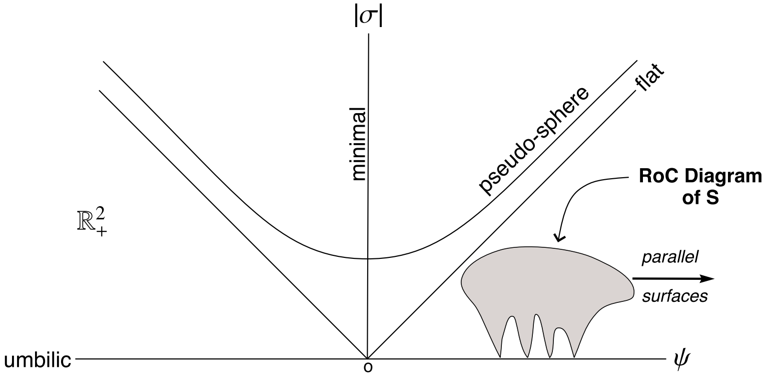

FIGURE 1. Radii of curvature space.

PROPOSITION 2.1. *The following identities hold*

$$e^{-i\phi}\partial\partial\psi = -\partial\bar\partial|\sigma| - i|\sigma|\partial\bar\partial\phi + ie^{-i\phi}\partial\phi\partial\psi - ie^{-i\phi}\partial|\sigma|\bar\partial\phi + \frac{2\xi\partial|\sigma|}{1+\xi\bar\xi} + \frac{2|\sigma|}{(1+\xi\bar\xi)^2},$$

$$e^{-i\phi}\partial\partial|\sigma| = -\partial\bar\partial\psi + i|\sigma|e^{-i\phi}\partial\partial\phi + 2ie^{-i\phi}\partial\phi\partial|\sigma| + |\sigma|e^{-i\phi}(\partial\phi)^2 + \frac{2\bar\xi e^{-i\phi}(\partial|\sigma| - i|\sigma|\partial\phi)}{1+\xi\bar\xi} - \frac{2\bar\xi^2|\sigma|e^{-i\phi}}{(1+\xi\bar\xi)^2}.$$

PROOF. The first of these equations is a rearrangement of ∂ of $e^{-i\phi}$ times Equation (2.5) and the second comes from $\bar\partial$ of Equation (2.5). □

2.2. The RoC diagram. For a convex surface S in $\mathbb{R}^3$, the RoC give a map $(r_1, r_2) : S \to \mathbb{R}^2_+$. In this context, we refer to $\mathbb{R}^2_+$ as the *RoC space*. We call the image of the map $f : S^2 \to \mathbb{R}^2_+$ given by $f(\xi,\bar\xi) = (\psi(\xi,\bar\xi), |\sigma(\xi,\bar\xi)|)$ the *RoC diagram* or *RoC diagram* of S. Many geometric properties of S can be read off its RoC diagram, including convexity and the number of umbilic points.

Figure 1 is the RoC diagram of a generic closed convex surface S: it is convex with outward pointing normal and so it lies below the diagonal in the first quadrant. The umbilic points on S map to the boundary of the upper half-plane since $|\sigma| = 0$ at such points (see the second of Equations (2.4)). For a convex surface with inward pointing normals, the RoC diagram would lie below the diagonal in the second quadrant.

DEFINITION 2.2. A surface is *Weingarten* if there exists a functional relationship between the RoC: $\mathcal{K}(\psi, |\sigma|) = 0$.

A convex surface S is Weingarten if and only if the infinitesimal area of its RoC diagram is zero. Different types of Weingarten surface give rise to different one-dimensional RoC diagrams, examples of which are also indicated in the figure (flat, pseudospherical, minimal surfaces). Round spheres have a single point at the boundary as their RoC diagram.

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The trio of Weingarten, real analyticity and the behaviour at umbilic points was investigated in the middle of the last century by a number of authors [3, 4, 14–16, 24]—see also Kuhnel and Steller [18]. The curvatures, rather than the RoC, were used in these works and the curvature equivalent of the RoC diagram, called the *W-diagram*, was introduced.

The boundary of the image of $f_t(S) \subset \mathbb{R}^2_+$ is continuous, but in general not smooth at an umbilic point. Let $p_0 \in S$ be an isolated umbilic point and consider

$$\kappa(p_0) = \lim_{p \to p_0} \frac{\psi(p) - \psi(p_0)}{|\sigma(p)|}.$$

In general, this limit is not well defined, as it depends on the direction of approach to the umbilic. As a result, in general the RoC diagram at an umbilic point is a solid wedge.

If, however, S is Weingarten, then κ is well defined. Moreover, for real analytic Weingarten surfaces, the slope at which the RoC curve strikes the boundary is quantized.

Theorem 2.3 [15]. *Let S be a real analytic Weingarten surface. Then if p_0 is an isolated umbilic point, $\kappa(p_0)$ takes one of the following values:*

$$-1, \quad 1, \quad 0, \quad -\Big(\frac{m+1}{m}\Big)^{\pm 1},$$

for $m \in \mathbb{N}$.

Definition 2.4. A Weingarten surface is *special Weingarten* if, at its umbilic points,

$$\frac{\partial \mathcal{K}}{\partial \lambda_1} \cdot \frac{\partial \mathcal{K}}{\partial \lambda_2} > 0,$$

where λ_1 and λ_2 are the principal curvatures.

As will be seen in the Note in Section 4.1, this is the condition that the Weingarten relation is elliptic at the umbilic points.

Weingarten spheres often turn out to be round.

Theorem 2.5 [14]. *Let S be a closed special Weingarten surface of genus zero, which is C^3-embedded in Euclidean space. Then S is a round sphere.*

In contrast to such nonexistence results, the Hopf spheres are a 2-parameter family of nonspherical Weingarten surfaces and therefore satisfy a nonelliptic relationship. For certain integer values of these parameters, the solutions are real analytic, while for nonintegers they are smooth nonreal analytic surfaces [18].

The hyperbolic and anti-de Sitter metrics on $\mathbb{R}^2_+$ play an interesting role in the RoC space. Given a surface S, movement at constant speed along the normal lines of the surface (to parallel surfaces) induces a translation in $\mathbb{R}^2_+$ parallel to the boundary, which is a hyperbolic and an anti-de Sitter isometry.

THEOREM 2.6. *Let $f : S^2 \to \mathbb{R}^2_+$ be the RoC diagram of S and $l : S^2 \to T\mathbb{S}^2$ be the map that takes a point on the surface to its oriented normal line, considered as a surface in the space of all oriented lines, $T\mathbb{S}^2$.*

Let $d_{\mathbb{H}^2}A$ be the hyperbolic or anti-de Sitter area on $\mathbb{R}^2_+$ and $\mathbb{G}$ be the canonical invariant neutral metric on $T\mathbb{S}^2$. Denote the curvature 2-form of the Lorentz metric induced on $l(S^2)$ by $\mathbb{G}$ by $\Omega_{\mathbb{G}}$.

Then

$$l_*\Omega_{\mathbb{G}} = f_* d_{\mathbb{H}^2}A.$$

PROOF. Compare with the curvature expressions in the proof of Guilfoyle and Klingenberg [11, Main Theorem 3]. In particular, with a slight shift of notation ($\sigma \leftrightarrow \sigma_0$ and $\psi \leftrightarrow r + \rho_0$), the curvature of the induced metric $\mathbb{G}$ on the Lagrangian section is

$$\mathbb{K} = \frac{(1+\xi\bar{\xi})^2}{8|\sigma|^3}(\partial\psi\bar{\partial}|\sigma| - \bar{\partial}\psi\partial|\sigma|),$$

and the area form is

$$d_{\mathbb{G}}A = \frac{8|\sigma|\, d\xi \wedge d\bar{\xi}}{(1+\xi\bar{\xi})^2}.$$

Thus the curvature 2-form is

$$l_*\Omega_{\mathbb{G}} = l_*\mathbb{K}\, d_{\mathbb{G}}A = f_*\frac{d\psi \wedge d|\sigma|}{|\sigma|^2},$$

which is the pullback of the hyperbolic or anti-de Sitter area 2-form on RoC space, as claimed. □

Geometrizing the RoC space as the hyperbolic or anti-de Sitter plane has the added advantage that it places the umbilic points at infinity. Given that many results of classical surface theory hold 'away from umbilic points', such results can be viewed as holding in the interior of the upper half-plane. Moreover, this suggests that properties of umbilics may be amenable to exploration using asymptotic methods of the upper half-plane.

It is also worth noting that the divergence between smoothness and real analyticity at umbilic points is further in evidence in the local version of the Carathéodory conjecture [12].

3. Classical curvature flows

3.1. Evolution equations. Consider a classical curvature flow $\vec{X} : S^2 \times [0, t_1) \to \mathbb{R}^3$, such that

$$\frac{\partial \vec{X}^{\perp}}{\partial t} = -\mathcal{K}(\psi, |\sigma|)\, \vec{n},$$
$$\vec{X}(S^2, 0) = S_0,$$

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where $\mathcal{K}$ is a given function of the RoC, $\vec{n}$ is the unit normal vector to the flowing surface and S_0 is an initial convex surface. The flow is *contracting* if $\mathcal{K} \geq 0$ everywhere and *expanding* if $\mathcal{K} \leq 0$ everywhere.

PROPOSITION 3.1. *The support function of S evolves under a classical curvature flow by*

$$\frac{\partial r}{\partial t} = -\mathcal{K}.$$

PROOF. Differentiating Equation (2.2) in time

$$\frac{\partial}{\partial t}(x^1 + ix^2) = \frac{2}{(1+\xi\bar{\xi})^2}\frac{\partial \eta}{\partial t} - \frac{2\xi^2}{(1+\xi\bar{\xi})^2}\frac{\partial \bar{\eta}}{\partial t} + \frac{2\xi}{1+\xi\bar{\xi}}\frac{\partial r}{\partial t},$$
$$\frac{\partial}{\partial t}x^3 = -\frac{2\bar{\xi}}{(1+\xi\bar{\xi})^2}\frac{\partial \eta}{\partial t} - \frac{2\xi}{(1+\xi\bar{\xi})^2}\frac{\partial \bar{\eta}}{\partial t} + \frac{1-\xi\bar{\xi}}{1+\xi\bar{\xi}}\frac{\partial r}{\partial t}.$$

Projecting onto the normal direction $\vec{n}$ gives

$$\frac{\partial \vec{X}}{\partial t}^{\perp} = \frac{\partial r}{\partial t}\vec{n} = -\mathcal{K}\vec{n}.$$

This yields the stated flow for the support function. □

We now compute the evolution of the functions ψ and $|\sigma|$. For $\mathcal{K} = \mathcal{K}(\psi, |\sigma|)$ denote the derivatives of $\mathcal{K}$ with respect to its arguments by ordered subscripts, so that

$$\mathcal{K}_{10} = \frac{\partial \mathcal{K}}{\partial \psi}, \quad \mathcal{K}_{01} = \frac{\partial \mathcal{K}}{\partial |\sigma|}, \quad \mathcal{K}_{11} = \frac{\partial^2 \mathcal{K}}{\partial \psi \partial |\sigma|} \quad \text{and so on.}$$

PROPOSITION 3.2. *The quantities ψ and $|\sigma|$ flow as*

$$\frac{\partial \vec{V}}{\partial t} = \triangle_\phi \vec{V} + Q(d\vec{V}) + \mathcal{Z}(\vec{V}),$$

where

$$\vec{V} = (\psi, |\sigma|),$$
$$\triangle_\phi = \tfrac{1}{2}(1+\xi\bar{\xi})^2[-\mathcal{K}_{10}\partial\bar{\partial} + \tfrac{1}{2}\mathcal{K}_{01}(e^{-i\phi}\partial\partial + e^{i\phi}\bar{\partial}\bar{\partial})],$$
$$Q(d\vec{V}) = \begin{bmatrix} Q_1(d\vec{V}) \\ Q_2(d\vec{V}) \end{bmatrix}, \quad \mathcal{Z}\begin{bmatrix} \psi \\ |\sigma| \end{bmatrix} = \begin{bmatrix} -\mathcal{K} - |\sigma|\mathcal{K}_{01} \\ |\sigma|\mathcal{K}_{10} \end{bmatrix},$$

and

$$Q_1 = \frac{1}{2}(1+\xi\bar{\xi})^2\Big[-\frac{1}{|\sigma|}(\mathcal{K}_{01} + |\sigma|\mathcal{K}_{20})\partial\psi\bar{\partial}\psi - \frac{1}{|\sigma|}\mathcal{K}_{01}(e^{-i\phi}\partial\psi\partial|\sigma| + e^{i\phi}\bar{\partial}\psi\bar{\partial}|\sigma|)$$
$$- \mathcal{K}_{11}(\partial\psi\bar{\partial}|\sigma| + \bar{\partial}\psi\partial|\sigma|) - \frac{1}{|\sigma|}(\mathcal{K}_{01} + |\sigma|\mathcal{K}_{02})\partial|\sigma|\bar{\partial}|\sigma|\Big]$$
$$+ \frac{1}{2}(1+\xi\bar{\xi})\mathcal{K}_{01}(\bar{\xi}e^{-i\phi}\partial\psi + \xi e^{i\phi}\bar{\partial}\psi),$$

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$$\begin{aligned}\mathcal{Q}_2 = \frac{1}{2}(1+\xi\bar{\xi})^2\Big[&\frac{1}{|\sigma|}\mathcal{K}_{10}\partial\psi\bar{\partial}\psi + \frac{1}{2}\mathcal{K}_{20}(e^{-i\phi}(\partial\psi)^2 + e^{-i\phi}(\bar{\partial}\psi)^2)\\
&+ \frac{1}{|\sigma|}(\mathcal{K}_{10} + |\sigma|\mathcal{K}_{11})(e^{-i\phi}\partial\psi\partial|\sigma| + e^{i\phi}\bar{\partial}\psi\bar{\partial}|\sigma|)\\
&+ \frac{1}{|\sigma|}\mathcal{K}_{10}\partial|\sigma|\bar{\partial}|\sigma| + \frac{1}{2}\mathcal{K}_{02}(e^{-i\phi}(\partial|\sigma|)^2 + e^{i\phi}(\bar{\partial}|\sigma|)^2)\Big]\\
&+ \frac{1}{2}(1+\xi\bar{\xi})\mathcal{K}_{01}(\bar{\xi}e^{-i\phi}\partial|\sigma| + \xi e^{i\phi}\bar{\partial}|\sigma|).\end{aligned}$$

Proof. By Equation (2.1) and Proposition 3.1,

$$\frac{\partial F}{\partial t} = \frac{1}{2}(1+\xi\bar{\xi})^2\bar{\partial}\Big(\frac{\partial r}{\partial t}\Big) = -\frac{1}{2}(1+\xi\bar{\xi})^2\bar{\partial}\mathcal{K} = -\frac{1}{2}(1+\xi\bar{\xi})^2(\mathcal{K}_{10}\bar{\partial}\psi + \mathcal{K}_{01}\bar{\partial}|\sigma|).$$

Now using the first definition in Equation (2.3)

$$\begin{aligned}\frac{\partial\psi}{\partial t} &= \frac{\partial r}{\partial t} + \partial\Big(\frac{\partial F}{\partial t}\Big) - \frac{2\bar{\xi}}{1+\xi\bar{\xi}}\frac{\partial F}{\partial t}\\
&= -\frac{1}{2}(1+\xi\bar{\xi})^2[\mathcal{K}_{10}\partial\bar{\partial}\psi + \mathcal{K}_{01}\partial\bar{\partial}|\sigma| + \mathcal{K}_{20}\partial\psi\bar{\partial}\psi + \mathcal{K}_{11}(\partial|\sigma|\bar{\partial}\psi + \bar{\partial}|\sigma|\partial\psi)\\
&\quad + \mathcal{K}_{02}\partial|\sigma|\bar{\partial}|\sigma|] - \mathcal{K}.\end{aligned}$$

Replace the expression for $\partial\bar{\partial}|\sigma|$ by the one from the real part of the derived Codazzi–Mainardi Equation (2.7) and use Equation (2.6) to remove all derivatives of ϕ. The result is

$$\begin{aligned}\frac{\partial}{\partial t}\psi = \frac{1}{2}(1+\xi\bar{\xi})^2\Big[&-\mathcal{K}_{10}\partial\bar{\partial}\psi + \frac{1}{2}\mathcal{K}_{01}(e^{-i\phi}\partial\partial\psi + e^{i\phi}\bar{\partial}\bar{\partial}\psi) + \frac{1}{|\sigma|}(\mathcal{K}_{01} + |\sigma|\mathcal{K}_{20})\partial\psi\bar{\partial}\psi\\
&- \frac{1+\xi\bar{\xi}}{|\sigma|}\mathcal{K}_{01}\Big(e^{-i\phi}\partial\psi\partial\Big(\frac{|\sigma|}{1+\xi\bar{\xi}}\Big) + e^{i\phi}\bar{\partial}\psi\bar{\partial}\Big(\frac{|\sigma|}{1+\xi\bar{\xi}}\Big)\Big)\\
&- \mathcal{K}_{11}(\partial\psi\bar{\partial}|\sigma| + \bar{\partial}\psi\partial|\sigma|) - \Big(\mathcal{K}_{02} + \frac{\mathcal{K}_{01}}{|\sigma|}\Big)\partial|\sigma|\bar{\partial}|\sigma|\Big]\\
&- \mathcal{K} - |\sigma|\mathcal{K}_{01}.\end{aligned}$$

This is identical to the expression for the flow of ψ in the statement of Proposition 3.2, given the definitions of $\triangle_\phi$, $\mathcal{Q}_1$ and $\mathcal{Z}$.

Turning to the second definition in Equation (2.3):

$$\begin{aligned}e^{-i\phi}\frac{\partial\sigma}{\partial t} &= -e^{-i\phi}\partial\Big(\frac{\partial\bar{F}}{\partial t}\Big)\\
&= -\frac{1}{2}(1+\xi\bar{\xi})^2e^{-i\phi}[\mathcal{K}_{10}\partial\partial\psi + \mathcal{K}_{01}\partial\partial|\sigma| + \mathcal{K}_{20}(\partial\psi)^2\\
&\quad + 2\mathcal{K}_{11}\partial|\sigma|\partial\psi + \mathcal{K}_{20}(\partial|\sigma|)^2]\\
&\quad + \frac{1}{2}(1+\xi\bar{\xi})e^{-i\phi}\bar{\xi}(\mathcal{K}_{10}\partial\psi + \mathcal{K}_{01}\partial|\sigma|).\end{aligned}$$

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Now replace the expression for $\partial\bar{\partial}\psi$ with the one from the derived Codazzi–Mainardi Equation (2.7) and use Equation (2.6) to remove all derivatives of ϕ. The result, taking the real part, is

$$
\begin{aligned}
\frac{\partial}{\partial t}|\sigma| = \frac{1}{2}(1+\xi\bar{\xi})^2\Big[&-\mathcal{K}_{10}\partial\bar{\partial}|\sigma| + \frac{1}{2}\mathcal{K}_{01}(e^{-i\phi}\partial\partial|\sigma| + e^{i\phi}\bar{\partial}\bar{\partial}|\sigma|) + \frac{1}{|\sigma|}\mathcal{K}_{10}\partial\psi\bar{\partial}\psi \\
&+ \frac{1}{2}\mathcal{K}_{20}(e^{-i\phi}(\partial\psi)^2 + e^{-i\phi}(\bar{\partial}\psi)^2) \\
&+ \frac{1}{|\sigma|}(\mathcal{K}_{10} + |\sigma|\mathcal{K}_{11})(e^{-i\phi}\partial\psi\partial|\sigma| + e^{i\phi}\bar{\partial}\psi\bar{\partial}|\sigma|) \\
&+ \frac{1}{2}\mathcal{K}_{02}(e^{-i\phi}(\partial|\sigma|)^2 + e^{i\phi}(\bar{\partial}|\sigma|)^2) + \frac{\mathcal{K}_{10}}{|\sigma|}\partial|\sigma|\bar{\partial}|\sigma|\Big] \\
&+ \frac{1}{2}(1+\xi\bar{\xi})\mathcal{K}_{01}(\bar{\xi}e^{-i\phi}\partial|\sigma| + \xi e^{i\phi}\bar{\partial}|\sigma|) + |\sigma|\mathcal{K}_{10},
\end{aligned}
$$

which is the claimed flow for $|\sigma|$. □

3.2. The classical curvature ODE. By Proposition 3.2, a classical curvature flow gives rise to a second-order system of partial differential equations in RoC space, which decouples to top order. Necessary and sufficient conditions for parabolicity arise from the second-order terms. If we can furthermore put a sign on the quadratic first-order terms, we can then compare the evolution with the zeroth-order terms.

That is, we are led to study the behaviour of the following ODE, which we refer to as the *classical curvature flow ODE*:

$$
\frac{\partial}{\partial t}\psi = -\mathcal{K} - |\sigma|\mathcal{K}_{01}, \qquad \frac{\partial}{\partial t}|\sigma| = |\sigma|\mathcal{K}_{10}.
$$

THEOREM 3.3. *The classical curvature flow ODE is Hamiltonian, with conserved quantity $\mathcal{I} = |\sigma|\mathcal{K}$ and canonical coordinates ψ and $|\sigma|$.*

PROOF. This follows from noting that

$$
\frac{\partial}{\partial t}\psi = -\frac{\partial}{\partial|\sigma|}\mathcal{I}, \qquad \frac{\partial}{\partial t}|\sigma| = \frac{\partial}{\partial\psi}\mathcal{I}.
$$

This concludes the proof. □

Thus, if the flow behaves well, it should converge to this flow. Conversely, the curvature ODE yields insight into the behaviour of the full flow and may suggest quantities that are conserved.

COROLLARY 3.4. *The flow of any function $\mathcal{H}\circ f_t$ is of the form*

$$
\frac{\partial}{\partial t}\mathcal{H}\circ f_t = \triangle_\phi\mathcal{H} + \tilde{Q}(d\mathcal{H}) - \{\mathcal{H},\mathcal{I}\},
$$

where $\{,\}$ are the Poisson brackets associated with the canonical coordinates $(\psi,|\sigma|)$.

The explicit expression will be given next.

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4. *A-priori* estimates

4.1. The main estimate. In this section, we study the behaviour of parabolic classical curvature flows and extract *a-priori* estimates under mild assumptions on the functional $\mathcal{K}$.

PROPOSITION 4.1. *A classical curvature flow is parabolic if and only if*

$$-\mathcal{K}_{10} > |\mathcal{K}_{01}|.$$

PROOF. For parabolicity, we compute the symbol of the operator $\triangle_\phi$ as follows. Introduce real variables $\xi = x + iy$, so that

$$\partial\bar{\partial} = \tfrac{1}{4}(\partial_x^2 + \partial_y^2), \quad \partial\partial = \tfrac{1}{4}(\partial_x^2 - \partial_y^2 - 2i\partial_x\partial_y).$$

Then the symbol of

$$\triangle_\phi = \tfrac{1}{2}(1 + \xi\bar{\xi})^2[-\mathcal{K}_{10}\partial\bar{\partial} + \tfrac{1}{2}\mathcal{K}_{10}(e^{-i\phi}\partial\partial + e^{i\phi}\bar{\partial}\bar{\partial})],$$

written in real coordinates is

$$P = \frac{1}{2}(1 + \xi\bar{\xi})^2 \begin{bmatrix} -\mathcal{K}_{10} + \mathcal{K}_{01}\cos\phi & -\mathcal{K}_{01}\sin\phi \\ -\mathcal{K}_{01}\sin\phi & -\mathcal{K}_{10} - \mathcal{K}_{01}\cos\phi \end{bmatrix}.$$

This is elliptic if $P(X, X) = 0$ implies that $X = 0$. In other words,

$$\det P = \tfrac{1}{4}(1 + \xi\bar{\xi})^4(\mathcal{K}_{10}^2 - \mathcal{K}_{01}^2) \neq 0.$$

For the operator $\partial/\partial t - \triangle_\phi$ to be parabolic, we must also require that $-\mathcal{K}_{10} > 0$. □

Note. This definition of parabolicity agrees with the usual definition of ellipticity for fully nonlinear second-order partial differential equations. That is, an equation involving the second derivatives of r

$$\mathcal{F}(\partial_x^2 r, \partial_x\partial_y r, \partial_y^2 r) = 0,$$

is *elliptic* if $\mathcal{F}_1\mathcal{F}_3 - \mathcal{F}_2^2 > 0$, where a subscript represents differentiation with respect to the corresponding arguments of $\mathcal{F}$.

In our case (see Equations (2.1) and (2.3), defining ψ and $|\sigma|$ in terms of r, and switching to real variables as before),

$$\mathcal{F}(\partial_x^2 r, \partial_x\partial_y r, \partial_y^2 r) = \mathcal{K}\left(\partial_x^2 r + \partial_y^2 r, \sqrt{\partial_x^2 r - \partial_y^2 r + (\partial_x\partial_y r)^2}\right) = 0,$$

for which $\mathcal{F}_1\mathcal{F}_3 - \mathcal{F}_2^2 > 0$ is easily found to be equivalent to $\mathcal{K}_{10}^2 - \mathcal{K}_{01}^2 > 0$.

As we saw in Section 2.2, a Weingarten surface is *special Weingarten* if at its umbilic points

$$\frac{\partial\mathcal{K}}{\partial\lambda_1} \cdot \frac{\partial\mathcal{K}}{\partial\lambda_2} > 0,$$

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where λ_1 and λ_2 are the curvatures. It is not hard to see that, in terms of the canonical coordinates $(\psi, |\sigma|)$,

$$\frac{\partial \mathcal{K}}{\partial \lambda_1} \cdot \frac{\partial \mathcal{K}}{\partial \lambda_2} = \frac{1}{4}(\psi^2 - |\sigma|^2)(\mathcal{K}_{10}^2 - \mathcal{K}_{01}^2),$$

and so *special Weingarten* for a convex Weingarten surface is equivalent to *elliptic at the umbilic points*.

We now prove the main estimate.

Main Theorem. *Consider a classical parabolic curvature flow with induced flow of RoC $f_t : S^2 \to \mathbb{R}^2_+$. For $\mathcal{H} : \mathbb{R}^2_+ \to \mathbb{R}$ define*

$$A = (\mathcal{H}_{10}^2 + \mathcal{H}_{01}^2)\{\mathcal{H}, \mathcal{K}\} - |\sigma|(\mathcal{K}_{10}\mathrm{Hess}_{\mathcal{H}}(d\mathcal{H}) - \mathcal{H}_{10}\mathrm{Hess}_{\mathcal{K}}(d\mathcal{H})), \tag{4.1}$$

$$B = -2\mathcal{H}_{10}\mathcal{H}_{01}\{\mathcal{H}, \mathcal{K}\} + |\sigma|(\mathcal{K}_{01}\mathrm{Hess}_{\mathcal{H}}(d\mathcal{H}) - \mathcal{H}_{01}\mathrm{Hess}_{\mathcal{K}}(d\mathcal{H})) \tag{4.2}$$

and

$$\mathrm{Hess}_{\mathcal{K}}(d\mathcal{H}) = \mathcal{K}_{20}\mathcal{H}_{01}^2 - 2\mathcal{K}_{11}\mathcal{H}_{01}\mathcal{H}_{10} + \mathcal{K}_{02}\mathcal{H}_{10}^2. \tag{4.3}$$

Suppose that $\mathcal{H}$ satisfies:

(1a) $\{\mathcal{H}, \mathcal{I}\} \geq 0$;

(2a) $A \geq |B|$.

Then the following a-priori estimate holds:

$$\mathcal{H} \circ f_t \leq \max_{S^2} \mathcal{H} \circ f_0.$$

Conversely, if $\mathcal{H}$ satisfies:

(1b) $\{\mathcal{H}, \mathcal{I}\} \leq 0$;

(2b) $A \leq -|B|$;

then

$$\mathcal{H} \circ f_t \geq \min_{S^2} \mathcal{H} \circ f_0.$$

Proof. The flow of the function $\mathcal{H} \circ f_t$ is computed as

$$\begin{aligned}
\frac{\partial}{\partial t}\mathcal{H} \circ f_t &= \mathcal{H}_{10}\frac{\partial}{\partial t}\psi + \mathcal{H}_{01}\frac{\partial}{\partial t}|\sigma| \\
&= \mathcal{H}_{10}(\triangle_\phi \psi + Q_1 - \mathcal{K} - |\sigma|\mathcal{K}_{01}) + \mathcal{H}_{01}(\triangle_\phi|\sigma| + Q_2 + |\sigma|\mathcal{K}_{10}) \\
&= \mathcal{H}_{10}(\triangle_\phi \psi + Q_1) + \mathcal{H}_{01}(\triangle_\phi|\sigma| + Q_2) - \{\mathcal{H}, \mathcal{I}\} \\
&= \triangle_\phi \mathcal{H} + Q_3(d\mathcal{H}) - \mathcal{Z}_2 - \{\mathcal{H}, \mathcal{I}\},
\end{aligned}$$

where on the second line we have used the flow equations in Proposition 3.2. Here, $Q_3(d\mathcal{H}) = 0$ if $d\mathcal{H} = 0$ and

$$\mathcal{Z}_2 = \frac{(1 + \xi\bar{\xi})^2}{2|\sigma|\mathcal{H}_{10}^2}\Big[A\partial|\sigma|\bar{\partial}|\sigma| + \frac{1}{2}B(e^{-i\phi}(\partial|\sigma|)^2 + e^{i\phi}(\bar{\partial}|\sigma|)^2)\Big],$$

with A and B given by Equations (4.1) to (4.3).

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Thus, at the maximum or minimum value of $\mathcal{H}$,

$$\Big(\frac{\partial}{\partial t} - \triangle_\phi\Big)\mathcal{H} \circ f_t = -\mathcal{Z}_2 - \{\mathcal{H}, \mathcal{I}\}.$$

If $A \geq |B|$ then $\mathcal{Z}_2 \geq 0$; if in addition $\{\mathcal{H}, \mathcal{I}\} \geq 0$, the estimate follows by the parabolic maximum principle.

If $A \leq -|B|$ then $\mathcal{Z}_2 \leq 0$; if $\{\mathcal{H}, \mathcal{I}\} \leq 0$, the estimate follows. □

4.2. Applications. Geometric assumptions can now be used to obtain *a-priori* bounds.

Theorem 4.2. *Consider a classical curvature flow with induced flow of RoC:* $f_t : S^2 \to \mathbb{R}^2_+$.

If the flow is contracting ($\mathcal{K} \geq 0$) *and the function satisfies:*

(a) *parabolicity:* $-\mathcal{K}_{10} > |\mathcal{K}_{01}|$;
(b) *convexity:* $[\mathrm{Hess}(\mathcal{K})] \geq 0$;

then for any function $\mathcal{H} : \mathbb{R}^2_+ \to \mathbb{R}$ *satisfying:*

(i) *ellipticity:* $\mathcal{H}_{10} \geq |\mathcal{H}_{01}|$;
(ii) *convexity:* $[\mathrm{Hess}(\mathcal{H})] \geq 0$;
(iii) *Poisson:* $\{\mathcal{H}, \mathcal{K}\} \geq 0$;

the following a-priori estimate holds for $\mathcal{H} \circ f_t : S^2 \to \mathbb{R}$:

$$\mathcal{H} \circ f_t \leq \max_{S^2} \mathcal{H} \circ f_0.$$

If, conversely, the flow is expanding ($\mathcal{K} \leq 0$) *and the function satisfies:*

(a) *parabolicity:* $-\mathcal{K}_{10} > |\mathcal{K}_{01}|$;
(b) *concavity:* $[\mathrm{Hess}(\mathcal{K})] \leq 0$;

then for any function $\mathcal{H} : \mathbb{R}^2_+ \to \mathbb{R}$ *satisfying:*

(i) *ellipticity:* $\mathcal{H}_{10} \geq |\mathcal{H}_{01}|$;
(ii) *concavity:* $[\mathrm{Hess}(\mathcal{H})] \leq 0$;
(iii) *Poisson:* $\{\mathcal{H}, \mathcal{K}\} \leq 0$;

the following a-priori estimate holds:

$$\mathcal{H} \circ f_t \geq \min_{S^2} \mathcal{H} \circ f_0.$$

Proof. Assume that the flow is contracting, so that $\mathcal{K} \geq 0$, and assume that conditions (a), (b), (i), (ii) and (iii) of Theorem 1.1 hold. Then, by conditions (i) and (iii)

$$\{\mathcal{H}, \mathcal{I}\} = \mathcal{H}_{10}\mathcal{K} + |\sigma|\{\mathcal{H}, \mathcal{K}\} \geq 0,$$

so that condition (1a) in the Main Theorem holds.

To see that condition (2a) also holds, compare the expressions for A and B in Equations (4.1) and (4.2) term by term. Note that the first term of A is positive by

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condition (iii) and dominates the first term of B. By condition (a), the second term of A is also positive and dominates the second term of B.

By conditions (a) and (ii), the third term of A is positive and dominates the third term of B. The final term in A is positive and dominates the final term of B, owing to conditions (i) and (b).

Thus conditions (1a) and (2a) of the Main Theorem hold, and we can apply it to yield the stated result.

The proof of the expanding case is analogous, with opposite inequalities. □

THEOREM 4.3. *For a parabolic classical curvature flow on* $[0, t_1) \times S^2$, *the following estimate holds:*

$$|\mathcal{K}(t)| \geq \min_{S^2} |\mathcal{K}(0)|.$$

PROOF. Let $\mathcal{H} = -\mathcal{K}$, then compute that $A = B = 0$ and $\{\mathcal{H}, \mathcal{I}\} = -\mathcal{K}\mathcal{K}_{10}$. Thus for a contracting parabolic flow $\{\mathcal{H}, \mathcal{I}\} \geq 0$; conditions (1a) and (2a) hold and, applying the Main Theorem, we obtain $\mathcal{K}(t) \geq \min_{S^2} \mathcal{K}(0)$.

In the expanding parabolic case $\{\mathcal{H}, \mathcal{I}\} \leq 0$ and so conditions (1b) and (2b) hold and again we apply the Main Theorem. □

Note that this lower bound is a tautology unless $\mathcal{K}$ has a fixed sign.

THEOREM 4.4. *For a parabolic classical curvature flow with* $\mathcal{K} + |\sigma|\mathcal{K}_{01} \geq 0$ *and* $\mathcal{K}_{01} + |\sigma|\mathcal{K}_{02} \geq 0$ *on* $[0, t_1) \times S^2$,

$$\psi(t) \leq \max_{S^2} \psi(0).$$

For a parabolic classical curvature flow with $\mathcal{K} + |\sigma|\mathcal{K}_{01} \leq 0$ *and* $\mathcal{K}_{01} + |\sigma|\mathcal{K}_{02} \leq 0$ *on* $[0, t_1) \times S^2$,

$$\psi(t) \geq \min_{S^2} \psi(0).$$

PROOF. This follows from the Main Theorem by setting $\mathcal{H} = \psi$ and noting that $A = \mathcal{K}_{01} + |\sigma|\mathcal{K}_{02}$, $B = 0$ and $\{\mathcal{H}, \mathcal{I}\} = \mathcal{K} + |\sigma|\mathcal{K}_{01}$, so that conditions (1a) and (2a) hold for contracting flows under the stated conditions, while conditions (1b) and (2b) hold for expanding flows. □

An upper bound on the deviation from roundness can also be found.

THEOREM 4.5. *For a parabolic curvature flow with* $-\mathcal{K}_{10} > \epsilon \geq 0$ *and* $-\mathcal{K}_{10} \geq |\sigma||\mathcal{K}_{20}|$ *on* $[0, t_1) \times S^2$,

$$|\sigma(t)| \leq \max_{S^2} |\sigma(0)| e^{-\epsilon t}.$$

PROOF. This follows from the Main Theorem by setting $\mathcal{H} = |\sigma|$ and noting that $A = -\mathcal{K}_{10}$, $B = -|\sigma|\mathcal{K}_{20}$ and $\{\mathcal{H}, \mathcal{I}\} = -|\sigma|\mathcal{K}_{10}$. □

Thus, strictly parabolic flows that satisfy $\mathcal{K}_{10}^2 \geq |\sigma|^2\mathcal{K}_{20}^2$ and exist for all time tend to a totally umbilic surface. For expanding flows, this would be a plane at infinity; for contracting flows, this would be a round sphere.

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4.3. Non-existence of homothetic solitons. A homothetic soliton is a surface S that satisfies the equation

$$\lambda r = \mathcal{K}, \tag{4.4}$$

for some constant λ, which is positive if the flow is contracting and negative if the flow is expanding, such that if we flow S by $\mathcal{K}$, then it simply scales the surface about the origin.

THEOREM 4.6. *If at $t = 0$ the surface S is a round sphere, then for as long as the classical curvature flow exists, it remains a round sphere with radius $R(t)$ evolving by*

$$\frac{dR}{dt} = -\mathcal{K}(R, 0).$$

PROOF. It is well known that the only convex totally umbilic surface is a round sphere. That is, S is umbilic if and only if $|\sigma| = 0$, which by Equation (2.6) implies that $\psi = \text{constant} = R$.

By a translation, let the centre of the initial sphere lie at the origin, so that $F(0) = 0$ and $r(0) = R_0$. By the evolution equations, we see that

$$\frac{\partial F}{\partial t}(0) = 0,$$

and so, by uniqueness of solutions to ODEs, it remains a round sphere centred at the origin, $r = R(t)$ and the flow of the radius is as stated. □

We now prove a nonexistence result for homothetic solitons.

THEOREM 4.7. *The only homothetic soliton for a contracting parabolic classical curvature flow with $\mathcal{K} + |\sigma|\mathcal{K}_{01} \geq 0$ and $\mathcal{K}_{01} + |\sigma|\mathcal{K}_{02} \geq 0$ is the evolving round sphere given in Theorem 4.6.*

Similarly, the only homothetic soliton for an expanding parabolic classical curvature flow with $\mathcal{K} + |\sigma|\mathcal{K}_{01} \leq 0$ and $\mathcal{K}_{01} + |\sigma|\mathcal{K}_{02} \leq 0$ is the evolving round sphere.

PROOF. Differentiating Equation (4.4) in a manner similar to the computations in Proposition 3.2,

$$\begin{aligned}
\lambda\psi = \frac{1}{2}(1 + \xi\bar{\xi})^2\Big[&-\mathcal{K}_{10}\partial\bar{\partial}\psi + \frac{1}{2}\mathcal{K}_{01}(e^{-i\phi}\partial\partial\psi + e^{i\phi}\bar{\partial}\bar{\partial}\psi) + \frac{1}{|\sigma|}(\mathcal{K}_{01} + |\sigma|\mathcal{K}_{20})\partial\psi\bar{\partial}\psi \\
&- \frac{1 + \xi\bar{\xi}}{|\sigma|}\mathcal{K}_{01}\Big(e^{-i\phi}\partial\psi\partial\Big(\frac{|\sigma|}{1 + \xi\bar{\xi}}\Big) + e^{i\phi}\bar{\partial}\psi\bar{\partial}\Big(\frac{|\sigma|}{1 + \xi\bar{\xi}}\Big)\Big) \\
&- \mathcal{K}_{11}(\partial\psi\bar{\partial}|\sigma| + \bar{\partial}\psi\partial|\sigma|) - \Big(\mathcal{K}_{02} + \frac{\mathcal{K}_{01}}{|\sigma|}\Big)\partial|\sigma|\bar{\partial}|\sigma|\Big] \\
&- \mathcal{K} - |\sigma|\mathcal{K}_{01}.
\end{aligned}$$

At the maximum and minimum value of ψ, therefore,

$$\lambda\psi = \triangle_\phi\psi - \frac{1}{2}(1 + \xi\bar{\xi})^2\Big(\mathcal{K}_{02} + \frac{\mathcal{K}_{01}}{|\sigma|}\Big)\partial|\sigma|\bar{\partial}|\sigma| - \mathcal{K} - |\sigma|\mathcal{K}_{01},$$

so, under the assumptions of Theorem 1.5, at a maximum of a contracting flow or a minimum of an expanding flow, $\psi \leq 0$, which is impossible. □

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5. Examples

In this section, we consider the following classical curvature flows:

(1) powers of mean curvature flow

$$\mathcal{K} = \pm H^n = \pm\Big(\frac{r_1 + r_2}{r_1 r_2}\Big)^n = \pm\frac{\psi^n}{(\psi^2 - |\sigma|^2)^n};$$

(2) powers of Gauss curvature flow

$$\mathcal{K} = \pm K^n = \pm\frac{1}{(r_1 r_2)^n} = \pm\frac{1}{(\psi^2 - |\sigma|^2)^n};$$

(3) power of mean radius of curvature flow

$$\mathcal{K} = \pm\Big(\frac{H}{K}\Big)^n = \pm(r_1 + r_2)^n = \pm\psi^n;$$

(4) linear Weingarten flow

$$\mathcal{K} = a + 2bH + cK = a + \frac{c + 2b(r_1 + r_2)}{r_1 r_2} = a + \frac{2b\psi + c}{\psi^2 - |\sigma|^2}.$$

We take the positive sign on the first three flows for $n > 0$ (and the negative sign for $n < 0$), and a, b, c are positive constants. In fact, the linear Weingarten flow is used as an abrasion model under the assumptions that $a = 1$ and $b^2 > c > 0$, when it is called the Bloore flow [2, 6, 8]. We do not require this restriction and our results hold for the Bloore flow.

THEOREM 5.1. *Consider these flows: powers of mean curvature, Gauss curvature, powers of mean radius of curvature and the linear Weingarten flow.*

These flows are all parabolic. Linear Weingarten flow is convex, as are powers of mean curvature for $n \geq -1$, powers of Gauss curvature for $n \geq \frac{1}{2}$ and powers of mean radius of curvature for $n \geq -1$. Powers of mean radius of curvature are concave for $n \leq -1$.

For each of these flows, we have the following estimate:

$$|\mathcal{K}(t)| \geq \min_{S^2} |\mathcal{K}(0)|,$$

while for the first three flows with $n > 0$ and the last flow for all positive a, b and c,

$$\psi(t) \leq \max_{S^2} \psi(0).$$

For these values, the flows do not admit homothetic solitons, other than round spheres.

For negative powers of mean radius of curvature we have $\psi(t) \geq \min_{S^2} \psi(0)$ and there are no homothetic solitons, other than round spheres.

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PROOF. By direct computation we find

	Power of mean curvature	Power of Gauss curvature	Power of mean RoC	Linear Weingarten
$\mathcal{K}$	$\pm\dfrac{\psi^n}{(\psi^2-\lvert\sigma\rvert^2)^n}$	$\pm\dfrac{1}{(\psi^2-\lvert\sigma\rvert^2)^n}$	$\pm\dfrac{1}{\psi^n}$	$a+\dfrac{2b\psi+c}{\psi^2-\lvert\sigma\rvert^2}$
$\mathcal{K}_{10}$	$-\dfrac{\lvert n\rvert\psi^{n-1}(\psi^2+\lvert\sigma\rvert^2)}{(\psi^2-\lvert\sigma\rvert^2)^{n+1}}$	$-\dfrac{2\lvert n\rvert\psi}{(\psi^2-\lvert\sigma\rvert^2)^{n+1}}$	$-\dfrac{\lvert n\rvert}{\psi^{n+1}}$	$-\dfrac{2(b(\psi^2+\lvert\sigma\rvert^2)+c\psi)}{(\psi^2-\lvert\sigma\rvert^2)^2}$
$\mathcal{K}_{01}$	$\dfrac{2\lvert n\rvert\lvert\sigma\rvert\psi^n}{(\psi^2-\lvert\sigma\rvert^2)^{n+1}}$	$\dfrac{2\lvert n\rvert\lvert\sigma\rvert}{(\psi^2-\lvert\sigma\rvert^2)^{n+1}}$	0	$\dfrac{2\lvert\sigma\rvert(c+2b\psi)}{(\psi^2-\lvert\sigma\rvert^2)^2}$

In each case ,$-\mathcal{K}_{10} > |\mathcal{K}_{01}|$ and so, as long as the surface remains convex, the flows are parabolic. Thus the first estimate follows from Theorem 1.2.

Moving to second derivatives,

	Power of mean curvature	Power of Gauss curvature
$\mathcal{K}_{11}$	$-\dfrac{2\lvert n\rvert\lvert\sigma\rvert\psi^{n-1}[(n+2)\psi^2+n\lvert\sigma\rvert^2]}{(\psi^2-\lvert\sigma\rvert^2)^{n+2}}$	$-\dfrac{4\lvert n\rvert(n+1)\lvert\sigma\rvert}{(\psi^2-\lvert\sigma\rvert^2)^{n+2}}$
$\mathcal{K}_{20}$	$\dfrac{\lvert n\rvert\psi^{n-2}[(n+1)\psi^4+2(n+2)\psi^2\lvert\sigma\rvert^2+(n-1)\lvert\sigma\rvert^4]}{(\psi^2-\lvert\sigma\rvert^2)^{n+2}}$	$\dfrac{2\lvert n\rvert[(2n+1)\psi^2+\lvert\sigma\rvert^2]}{(\psi^2-\lvert\sigma\rvert^2)^{n+2}}$
$\mathcal{K}_{02}$	$\dfrac{2\lvert n\rvert\psi^n(\psi^2+(2n+1)\lvert\sigma\rvert^2)}{(\psi^2-\lvert\sigma\rvert^2)^{n+2}}$	$\dfrac{2\lvert n\rvert[\psi^2+(2n+1)\lvert\sigma\rvert^2]}{(\psi^2-\lvert\sigma\rvert^2)^{n+2}}$
$\lvert\mathrm{Hess}(\mathcal{K})\rvert$	$\dfrac{2n^2(n+1)\psi^{2n-2}}{(\psi^2-\lvert\sigma\rvert^2)^{2n+1}}$	$\dfrac{4n^2(2n+1)}{(\psi^2-\lvert\sigma\rvert^2)^{2n+2}}$

and

	Power of mean RoC	Linear Weingarten
$\mathcal{K}_{11}$	0	$-\dfrac{4\lvert\sigma\rvert[b(3\psi^2+\lvert\sigma\rvert^2)+2c\psi]}{(\psi^2-\lvert\sigma\rvert^2)^3}$
$\mathcal{K}_{20}$	$\dfrac{\lvert n\rvert(n+1)}{\psi^{n+2}}$	$\dfrac{2[2b\psi(\psi^2+3\lvert\sigma\rvert^2)+c(3\psi^2+\lvert\sigma\rvert^2)]}{(\psi^2-\lvert\sigma\rvert^2)^3}$
$\mathcal{K}_{02}$	0	$\dfrac{2[2b\psi(\psi^2+3\lvert\sigma\rvert^2)+c(\psi^2+3\lvert\sigma\rvert^2)]}{(\psi^2-\lvert\sigma\rvert^2)^3}$
$\lvert\mathrm{Hess}(\mathcal{K})\rvert$	0	$\dfrac{4[4b^2(\psi^2-\lvert\sigma\rvert^2)+8bc\psi\lvert\sigma\rvert+3c^2]}{(\psi^2-\lvert\sigma\rvert^2)^4}$

Convexity for the stated values of n follows from checking that $|\mathrm{Hess}(\mathcal{K})| \geq 0$ and $\mathcal{K}_{20} \geq 0$ for each flow.

Finally, the second estimate and the nonexistence of homothetic solutions follows from Theorems 1.3 and 1.5, respectively, as long as the conditions $\mathcal{K} + |\sigma|\mathcal{K}_{01} \geq 0$ and $\mathcal{K}_{01} + |\sigma|\mathcal{K}_{02} \geq 0$ for the contracting flows and $\mathcal{K} + |\sigma|\mathcal{K}_{01} \leq 0$ and $\mathcal{K}_{01} + |\sigma|\mathcal{K}_{02} \leq 0$ for the expanding flows. □

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