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ON THE COALGEBRAIC RING AND BOUSFIELD–KAN SPECTRAL SEQUENCE FOR A LANDWEBER EXACT SPECTRUM

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Abstract We construct a Bousfield–Kan (unstable Adams) spectral sequence based on an arbitrary (and not necessarily connective) ring spectrum E with unit and which is related to the homotopy groups of a certain unstable E completion $X^{\wedge}_{E}$ of a space X. For E an $\mathbb{S}$-algebra this completion agrees with that of the first author and Thompson. We also establish in detail the Hopf algebra structure of the unstable cooperations (the coalgebraic module) $E_*(\underline{E}_*)$ for an arbitrary Landweber exact spectrum E, extending work of the second author with Hopkins and with Turner and giving basis-free descriptions of the modules of primitives and indecomposables. Taken together, these results enable us to give a simple description of the E_2-page of the E-theory Bousfield–Kan spectral sequence when E is any Landweber exact ring spectrum with unit. This extends work of the first author and others and gives a tractable unstable Adams spectral sequence based on a v_n-periodic theory for all n.

Keywords: Bousfield–Kan spectral sequence; unstable Adams spectral sequence; Hopf rings; coalgebraic algebra; Landweber exact; unstable completions

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1. Introduction

An unstable Adams spectral sequence computes homotopy-theoretic information for a space X from homological information. More specifically, such a spectral sequence based on a homology theory $E_*(-)$ seeks, under certain hypotheses, to compute the homotopy of an appropriate E-completion of X from an Ext group (in a suitable category) involving $E_*(X)$. This paper identifies, for E a general ring spectrum with unit, an unstable E-completion $X^{\wedge}_{E}$ of X and an associated E-theory Bousfield–Kan spectral sequence with E_2-term the homology of a certain unstable cobar complex. When E is an arbitrary Landweber exact spectrum [**23**] we obtain a more tractable description of the E_2-term, and, when E additionally has the structure of an $\mathbb{S}$-algebra in the sense of [**14**], our completion $X^{\wedge}_{E}$ and spectral sequence agree with those of [**5**]. In order to obtain the description of the E_2-term we prove a number of results on the generalized homology of the spaces in the Ω spectrum for a Landweber exact theory E, that is, on the coalgebraic ring $F_*(\underline{E}_*)$. These results are of independent interest (see, for example, [**15**]).

The first example of an unstable Adams spectral sequence based on a theory E other than ordinary homology was that of the first author with Curtis and Miller [**6**] which considered the case of a connective theory E and concentrated in particular on the case of BP-theory. This provided a sequence that converged to the p-localization of the unstable homotopy of an odd-dimensional sphere and identified the E_2-term as an Ext group in a non-abelian category of unstable $BP_*(BP)$-coalgebras. Using results of Wilson [**26**], the E_2-term was given a simpler, and more computationally practical, interpretation as the homology of an unstable cobar complex, which could be further considered [**4**] as the homology of a certain sub-complex of the *stable* cobar complex. This spectral sequence and subsequent variations were generalizations of that of Bousfield and Kan [**9**] and we refer throughout to all these models as Bousfield–Kan spectral sequences.

A more mysterious gadget, however, is a Bousfield–Kan spectral sequence based on a *periodic* theory E. Theories E that one would naturally wish to consider include complex K-theory, the Johnson–Wilson theories $E(n)$ and the Morava E- and K-theories; for technical reasons one is probably going to make easiest headway with those theories E which are also Landweber exact, as in the connective example BP successfully dealt with by [**6**] and subsequent papers. With Thompson, the first author has developed a framework [**5**] to define and study sequences based on periodic theories. The requirements on E to set such a sequence up, to identify the E_2-term in a practical and computable manner, and to prove convergence to an identifiable object are, however, significant. In brief, convergence is proved, in appropriate cases, to a certain 'unstable E-completion' of the underlying space X, where this completion is defined as Tot of a certain cosimplicial space, and is defined only when E is represented by an $\mathbb{S}$-algebra in the sense of [**14**]. Of the example theories E listed above, to our knowledge this rules out all but complex K-theory and the Morava E-theories.

The understanding of the E_2-page of any of these Bousfield–Kan spectral sequences involves, in large part, having both good and well-understood structure in the unstable E-theory cooperation algebras, that is, in the *coalgebraic ring* [**20**] or *Hopf ring* [**25**] $E_*(\underline{E}_*)$, where the $\underline{E}_r$ are the spaces in the Ω-spectrum for E-theory. In [**5**] the E_2-term was identified in a practical manner under the hypotheses that each $E_*(\underline{E}_r)$ was free as an E_*-module and that the sub-module of primitives $PE_*(\underline{E}_r)$ inject under infinite stabilization in the stable cooperation ring $E_*(E)$. Work by the second author and Hopkins [**16**] showed that these hypotheses were satisfied for a Landweber exact theory E whose coefficients were 'not too large': this included the cases of K-theory and the Johnson–Wilson theories $E(n)$, but not the $\mathbb{S}$-algebra examples of the Morava E-theories. Between these two sets of requirements on E—for convergence and for the computation of the E_2-term—fully satisfactory results in [**5**] were obtainable only when E was taken as complex K-theory.

The main results of this paper fall into three sections. In § 2 we study in depth the homology and generalized homology of the spaces $\underline{E}_r$ in the Ω-spectrum for an arbitrary Landweber exact theory E, assuming only that the coefficients E_* are concentrated in even dimensions (an assumption which fails only in rather artificial examples). These results extend those of [**16**], removing the size restrictions on the coefficients, proving, for example (Theorem 2.4), that the algebras $F_*(\underline{E}_r)$ for a wide class of homology theories $F_*(-)$ are polynomial or exterior for r even or odd, respectively. However, they go further than the type of results in [**16**], also giving basis-free descriptions (Theorem 2.11) of the modules of primitives and indecomposables associated with the $F_*(\underline{E}_r)$. Unusually for results on the coalgebraic ring $F_*(\underline{E}_*)$ for theories F and E, these results give explicit descriptions of the individual Hopf algebras $F_*(\underline{E}_r)$, rather than just implicit descriptions in terms of the global object $F_*(\underline{E}_*)$. We also relate (Corollary 2.12) the modules $PF_*(\underline{E}_r)$ and $QF_*(\underline{E}_r)$ to the primitives and indecomposables of the universal example $MU_*(\underline{MU}_*)$. These results are of independent interest in the study of the homology of Ω-spectra, having applications, for example, to group cohomology (see [**15**] and [**19**]). In the case of certain completed spectra, such as the Morava E-theories and the Baker–Würgler completions $\widehat{E(n)}$ [**3**], these results have parallels with those of [**17**], where the homological effects of completion on Ω-spectra are examined using rather different methods.

In § 3 we suppose E merely to be a ring spectrum with a unit. For a space X we define a notion (Definition 3.2) of E-completion of X, denoted $X^\wedge_E$. If E is an $\mathbb{S}$-algebra, then the space $X^\wedge_E$ turns out to be homotopy equivalent to the E-completion of X as defined in [**5**]. The E-theory Bousfield–Kan spectral sequence related to the homotopy groups of this space $X^\wedge_E$ is introduced and we identify (Theorem 3.8) the E_2-page as the homology of an unstable cobar complex.

The results of § 3 are very general but as they stand offer small hope for specific computation. In § 4 we build on them in the special case of a Landweber exact ring spectrum (with unit), using the work of § 2 on the coalgebraic ring for such a spectrum. The main result here is a 'change of rings' theorem that identifies (Theorem 4.1) the E_2-page of the E-theory Bousfield–Kan spectral sequence of § 3 as an Ext group in a convenient, moreover abelian, category. This applies to spaces X such as torsion free H-spaces and odd-dimensional spheres. We note also (Remark 4.11) that a similar result holds for spaces such as ΩS^{2n+1}, though care is needed for such examples as, by the work of § 2, the relevant Hopf algebras $E_*(\underline{E}_{2r})$ in the computation are not primitively generated. Taken together, the results of this article allow for the construction and description of an unstable Adams spectral sequence based on a v_n-periodic theory for any positive integer n, extending the framework of [**5**] which established the v_1-periodic case.

Notation

The convention we use for denoting spaces, spectra, etc., related to a theory E is as follows. For a theory E we write $E_*(-)$ and $E^*(-)$ for the generalized E-homology and cohomology, $\boldsymbol{E}$ for the associated spectrum when we wish to consider it as an explicit object in the stable category, and $\underline{E}_r$ and $\underline{E}_*$ for the spaces in the Ω-spectrum and for the Ω-spectrum itself. Thus the space $\underline{E}_r$ represents the cohomological functor $E^r(-)$ in the sense that $E^r(X) = [X, \underline{E}_r]$ for any space X. The $\underline{E}_r$ are related by equivalences $\Omega\underline{E}_{r+1} \simeq \underline{E}_r$.

For the theories E considered in this article, the spaces $\underline{E}_r$ are frequently not path connected. It is convenient to have a notation for a single connected component, and we shall follow the usual convention of writing $\underline{E}'_r$ for this space. We then have the relation $\underline{E}_r = \underline{E}'_r \times E^r$, where E^r is the rth E-cohomology group of a point. This in turn leads to the homological relation $F_*(\underline{E}_r) = F_*(\underline{E}'_r) \otimes_{F_*} F_*[E^r]$ for a homology theory $F_*(-)$.

2. The coalgebraic module $F_*(\underline{E}_*)$

Throughout this section we shall assume that $\underline{E}_*$ is an Ω-spectrum representing a Landweber exact cohomology theory [**23**]. Such theories include the examples of complex cobordism MU and the Brown–Peterson theories BP [**1**], the Johnson–Wilson theories $E(n)$ [**22**] and their I_n-adic completions $\widehat{E(n)}$ [**3**] as well as Morava E-theory, complex K-theory, various forms of elliptic cohomology [**24**] and their completions. For simplicity in the statement of our results, we assume the coefficients E_* are concentrated in even degrees; this is satisfied by all standard examples including those just mentioned. As $\boldsymbol{E}$ is necessarily a module spectrum over $\boldsymbol{MU}$, the mod p homology $H_*(\underline{E}_*; \mathbb{F}_p)$ will be a *coalgebraic module* over both $H_*(\underline{MU}_*; \mathbb{F}_p)$ and $\mathbb{F}_p[MU^*]$ in the sense of [**20**]; if, as will in fact generally be the case, $\boldsymbol{E}$ is a ring spectrum, $H_*(\underline{E}_*; \mathbb{F}_p)$ will be a coalgebraic ring (Hopf ring) and a coalgebraic algebra over these objects as well. If $\boldsymbol{E}$ is a p-local spectrum, then similar statements hold on replacement of MU by BP. We shall adopt the notation and results on $H_*(\underline{MU}_*; \mathbb{F}_p)$ and $H_*(\underline{MU}_*; \mathbb{Z}_{(p)})$ that are found in [**25**] and the notions of coalgebraic algebra, including a discussion of the algebraic structures of 'ring–rings' such as $F_*[E^*]$, in [**20**, **25**].

We shall use in particular the facts that the algebras $H_*(\underline{MU}'_r; \mathbb{F}_p)$ and $H_*(\underline{MU}'_r; \mathbb{Z}_{(p)})$ are polynomial or exterior as r is even, respectively odd, and that the p-local groups $H_*(\underline{MU}_r; \mathbb{Z}_{(p)})$ are torsion free. These results were essentially first proved (for the analogous case of BP) by Wilson in [**26**], and re-established using Hopf ring technology by Ravenel and Wilson in [**25**]; both these approaches were heavily basis dependent. A more direct, basis-independent proof of these facts was subsequently published by Chan [**11**] (see also [**27**]). The arguments of [**11**] are sufficient for the results of this paper.

The work of [**20**] establishes, in particular, a tensor product $\bar{\otimes}$ in the category of $\mathbb{F}_p[MU^*]$ coalgebraic modules; note that this is quite distinct from the tensor product of the underlying $\mathbb{F}_p$ coalgebras. The main theorem of [**21**] tells us the following.

Theorem 2.1. $H_*(\underline{E}_*; \mathbb{F}_p) \cong H_*(\underline{MU}_*; \mathbb{F}_p) \underset{\mathbb{F}_p[MU^*]}{\bar{\otimes}} \mathbb{F}_p[E^*]$.

Corollary 2.2. *$H_*(\underline{E}_{2r+1}; \mathbb{F}_p)$ is an exterior algebra.*

Proof. Consider the indecomposable quotient $QH_*(\underline{E}_{2r+1}; \mathbb{F}_p)$. Unwinding the definition of $\bar{\otimes}$, elements in this quotient are represented by sums of $\circ$ products of elements of the form $q \,\bar{\otimes}\, x$, where q represents an indecomposable in an odd MU space and $x \in \mathbb{F}_p[E^*] = H_0(\underline{E}_*; \mathbb{F}_p)$ (this follows from the fact that E^* is concentrated in even dimensions). As $QH_*(\underline{MU}_s; \mathbb{F}_p)$ lies in odd homological dimensions if s is odd [**25**], we conclude that $QH_*(\underline{E}_{2r+1}; \mathbb{F}_p)$ lies in odd homological dimension.

Thus any finite-dimensional sub-algebra of $H_*(\underline{E}_{2r+1};\mathbb{F}_p)$ lies in a finite-dimensional sub-Hopf algebra generated by odd-dimensional elements, and so is an exterior algebra. As $H_*(\underline{E}_{2r+1};\mathbb{F}_p)$ is the colimit of its finite-dimensional sub-algebras, the result follows. $\square$

Corollary 2.3. *$H_*(\underline{E}'_{2r};\mathbb{F}_p)$ is a polynomial algebra and homology suspension induces an isomorphism $QH_*(\underline{E}_{2r};\mathbb{F}_p) \cong QH_*(\underline{E}_{2r+1};\mathbb{F}_p)$.*

Proof. This is an immediate consequence of Corollary 2.2 and the homology Eilenberg–Moore spectral sequence [**13**]

$$\mathrm{Cotor}^{H_*(\underline{E}_{2r+1};\mathbb{F}_p)}(\mathbb{F}_p,\mathbb{F}_p) \Longrightarrow H_*(\underline{E}_{2r};\mathbb{F}_p).$$

As $H_*(\underline{E}_{2r+1};\mathbb{F}_p)$ is exterior, the E^2-page is already polynomial in positive dimensions and is concentrated in even degrees. The sequence thus collapses and the result follows. $\square$

We require knowledge of $F_*(\underline{E}_*)$ for more general theories F than just mod p homology; for example, we need results with $F = E$ for the unstable homotopy spectral sequences later, but other examples are of importance too. For the remainder of this article we shall assume that F is a p-local ring spectrum, with coefficients torsion free and concentrated in even dimensions. We will also have occasion to consider a version of a theory F with coefficients reduced mod p; such homology of a space X will be denoted $F_*(X;\mathbb{F}_p)$.

Theorem 2.4. *Suppose $\underline{E}_*$ and F are as above. Then $F_*(\underline{E}'_s)$ is a free F_* module, with algebra structure polynomial for s even and exterior for s odd.*

Proof. Begin with the case $F = H\mathbb{Z}_{(p)}$. As $H_*(\underline{E}'_{2r};\mathbb{F}_p)$ is polynomial, and in even dimensions, its generators lift to polynomial generators of the torsion-free algebra $H_*(\underline{E}'_{2r};\mathbb{Z}_{(p)})$. From this we can deduce that the homology of the odd spaces $H_*(\underline{E}_{2r+1};\mathbb{Z}_{(p)})$ are torsion-free, exterior algebras, generated by the suspensions of generators of $H_*(\underline{E}_{2r};\mathbb{Z}_{(p)})$. The result for general F follows by a collapsing Atiyah–Hirzebruch spectral sequence argument. $\square$

Corollary 2.5. *For $\underline{E}_*$ and F as above, $F_*(\underline{E}_*)$ is an $F_*(\underline{MU}_*)$ coalgebraic module; if $\boldsymbol{E}$ is a ring spectrum, $F_*(\underline{E}_*)$ is also a coalgebraic ring.*

Proof. This result is a standard and formal argument (see, for example, [**25**]) and follows as soon as a Künneth theorem

$$F_*(\underline{E}_r \times \underline{E}_s) \cong F_*(\underline{E}_r) \underset{F_*}{\otimes} F_*(\underline{E}_s)$$

is established. This holds by the freeness result of (2.4). $\square$

For $\boldsymbol{E}$ a ring spectrum, recall the algebraic model coalgebraic rings $F^R_*(\underline{E}_*)$ and $F^Q_*(\underline{E}_*)$ constructed in [**25**] and [**18**], respectively. The former, $F^R_*(\underline{E}_*)$, is the free

$F_*[E^*]$ coalgebraic algebra generated by certain classes arising from the complex orientation on E, modulo specific relations arising from the interaction of the E and F formal group laws; the latter, $F_*^Q(\underline{E}_*)$, can be constructed as a certain sub-coalgebraic ring of the rational object $F\mathbb{Q}_*(\underline{E\mathbb{Q}}_*)$. There are natural maps

$$F_*(\underline{E}_*) \xleftarrow{\tau} F_*^R(\underline{E}_*) \to F_*^Q(\underline{E}_*).$$

Corollary 2.6. *For E and F as above, there are isomorphisms of coalgebraic rings*

$$F_*(\underline{E}_*) \cong F_*^R(\underline{E}_*) \cong F_*^Q(\underline{E}_*).$$

Proof. That τ is an isomorphism follows from [**21**] and the corresponding result for MU [**25**]. This, together with the fact that $F_*(\underline{E}_*)$ is torsion free by Theorem 2.4, gives the second isomorphism using [**18**, Corollary 6.3]. □

We seek now to describe the modules of primitives $PF_*(\underline{E}_s)$ and indecomposables $QF_*(\underline{E}_s)$ for the Hopf algebras $F_*(\underline{E}_s)$. One of the main ideas of [**18**] is that when $F_*(\underline{E}_*)$ is torsion free (as here), simple descriptions of its algebra structure can be obtained by identifying its image in the rational coalgebraic ring $F\mathbb{Q}_*(\underline{E\mathbb{Q}}_*)$. The following result allows analogous descriptions of $PF_*(\underline{E}_s)$ and $QF_*(\underline{E}_s)$ by embedding these modules in the *stable* object $F_*(E)$.

Proposition 2.7. *For E and F as above, homology suspension induces monomorphisms*

$$PF_*(\underline{E}_s) \longrightarrow F_{*-s}(E),$$
$$QF_*(\underline{E}_s) \longrightarrow F_{*-s}(E).$$

Proof. The proof is essentially given by the following commutative diagram:

$$\begin{tikzcd}
PF_*(\underline{E}_s) \arrow[r, "\iota"] \arrow[d, "P\rho_*"] & QF_*(\underline{E}_s) \arrow[r, "\sigma_s"] \arrow[d, "Q\rho_*"] & F_{*-s}(E) \arrow[d, "\rho_*"] \\
PF\mathbb{Q}_*(\underline{E}_s) \arrow[r, "\iota^{\mathbb{Q}}"] & QF\mathbb{Q}_*(\underline{E}_s) \arrow[r, "\sigma_s^{\mathbb{Q}}"] & F\mathbb{Q}_{*-s}(E)
\end{tikzcd}$$

The map ι is the natural map from primitives to indecomposables; as $F_*(\underline{E}_*)$ is torsion free, by (2.4), this is an inclusion. Infinite homology suspension from the sth space is denoted by σ_s and ρ indicates the rationalization map $F \to F\mathbb{Q}$. Then Theorem 2.4 tells us that the middle vertical map $Q\rho_*$ is a monomorphism, and the analysis of rational coalgebraic rings in [**18**] shows that the suspension $\sigma_s^{\mathbb{Q}} : QF\mathbb{Q}_*(\underline{E}_s) \to F\mathbb{Q}_{*-s}(E)$ is also monic. □

This result allows us to give a basis-free description of $PF_*(\underline{E}_s)$ and $QF_*(\underline{E}_s)$ as a sub-module of the stable module $F_*(E)$. This generalizes the construction of [**4**, Definition 2.13]. First though we need to recall some standard notation for elements in coalgebraic rings (see [**25**] for further details).

Notation 2.8. The published literature delivers us two incompatible meanings for the notation b_i. At the *unstable* level, there are classes $b_i \in F_{2i}(\underline{MU}_2)$ defined as the images in F-homology of certain generating classes $\beta_i \in F_{2i}(\mathbb{C}P^\infty)$ under the complex orientation $\mathbb{C}P^\infty \to \underline{MU}_2$ for complex cobordism. When localized at a prime p (as here) it is customary to denote the class b_{p^i} by $b_{(i)} \in F_{2p^i}(\underline{MU}_2)$. We shall use this latter notation throughout, reserving the names b_i (without brackets) for the infinite suspensions (see below) of the $b_{(i)}$, i.e. for elements of the *stable* module $F_*(E)$. Thus, in our notation, $b_i \in F_{2p^i-2}(E)$. There is also the suspension element $e \in F_1(\underline{MU}_1)$ with the relation $e \circ e = -b_{(0)}$. For v a homogeneous element of MU_*, say $v \in MU_{|v|} = \pi_0(\underline{MU}_{-|v|})$, we have the element $[v] \in F_0(\underline{MU}_{-|v|})$, its Hurewicz image; note that $v \in MU_{|v|} = MU^{-|v|}$ and $|v| \geqslant 0$. By [**25**], $H_*(\underline{MU}_*; \mathbb{Z}_{(p)})$ is generated as a coalgebraic ring by the classes $[v]$, e and $b_{(s)}$. The algebraic models $F_*^R(\underline{E}_*)$ are by definition generated over F_* by the analogous elements $[v]$, e and $b_{(s)}$ with the v homogeneous elements of E_*, and by Corollary 2.6 we know that the corresponding classes also generate $F_*(\underline{E}_*)$.

The suspension homomorphisms, $\sigma_s : F_*(\underline{E}_s) \to F_{*-s}(E)$ send $b_{(i)}$ to $b_i \in F_{2p^i-2}(E)$, kill $*$ products and take $\circ$ products to multiplication in $F_*(E)$; note, in particular, that $\sigma_2 : b_{(0)} \mapsto 1$. For v a homogeneous element of E_*, we denote also by v the image of $[v]$ under suspension in $F_{|v|}(E)$; this is additionally the image of v in $F_*(E)$ under the right unit $F_* \to F_*(E)$.

We shall denote the free E_* module generated by a class ι_n in dimension n by M_n, or, when we need to indicate the Ω spectrum being considered, by M_n^E. This is useful for keeping track of the domain of the stabilization map, $\sigma_s : F_*(\underline{E}_s) \to F_{*-s}(E)$ and is accomplished by defining the range of σ_s to be

$$F_{*-s}(E) \underset{E_*}{\otimes} M_s.$$

In this notation $b_{(i)} \in F_{2p^i}(\underline{E}_2)$ maps to $b_i \otimes \iota_2$ while a class such as $b_{(i)} \circ [v] \in F_{2p^i}(\underline{E}_{2-|v|})$ maps to $b_i \otimes v\iota_{2-|v|}$. Notice that the suspension homomorphisms now preserve dimension.

For each finite sequence of non-negative integers $I = (i_1, i_2, \ldots, i_n)$ we write b^I for the stable element

$$b^I = b_1^{i_1} b_2^{i_2} \cdots b_n^{i_n} \in F_*(E).$$

The *length* of I is the integer $l(I) = i_1 + \cdots + i_n$. Write $b^{\circ I}$ for the *unstable* element

$$b_{(1)}^{\circ i_1} \circ \cdots \circ b_{(n)}^{\circ i_n} \in F_*(\underline{E}_{2l(I)}).$$

Of course $\sigma_{2l(I)} : b^{\circ I} \mapsto b^I$. More generally, any element of the form $(b_{(0)}^{\circ r} \circ b^{\circ I} + \text{decomposables})$ suspends to b^I.

Definition 2.9. Let M be a free, graded E_*-module. Write $U_F(M)$ for the sub-F_*-module of

$$F_*(E) \underset{E_*}{\otimes} M$$

spanned by all elements of the form $b^I \otimes m$ where $2l(I) < |m|$.

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Definition 2.10. Let M be a free, graded E_*-module. Write $V_F(M)$ for the sub-F_*-module of

$$F_*(E) \underset{E_*}{\otimes} M$$

spanned by all elements of the form $b^I \otimes m$ where $2l(I) \leqslant |m|$.

The special case of the next result for $E = F = BP$ was proved in [**4**, **6**].

Theorem 2.11. *The image of the suspension homomorphism,*

$$\sigma_s : QF_*(\underline{E}_s) \to F_{*-s}(E) \underset{E_*}{\otimes} M_s \cong F_*(E)$$

lies in $V(M_s)$ *and*

$$\sigma_s : QF_*(\underline{E}_s) \to V(M_s)$$

is an isomorphism.

Furthermore, the image of $\sigma_s|_{PF_*(\underline{E}_s)}$ *lies in* $U(M_s)$ *and*

$$\sigma_s : PF_*(\underline{E}_s) \to U(M_s)$$

is an isomorphism.

Proof. We start with the identification of the image of the indecomposables $QF_*(\underline{E}_s)$ with $V_F(M_s)$. As σ_s on $QF_*(\underline{E}_s)$ is monomorphic this will prove the first statement. We begin with the case where s is even.

By Corollary 2.6 we know that any element of $QF_*(\underline{E}_s)$ can be written as an F_*-linear sum of elements of the form $b_{(0)}^{\circ r} \circ b^{\circ I} \circ [v]$ with $r \geqslant 0$. Such an element suspends to $b^I \otimes v\iota_s$. The condition that an element $b_{(0)}^{\circ r} \circ b^{\circ I} \circ [v]$ lies in the F-homology of the sth space $\underline{E}_s$ is that $2r + 2l(I) - |v| = s$, thus $2l(I) \leqslant |v| + s = |v\iota_s|$ and so the image of σ_s lies in $V_F(M_s)$.

Conversely, if $b^I \otimes v\iota_s$ lies in $V_F(M_s)$, then $2l(I) - |v| \leqslant s$ and so $b^I \otimes v\iota_s = \sigma_s(b_{(0)}^{\circ r} \circ b^{\circ I} \circ [v])$, where $2r = s + |v| - 2l(I) \geqslant 0$. Hence σ_s is onto $V_F(M_s)$ and the isomorphism for even spaces is proved.

The result for odd spaces is very similar; note that circle multiplication by e induces a one-to-one correspondence between $QF_*(\underline{E}_{2t})$ and $QF_*(\underline{E}_{2t+1})$.

The result for primitives is again similar and follows immediately after making the observation that $PF_*(\underline{E}_s)$ for even s is the F_*-linear span of elements of the form $b_{(0)}^{\circ r} \circ b^{\circ I} \circ [v]$ with $r > 0$. Also, for odd s there is an isomorphism $PF_*(\underline{E}_s) \cong QF_*(\underline{E}_s)$. $\square$

The F_*-module $QF_*(\underline{E}_s)$ is not, as it stands, an E_* module, but may be modified to be so. Looking at all the spaces together, the bigraded object $QF_*(\underline{E}_*)$ is an E_* module under the action $x \otimes v \mapsto x \circ [v]$ for $x \in QF_*(\underline{E}_*)$ and $v \in E_*$. (Verification that

$x \circ [v+w] = x \circ [v] + x \circ [w]$ in $QF_*(\underline{E}_*)$ is left as an exercise in coalgebraic modules: see the axioms listed in [**25**].)

We may modify the construction of $V_F(M_s)$ so as also to carry the action of E_* by considering the corresponding bigraded object $V_F(M_*)$ equipped with the action $(y \otimes v\iota_s) \otimes w \mapsto y \otimes vw\iota_{s-|w|}$. Define a global suspension map $\sigma : QF_*(\underline{E}_*) \to V_F(M_*)$ as σ_s on the component $QF_*(\underline{E}_s)$. With these definitions and the previous result it may easily be checked that σ is an F_*–E_* bimodule isomorphism. Similar constructions may be made and results established for the objects of primitives $PF_*(\underline{E}_*)$ and $U_F(M_*)$.

Our second description of the modules of primitives and indecomposables for $F_*(\underline{E}_s)$ can now be given in terms of a simple relation to those of the universal theories. As in the underlying philosophy of [**25**], etc., this requires us to consider all spaces $\underline{E}_s$ together.

Corollary 2.12. *Let E and F be as above. Then there are isomorphisms*

$$QF_*(\underline{E}_*) = F_* \underset{MU_*}{\otimes} QMU_*(\underline{MU}_*) \underset{MU_*}{\otimes} E_* = QF_*(\underline{MU}_*) \underset{MU_*}{\otimes} E_*,$$
$$PF_*(\underline{E}_*) = F_* \underset{MU_*}{\otimes} PMU_*(\underline{MU}_*) \underset{MU_*}{\otimes} E_* = PF_*(\underline{MU}_*) \underset{MU_*}{\otimes} E_*,$$

where $\otimes$ denotes the tensor product of modules in the standard sense. Assuming that E is p-local, analogous results hold on replacing MU by BP.

Proof. We prove the first line, concerning the indecomposable functor: the proof of the version involving the primitives is essentially identical. Note also that the equality

$$F_* \underset{MU_*}{\otimes} QMU_*(\underline{MU}_*) \underset{MU_*}{\otimes} E_* = QF_*(\underline{MU}_*) \underset{MU_*}{\otimes} E_*$$

follows immediately since each $MU_*(\underline{MU}_s)$ is a free (left) MU_* algebra and hence

$$F_* \underset{MU_*}{\otimes} QMU_*(\underline{MU}_*) = QF_*(\underline{MU}_*).$$

We show that

$$QF_*(\underline{E}_*) = QF_*(\underline{MU}_*) \underset{MU_*}{\otimes} E_*.$$

By Theorem 2.11 it suffices to show that

$$V_F(M_*^E) = V_F(M_*^{MU}) \underset{MU_*}{\otimes} E_*.$$

Of these, the left-hand side is the (bigraded) sub-F_*-module of

$$F_*(E) \underset{E_*}{\otimes} M_*^E$$

spanned in grading s by elements $b^I \otimes m\iota_s$ satisfying $2l(I) \leqslant |m\iota_s|$. As E is Landweber exact, and, by definition, M_*^E and M_*^{MU} are free over E_* and MU_*, respectively, in each grading,

$$F_*(E) \underset{E_*}{\otimes} M_*^E = F_*(MU) \underset{MU_*}{\otimes} E_* \underset{E_*}{\otimes} M_*^E = F_*(MU) \underset{MU_*}{\otimes} M_*^{MU} \underset{MU_*}{\otimes} E_*.$$

Under this equivalence, the element

$$b^I \otimes m\iota_s \in V_F(M_*^E) \subset F_*(E) \underset{E_*}{\otimes} M_*^E$$

is then identified with

$$b^I \otimes \iota_s \otimes m \in V_F(M_*^{MU}) \underset{MU_*}{\otimes} E_* \subset F_*(MU) \underset{MU_*}{\otimes} M_*^{MU} \underset{MU_*}{\otimes} E_*.$$

This is also onto

$$V_F(M_*^{MU}) \underset{MU_*}{\otimes} E_*,$$

as the map $MU \to E$ induces a left inverse. □

The constructions U_F and V_F may be extended to other E_* modules M. For an arbitrary non-negatively graded left E_*-module M let

$$F_1 \xrightarrow{f} F_0 \to M \to 0$$

be exact with F_0 and F_1 free over E_*. Then U_F may be extended to M by defining

$$U_F(M) = \operatorname{coker}(U_F(f) : U_F(F_1) \to U_F(F_0)).$$

V_F is similarly extended to such E_*-modules.

Proposition 2.13.

$$V_F(M_s \otimes \mathbb{Z}/p) \cong QF_*(\underline{E}_s; \mathbb{F}_p),$$
$$U_F(M_s \otimes \mathbb{Z}/p) \cong \operatorname{Im}(PF_*(\underline{E}_s; \mathbb{F}_p) \to QF_*(\underline{E}_s; \mathbb{F}_p)).$$

Proof. Since $F_*(\underline{E}_s)$ is a free algebra, there is a diagram with rows short exact:

$$\begin{tikzcd}
0 \arrow[r] & QF_*(\underline{E}_s) \arrow[r, "\times p"] \arrow[d, equal] & QF_*(\underline{E}_s) \arrow[r] \arrow[d, equal] & QF_*(\underline{E}_s; \mathbb{Z}/p) \arrow[r] & 0 \\
0 \arrow[r] & V_F(M_s) \arrow[r, "\times p"] & V_F(M_s) \arrow[r] & V_F(M_s \otimes \mathbb{Z}/p) \arrow[r] & 0
\end{tikzcd}$$

Hence there is an induced isomorphism $QF_*(\underline{E}_s; \mathbb{F}_p) \to V_F(M_s \otimes \mathbb{Z}/p)$. A similar proof gives the second isomorphism. □

Remark 2.14. Since in practice our cohomology theories tend to be $\mathbb{Z}_{(p)}$ local it can be advantageous to use BP generators. The generators $h_i = c(t_i)$, where the t_i are the standard generators for $BP_*(BP)$ and c denotes the canonical anti-isomorphism, have proven to be useful for unstable calculations. Following [**6**, 8.5] we may replace the generator b_i with h_i in Theorem 2.11.

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We conclude this section with an example to help clarify these definitions.

Example 2.15. Let $F = E = BP$. We claim that $ph_1 \otimes \iota_1$ defines a non-zero element in $V_{BP}(M_1 \otimes \mathbb{Z}/p) = QBP_*(\underline{BP}_1; \mathbb{F}_p)$, but which suspends to zero in $QBP_*(\underline{BP}_2; \mathbb{F}_p)$. To see that $ph_1 \otimes \iota_1$ is indeed in $V_{BP}(M_1 \otimes \mathbb{Z}/p)$, note that the right action formula tells us that $ph_1 = v_1 \cdot 1 - 1 \cdot v_1$. Thus

$$ph_1 \otimes \iota_1 = v_1 \otimes \iota_1 - 1 \otimes v_1 \cdot \iota_1,$$

an element of $V_{BP}(M_1)$. This element is not divisible by p in $V_{BP}(M_1)$, and so is not zero in $V_{BP}(M_1 \otimes \mathbb{Z}/p)$. On the other hand, $h_1 \otimes \iota_2$ is an element of $V_{BP}(M_1)$ and so $ph_1 \otimes \iota_2$ is p-divisible in $V_{BP}(M_1)$ and thus is zero in $V_{BP}(M_1 \otimes \mathbb{Z}/p)$. In general, $V_F(M_s \otimes \mathbb{Z}/p)$ is not a sub-module of $F_*(E)$: when working mod p the unstable classes do not necessarily inject into the stable module.

In a similar fashion the right action formula for ph_1 can be used to show that ph_1^n is a non-zero element in $V_{BP}(M_{2n-1} \otimes \mathbb{Z}/p)$ but one which suspends to zero in $V_{BP}(M_{2n} \otimes \mathbb{Z}/p)$.

Example 2.16. Consider the Araki generators $w_i \in BP_{2p^i-2}$, as in [**2**],

$$pm_n = \sum_{0 \leqslant j \leqslant n} m_i(w_{n-i}^{p^i}), \quad w_0 = p.$$

We prefer the Araki generators to the Hazewinkel generators because of the integral form of Ravenel's formulae

$$\sum^{F^*} h_j^{p^i} \cdot w_i = \sum^{F^*} w_j^{p^i} \cdot h_i.$$

Here $\sum^F c(\gamma_i)$ is the formal group sum, c is the canonical anti-isomorphism and $\sum^{F^*} \gamma_i = c(\sum^F c(\gamma_i))$ (i.e. $\sum^{F^*}$ looks like the usual formal group law, but the formal group coefficients act on the right). It is easy to check that the Ravenel formulae imply that $\sum^{F^*} h_j^{p^i} \cdot w_i = \sum^{F^*} w_j^{p^i} \cdot h_i$ also holds in $V_F(M_s^E)$ and $V_F(M_s^E \otimes \mathbb{Z}/p)$ with $s \geqslant 2$. There are similar formulae involving the Hazewinkel generators, but they are only true stably mod p. We do not know if the Hazewinkel generators satisfy similar, mod p formulae, unstably.

If $E = E(1)$, Adams's summand of p-local K-theory, or equivalently the first Johnson–Wilson theory and we take F to be H, integral homology, these formulae reduce to

$$\left(\sum^{F^*} h_j^p \cdot w_1\right) \otimes \iota_s = 0 \quad \text{in } V_{H_*}(M_s \otimes \mathbb{Z}/p) \text{ if } s \geqslant 2.$$

Using the grading, this implies that $h_j^p \cdot w_1 \otimes \iota_s = 0$ here. (Notice that $h_j^p \cdot w_1 \otimes \iota_2 = h_j^p \otimes w_1 \cdot \iota_2$ and $w_1 \cdot \iota_2$ has degree $2p$, so this class is defined.)

3. The Bousfield–Kan spectral sequence (BKSS) for $\boldsymbol{E}$-theory

Let $\mathcal{S}$ be the category of pointed CW complexes and suppose $\boldsymbol{E}$ is a ring spectrum with unit. Associated with $\boldsymbol{E}$ is a functor $T_E : \mathcal{S} \to \mathcal{S}$ given by sending X to $\Omega^\infty(\boldsymbol{E} \wedge \sum^\infty X)$.

There are natural transformations $\phi : 1_{\mathcal{S}} \to T_E$ and $\mu : T_E^2 = T_E \circ T_E \to T_E$ induced by the unit and the multiplication in E, respectively, and these make (T_E, ϕ, μ) a *triple* up to homotopy. See, for example, [**7**, § 2], [**5**, § 4] and [**8**] for details of the notions of triple, cotriple and their associated categories and derived functors, as used in this and the next section.

If $\boldsymbol{E}$ is an $\mathbb{S}$-algebra in the sense of [**14**] (for example, K-theory), it is shown in [**5**] that (T_E, ϕ, μ) is in fact a triple on the category $\mathcal{S}$. Following [**10**] there is then a cosimplicial space, $\boldsymbol{T}_E X$, with coface maps and codegeneracies denoted d^i and s^j, respectively. The *completion of* X *with respect to* $\boldsymbol{E}$ is taken as

$$X_E^\wedge = \mathrm{Tot}(\boldsymbol{T}_E X).$$

The E_2-page of the Bousfield–Kan spectral sequence associated with $X_E^\wedge$ is identified [**5**] with the homology of the *unstable cobar complex*,

$$E_2^s(X) = \pi^s \pi_* \boldsymbol{T}_E X = H^s(\pi_* \boldsymbol{T}_E X, \partial),$$

where $\pi_* \boldsymbol{T}_E X$ is considered as a cochain complex with coboundary map $\partial = \sum (-1)^i \pi_* d^i$.

We wish, however, to be able to consider an 'E-completion' of a space X and a corresponding E-theory Bousfield–Kan spectral sequence whenever E is an arbitrary ring spectrum with a unit. In this section we use the results of [**12**] to construct (3.2) a space $X_E^\wedge$ for any such E, and prove it to be homotopic to the construction in [**5**] if $\boldsymbol{E}$ is an $\mathbb{S}$-algebra. In Theorem 3.8 we identify the E_2-term of the E-theory Bousfield–Kan spectral sequence as an unstable cobar complex.

We recall the notion [**12**] of a *restricted cosimplicial space*, i.e. a 'cosimplicial space' without the codegeneracies.

Definition 3.1. Suppose that (T, ϕ) is an augmented functor on $\mathcal{S}$, i.e. a functor $T : \mathcal{S} \to \mathcal{S}$ equipped with a natural transformation $\phi : 1_{\mathcal{S}} \to T$. Let X be a space in $\mathcal{S}$. Define the *restricted cosimplicial space* $\hat{\boldsymbol{T}}X$ to be the restricted cosimplicial resolution with respect to T given by

$$(\hat{\boldsymbol{T}}X)^k = T^{k+1}X$$

in codimension k, and coface maps given by

$$((\hat{\boldsymbol{T}}X)^{k-1} \xrightarrow{d^i} (\hat{\boldsymbol{T}}X)^k) = \Big(T^k X \xrightarrow{T^i \phi T^{k-i}} T^{k+1} X\Big).$$

We may describe a restricted cosimplicial space as a diagram in $\mathcal{S}$ as follows. Let Δ_{rest} denote the *restricted simplicial category*, that is the category whose objects are finite ordered sets $[n] = \{0, 1, \dots, n\}$ $(n \geqslant 0)$ and whose morphisms are strictly monotone maps. A restricted, unaugmented, cosimplicial space $\boldsymbol{C}_{\mathrm{rest}}$ is equivalent to a functor

$$\boldsymbol{C}_{\mathrm{rest}} : \Delta_{\mathrm{rest}} \to \mathcal{S}.$$

In particular, $\hat{\boldsymbol{T}}_E X \in \mathcal{S}^{\Delta_{\mathrm{rest}}}$.

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The full simplicial category, Δ, is the category whose objects are the sets $[n]$ and whose morphisms are all weakly monotone maps. Then a cosimplicial space is a functor

$$\boldsymbol{C} : \Delta \to \mathcal{S}.$$

So $\boldsymbol{C} \in \mathcal{S}^{\Delta}$.

Let $J : \Delta_{\text{rest}} \to \Delta$ be the inclusion functor. Then there is a natural transformation

$$J^* : \mathcal{S}^{\Delta} \to \mathcal{S}^{\Delta_{\text{rest}}},$$

essentially the forgetful functor from cosimplicial spaces to restricted cosimplicial spaces.

Definition 3.2. For a general ring spectrum with unit $\boldsymbol{E}$, define $X_E^{\wedge}$, the E-completion of X, to be $\underleftarrow{\operatorname{holim}}\, \hat{\boldsymbol{T}}_E X$.

Strictly speaking, this definition only requires $\boldsymbol{E}$ to have a unit. However, we shall need $\boldsymbol{E}$ to have a ring structure directly after the next definition, which introduces an object lying between a cosimplicial space and a restricted cosimplicial space.

Definition 3.3. A *modified cosimplicial space* is a restricted cosimplicial space with codegeneracies that satisfy cosimplicial-like identities:

$$\begin{aligned}
d^j d^i &= d^i d^{j-1}, && i < j, \\
s^j d^i &\simeq d^i s^{j-1}, && i < j, \\
&\simeq \mathrm{id}, && i = j, j+1, \\
&\simeq d^{i-1} s^j, && i > j+1, \\
s^j s^i &\simeq s^{i-1} s^j, && i > j,
\end{aligned}$$

where the first identity is the usual cosimplicial identity, but the rest are required to hold only up to homotopy.

Remark 3.4. If $\boldsymbol{E}$ is a ring spectrum with unit, then, for $X \in \mathcal{S}$, the triple (T_E, ϕ, μ) induces a modified cosimplicial space which we also denote by $\boldsymbol{T}_E X$. Clearly, any cosimplicial space $\boldsymbol{C}$ is also a modified cosimplicial space and so if X is an $\mathbb{S}$-algebra the two objects denoted $\boldsymbol{T}_E X$ agree.

Remark 3.5. Corollary 3.9 of [**12**] proves that

$$\operatorname{Tot}(\boldsymbol{C}) = \underleftarrow{\operatorname{holim}}\,(\boldsymbol{C}_{\text{rest}}) \quad \text{when } \boldsymbol{C} = J^* \boldsymbol{C}_{\text{rest}}.$$

In particular, if $\boldsymbol{E}$ is an $\mathbb{S}$-algebra, the completion $X_E^{\wedge}$ defined in [**5**] agrees with that of Definition 3.2.

Remark 3.6. It is not possible to apply Tot to modified cosimplicial spaces. However, after applying π_* we obtain a cosimplicial group $\pi_* \boldsymbol{T}_E X$, which we view as a diagram $\pi_* \boldsymbol{T}_E X \in \mathcal{A}^{\Delta}$, where $\mathcal{A}$ is the category of abelian groups. Applying π_* to $\hat{\boldsymbol{T}}_E X$ gives an object in $\mathcal{A}^{\Delta_{\text{rest}}}$ which is $J^*(\pi_* \boldsymbol{T}_E X)$.

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For a wide class of diagrams $\underline{X} \in \mathcal{S}^I$, Bousfield and Kan [**10**, XI 7.1] define a spectral sequence related to the groups $\pi_* \underleftarrow{\operatorname{holim}}\, \underline{X}$.

Definition 3.7. For $X \in \mathcal{S}$ and $\boldsymbol{E}$ a ring spectrum with unit, define $E_r^{*,*}(X)$, the *E-theory Bousfield–Kan spectral sequence of X*, as the Bousfield–Kan spectral sequence for $\hat{\boldsymbol{T}}_E X \in \mathcal{S}^{\Delta_{\mathrm{rest}}}$.

Theorem 3.8. *$E_2^{s,*}(X)$ is isomorphic to the homology of the unstable cobar complex. That is to say $E_2^{s,*}(X) = \pi^s \pi_* \boldsymbol{T}_E X$.*

Remark 3.9. Recall that the cohomotopy $\pi^s \underline{A}$ of a cosimplicial abelian group $\underline{A}$ is defined [**10**, X 7.1] as the cohomology $H^s(\mathrm{ch}(\underline{A}), \partial)$, where $(\mathrm{ch}(\underline{A}), \partial)$ is the cochain complex given by $\mathrm{ch}(\underline{A})^n = \underline{A}^n$ and $\partial = \sum (-1)^i d^i$.

Proof of Theorem 3.8. Let I be either Δ or Δ_{rest}. For $\underline{X} \in \mathcal{S}^I$ the E_2-page is given by [**10**, p. 309]

$$E_2^{s,t} = \underleftarrow{\lim}^s \pi_t \underline{X}.$$

Since $\pi_* \boldsymbol{T}_E X$ is a cosimplicial group, $\underleftarrow{\lim}^s \pi_* \boldsymbol{T}_E X = \pi^s \pi_* \boldsymbol{T}_E X$ [**10**, XI 7.3 (i)] and it suffices to show that

$$\underleftarrow{\lim}^s \pi_* \hat{\boldsymbol{T}}_E X = \underleftarrow{\lim}^s \pi_* \boldsymbol{T}_E X.$$

For any fixed n, denote by $\underline{K}^I(n) \in \mathcal{S}^I$ the diagrams of Eilenberg–Mac Lane spaces $K(A, n)$, which correspond to $\pi_* \boldsymbol{T}_E X \in \mathcal{A}^{\Delta}$ and $\pi_* \hat{\boldsymbol{T}}_E X \in \mathcal{A}^{\Delta_{\mathrm{rest}}}$ for the respective I (see [**10**, XI 7.2]). Then, for $s \leqslant n$ (again from [**10**, XI 7.2]),

$$\begin{aligned} \underleftarrow{\lim}^s \pi_* \boldsymbol{T}_E X &= \pi_{n-s} \underleftarrow{\operatorname{holim}}\, \underline{K}^{\Delta}(n), \\ \underleftarrow{\lim}^s \pi_* \hat{\boldsymbol{T}}_E X &= \pi_{n-s} \underleftarrow{\operatorname{holim}}\, \underline{K}^{\Delta_{\mathrm{rest}}}(n). \end{aligned}$$

However, $J : \Delta_{\mathrm{rest}} \to \Delta$ is left cofinal [**12**, p. 193]. Thus

$$J^* : \underleftarrow{\operatorname{holim}}\, \underline{K}^{\Delta}(n) \to \underleftarrow{\operatorname{holim}}\, \underline{K}^{\Delta_{\mathrm{rest}}}(n)$$

is a homotopy equivalence. Since n was arbitrary, it follows that

$$\underleftarrow{\lim}^s \pi_* \boldsymbol{T}_E X = \underleftarrow{\lim}^s \pi_* \hat{\boldsymbol{T}}_E X \quad \text{for all } s.$$

□

4. The unstable cobar complex for $\boldsymbol{E}$-theory

Section 3 identifies the E_2-page of the Bousfield–Kan spectral sequence for a ring spectrum with unit $\boldsymbol{E}$ as the homology of the cochain complex $\mathrm{ch}(\pi_* \boldsymbol{T}_E X)$. However, for practical purposes, as in [**6**], [**5**], etc., it is important to be able to reinterpret this in terms of a more manageable object: in practice as the homology of a sub-complex of the *stable* cobar complex, i.e. as an Ext group over a more convenient (in particular, abelian) category.

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We suppose for this section that $\boldsymbol{E}$ is a Landweber exact ring spectrum with unit and (largely for convenience) that $\boldsymbol{E}$ is p-local with coefficients E_* concentrated in even dimensions. Let $\mathcal{M}$ be the category of free, graded E_*-modules. Drawing on the results of [**5**–**7**] and those of §§ 2 and 3, we introduce a certain associated abelian category $\mathcal{U}$. Our main theorem is the following.

Theorem 4.1. *Suppose $\boldsymbol{E}$ is a Landweber exact ring spectrum with unit. Suppose $M \in \mathcal{M}$ has E_*-module generators only in odd degrees and suppose that X is a space with $E_*(X) \cong \Lambda(M)$ as coalgebras, where $\Lambda(M)$ is the E_*-Hopf algebra defined by letting M be the sub-module of primitives, i.e. $\Lambda(M)$ is the exterior algebra on M. Then the E_2-term of the E-theory Bousfield–Kan spectral sequence of X can be identified as*

$$E_2^{s,t}(X) \cong \operatorname{Ext}^s_{\mathcal{U}}(E_*(S^t), M).$$

Example 4.2. Spaces X satisfying the hypotheses of the theorem include torsion free H-spaces and odd-dimensional spheres.

We begin by defining functors G and $U : \mathcal{M} \to \mathcal{M}$. Here and below we draw on a number of the results of § 2 with $F = E$, i.e. in this section we deal only with the coalgebraic ring $E_*(\underline{E}_*)$.

Definition 4.3. For a free E_*-module M define

(a) $G(M)$ to be $E_*(\underline{EM}_0)$, where $\boldsymbol{EM}$ denotes the spectrum realizing the homology theory $E_*(-) \underset{E_*}{\otimes} M$; and

(b) $U(M)$ to be $PG(M)$, the primitive elements in $G(M)$.

Both G and U are functorial; they take values in $\mathcal{M}$, the category of free E_* modules, by the results of § 3.

Remark 4.4.

(a) As M is a free E_*-module, it is helpful to observe that $\underline{EM}_0 = \Omega^\infty \boldsymbol{EM}$ is a product of spaces in the Ω spectrum associated with $\boldsymbol{E}$ indexed by a set of generators of M. In particular, if $\{g_i\}$ is a set of E_* generators of M with g_i in dimension $|g_i|$,

$$\underline{EM}_0 = \Omega^\infty \biggl(\bigvee_i \sum^{-|g_i|} \boldsymbol{E} \biggr) = \operatorname{colim} \prod_{\text{fin}} \underline{E}_{-|g_i|},$$

where the colimit is over finite products of the $\underline{E}_{-|g_i|}$. Moreover, with this notation,

$$M \cong \pi_* \biggl(\bigvee_i \sum^{-|g_i|} \boldsymbol{E} \biggr) = \pi_* \Bigl(\operatorname{colim} \prod_{\text{fin}} \underline{E}_{-|g_i|} \Bigr).$$

(b) Note that G is closely related to the functor $T_E : \mathcal{S} \to \mathcal{S}$ of § 3. For a space $X \in \mathcal{S}$ with $E_*(X) \in \mathcal{M}$, there is an isomorphism

$$G(E_*(X)) \cong E_*(T_E(X)).$$

(c) Note also that $U(M)$ is identical to the construction $U_E(M)$ of § 2. There is, of course, a similar functor $V : \mathcal{M} \to \mathcal{M}$ based on the indecomposable quotient of $G(M)$ and given by the construction $V_E(M)$ of § 2, but it will play no part in the proof of Theorem 4.1.

Proposition 4.5. *The unit and product in* $\boldsymbol{E}$*, respectively, induce natural transformations*

$$\delta^G : G \to G^2, \qquad \epsilon^G : G \to I,$$

making $(G, \delta^G, \epsilon^G)$ *a cotriple on the category* $\mathcal{M}$*. There are similar natural transformations* $\delta^U : U \to U^2$, $\epsilon^U : U \to I$ *making* $(U, \delta^U, \epsilon^U)$ *also a cotriple on* $\mathcal{M}$ *and a sub-cotriple of* $(G, \delta^G, \epsilon^G)$.

Proof. The proof is essentially as in §§ 6 and 7 of [**6**]; moreover, with the first observations of Remark 4.4, the maps δ^G and ϵ^G, for example, may be written explicitly. Alternatively, for Landweber exact $\boldsymbol{E}$, given the definition (2.9) and Theorem 2.11, the result on $(U, \delta^U, \epsilon^U)$ also follows from the coaction formulae for the b_i. □

Remark 4.6. As usual the cotriples define categories $\mathcal{G}$ and $\mathcal{U}$ of G, respectively U, coalgebras: writing C for either G or U, recall that a C coalgebra in $\mathcal{M}$ is an object $M \in \mathcal{M}$ with a map $\psi : M \to CM$ such that

$$\epsilon^C \psi = \mathrm{Id}_M : M \to M \quad \text{and} \quad \delta^C \psi = (C\psi)\psi : M \to C^2 M$$

(see [**6**, § 5] for details).

In particular, recall that if $M \in \mathcal{M}$, then CM is naturally a C coalgebra with map ψ on $CM \to C^2 M$ given by δ^C. There are adjoint functors

$$\mathcal{M} \underset{J}{\overset{C}{\rightleftarrows}} \mathcal{C},$$

where J denotes the forgetful functor. The adjunction gives natural isomorphisms

$$\mathrm{Hom}_{\mathcal{C}}(D, CM) \cong \mathrm{Hom}_{\mathcal{M}}(D, M)$$

for any $D \in \mathcal{C}$ (where we identify D with its image under the forgetful functor).

Strictly speaking, we shall abuse notation and write C not only for the functor $\mathcal{M} \to \mathcal{C}$ above, but also for the functor $JC : \mathcal{M} \to \mathcal{M}$ of the cotriple $(C, \delta^C, \epsilon^C)$ on $\mathcal{M}$ and for the other composite, $CJ : \mathcal{C} \to \mathcal{C}$, the functor of the adjoint *triple* (C, μ^C, η^C) on $\mathcal{C}$, as in [**6**, § 5].

For $\mathcal{C} = \mathcal{G}$ or $\mathcal{U}$ and objects $W \in \mathcal{C}$ we recall the notions of *cosimplicial resolution* over $\mathcal{C}$, as in [**8**, 2.5] and [**7**, 2.2] and the resulting derived functors $\mathrm{Ext}_{\mathcal{C}}(E_*, W)$.

Definition 4.7. A *cosimplicial resolution,* $\boldsymbol{N}$*, over* $\mathcal{C}$, of $W \in \mathcal{C}$ consists of objects $N^n \in \mathcal{C}$ for $n \geqslant -1$ and, for every pair of integers (i, n) with $0 \leqslant i \leqslant n$, coface and codegeneracy maps (in $\mathcal{C}$)

$$d^i : N^{n-1} \to N^n, \qquad s^i : N^{n+1} \to N^n$$

satisfying the usual cosimplicial identities (cf. Definition 3.3) and such that

(a) $N^{-1} = W$;

(b) for $n \geqslant 0$ there is an $M_n \in \mathcal{M}$ with $N^n = CM_n$;

(c) $H^n(J\boldsymbol{N}) = 0$ for $n \geqslant -1$.

Here $J : \mathcal{C} \to \mathcal{M}$ is the forgetful functor and the homology of $J\boldsymbol{N}$ is the homology of the cochain complex with groups JN^n and boundary maps $\sum(-1)^i Jd^i$.

The Ext groups $\mathrm{Ext}_{\mathcal{C}}(E_*, W)$ are then defined as the homology of the chain complex associated with $\mathrm{Hom}_{\mathcal{C}}(E_*, \widetilde{J\boldsymbol{N}})$, where $\widetilde{J\boldsymbol{N}}$ denotes the unaugmented complex

$$0 \to JN_0 \to JN_1 \to JN_2 \to \cdots .$$

These are the derived functors of $\mathrm{Hom}_{\mathcal{C}}(E_*, -)$ by [**8**].

Example 4.8. The $\mathcal{C}$ *cobar complex* provides a standard example of a cosimplicial resolution. We illustrate it for $\mathcal{C} = \mathcal{U}$; the case of $\mathcal{G}$ is similar.

For $W \in \mathcal{U}$, consider the resolution with qth module $U^{q+1}(W)$. The maps in the $\mathcal{U}$ resolution are displayed in the diagram of E_*-modules,

$$W \xrightarrow{d^0} U(W) \begin{array}{c} \xrightarrow{d^0} \\ \xrightarrow{d^1} \\ \xleftarrow{s^0} \end{array} \cdots ,$$

and are defined in terms of the triple (U, μ^U, η^U) by

$$\begin{aligned} d^i &= U^i \eta^U U^{n-i} : U^n(W) \to U^{n+1}(W), && 0 \leqslant i \leqslant n, \\ s^i &= U^i \mu^U U^{n-i} : U^{n+2}(W) \to U^{n+1}(W), && 0 \leqslant i \leqslant n. \end{aligned}$$

The $\mathcal{U}$ cobar complex is then the complex

$$W \xrightarrow{\partial} U(W) \xrightarrow{\partial} U^2(W) \xrightarrow{\partial} \cdots ,$$

where

$$\partial = \sum_{i=0}^{n} (-1)^n d^i : U^n(W) \to U^{n+1}(W).$$

The embedding of the primitives in the stable cooperations, (2.7) and (2.11), shows that the acyclicity condition is satisfied since there is an extra codegeneracy $s^{-1} : U^{q+1}(W) \to U^q(W)$ induced by the counit in $E_*(E)$:

$$U^{q+1}(C) \to E_*(E) \otimes U^q(C) \xrightarrow{\epsilon \otimes 1} U^q(C).$$

In particular, again by (2.11), $\mathrm{Ext}_{\mathcal{U}}(E_*, W)$ is the homology of a sub-complex of the *stable* cobar complex.

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These constructions and the link between the functors G and $\boldsymbol{T}_E$ of Remark 4.4 (b) allow us to rewrite Theorem 3.8 as follows.

Theorem 4.9. *For $\boldsymbol{E}$ a ring spectrum with unit and $X \in \mathcal{S}$ such that $E_*(X) \in \mathcal{M}$, there is a natural isomorphism*

$$E_2^{s,t}(X) = \operatorname{Ext}^s_{\mathcal{G}}(E_*(S^t), E_*(X)).$$

Theorem 4.1 will now follow upon proving the following theorem.

Theorem 4.10. *Suppose $\boldsymbol{E}$ is a Landweber exact ring spectrum with unit. For $M \in \mathcal{M}$ with generators in odd degree and $\Lambda(M)$ denoting the exterior algebra on M with $M \subset \Lambda(M)$ the sub-module of primitives, there is a natural isomorphism*

$$\operatorname{Ext}^s_{\mathcal{G}}(E_*(S^t), \Lambda(M)) \cong \operatorname{Ext}^s_{\mathcal{U}}(E_*(S^t), M).$$

Proof. Let us write $\boldsymbol{U}M$ for the $\mathcal{U}$ cobar complex as in Example 4.8, i.e. with qth space $U^{q+1}(M)$. Applying the functor $\Lambda(-)$ gives a complex

$$\Lambda\boldsymbol{U}M: \quad \Lambda(M) \to \Lambda(U(M)) \to \Lambda(U^2(M)) \to \cdots.$$

Now let

$$Y^q = G(U^q(M))$$

for $q \geqslant 0$. Since M is concentrated in odd degrees the same is true for $U^q(M)$. By Theorems 2.4 and 2.11 we have natural isomorphisms

$$G(U^q(M)) \cong \Lambda(U^{q+1}(M))$$

and we can identify the complex $\Lambda\boldsymbol{U}M$ as a complex

$$\boldsymbol{Y}: \quad \Lambda(M) \to G(M) \to G(U(M)) \to G(U^2(M)) \to \cdots.$$

The maps in $\boldsymbol{Y}$ are coalgebra maps and $E_*(E)$-comodule maps. By [**6**, 7.3] the maps are in $\mathcal{G}$ (note that [**6**, 7.3] does not require the assumption [**6**, 7.7] that the homology of the spaces in the Ω-spectrum be cofree coalgebras—this is not satisfied in general). The extra codegeneracy in the $\mathcal{U}$ cobar complex passes via Λ to an extra codegeneracy in $\boldsymbol{Y}$, showing $\boldsymbol{Y}$ to be acyclic. Thus $\boldsymbol{Y}$ is a $\mathcal{G}$-resolution of $\Lambda(M)$.

The Ext groups $\operatorname{Ext}^s_{\mathcal{G}}(E_*(S^t), \Lambda(M))$ can be obtained using the complex $\boldsymbol{Y}$ by computing the homology of the complex

$$\operatorname{Hom}_{\mathcal{G}}(E_*(S^t), Y^s) = \operatorname{Hom}_{\mathcal{G}}(E_*(S^t), G(U^s(M))).$$

However, by the adjunction isomorphism mentioned in Remark 4.6 (applied twice), we have

$$\begin{aligned} \operatorname{Hom}_{\mathcal{G}}(E_*(S^t), G(U^s(M))) &= \operatorname{Hom}_{\mathcal{M}}(E_*(S^t), U^s(M)) \\ &= \operatorname{Hom}_{\mathcal{U}}(E_*(S^t), U^{s+1}(M)). \end{aligned}$$

Thus $\operatorname{Ext}_{\mathcal{G}}(E_*(S^t), \Lambda(M))$ is isomorphic to the homology of the $\mathcal{U}$-cobar complex, which by definition is precisely $\operatorname{Ext}_{\mathcal{U}}(E_*(S^t), M)$. $\square$

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Remark 4.11. The results of § 2 on the algebra structure of $E_*(\underline{E}_*)$ allow further results to follow. For example, suppose for $M \in \mathcal{M}$ we write $\sigma^{-1}M$ for the isomorphic E_*-module with degrees shifted downward by one, i.e. we let $\sigma^{-1}M_t = M_{t+1}$. Then Theorem 2.4 and its proof show that

$$\sigma^{-1}U(M) = QG(\sigma^{-1}M).$$

If we take $M = E_*(S^{2n+1})$, then $E_*(\Omega S^{2n+1}) = \sigma^{-1}M$ and an argument similar to that for BP-theory in [**7**, § 6] shows that the complex $\boldsymbol{Y}$ used in the proof of Theorem 4.10 may also be used to compute the E_2-page of the E-theory Bousfield–Kan spectral sequence for ΩS^{2n+1}: for any odd-dimensional sphere S^{2n+1} there is an isomorphism

$$E_2^{s,t-1}(\Omega S^{2n+1}) \cong E_2^{s,t}(S^{2n+1}).$$

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References

1. J. F. Adams, *Stable homotopy and generalised homology*, Chicago Lectures in Mathematics (Chicago University Press, 1974).
2. S. Araki, *Typical formal groups in complex cobordism and K-theory* (Kinkokuniya, Tokyo, 1973).
3. A. J. Baker and U. Würgler, Liftings of formal groups and the Artinian completion of $v_n^{-1}BP$, *Math. Proc. Camb. Phil. Soc.* **106** (1989), 511–530.
4. M. Bendersky, Some calculations in the unstable Adams–Novikov spectral sequence, *Publ. RIMS Kyoto* **16** (1980), 739–766.
5. M. Bendersky and R. D. Thompson, The Bousfield–Kan spectral sequence for periodic homology theories, *Am. J. Math.* **122** (2000), 599–635.
6. M. Bendersky, E. B. Curtis and H. R. Miller, The unstable Adams spectral sequence for generalized homology, *Topology* **17** (1978), 229–248.
7. M. Bendersky, E. B. Curtis and D. Ravenel, EHP sequences in BP theory, *Topology* **21** (1982), 373–391.
8. A. K. Bousfield, Nice homology coalgebras, *Trans. Am. Math. Soc.* **148** (1970), 473–489.
9. A. K. Bousfield and D. M. Kan, The homotopy spectral sequence of a space with coefficients in a ring, *Topology* **11** (1972), 79–106.
10. A. K. Bousfield and D. M. Kan, *Homotopy limits, completions and localizations*, Lecture Notes in Mathematics, vol. 304 (Springer, 1972).
11. K. Chan, A simple proof that the unstable (co)homology of the Brown–Peterson spectrum is torsion free, *J. Pure Appl. Alg.* **26** (1982), 155–157.
12. E. Dror and W. G. Dwyer, A long homology localization tower, *Comment. Math. Helv.* **52** (1977), 185–210.

13. S. Eilenberg and J. C. Moore, Homology and fibrations, I, Coalgebras, cotensor product and its derived functors, *Comment. Math. Helv.* **40** (1966), 199–236.
14. A. O. Elmendorf, I. Kriz, M. A. Mandall and J. P. May, Modern foundations for stable homotopy theory, in *Handbook of algebraic topology* (ed. I. M. James), pp. 213–253 (North-Holland, Amsterdam, 1995).
15. D. J. Green, J. R. Hunton and B. Schuster, Chromatic characteristic classes in ordinary group cohomology, *Topology* **42** (2003), 243–263.
16. M. J. Hopkins and J. R. Hunton, On the structure of spaces representing a Landweber exact cohomology theory, *Topology* **34** (1995), 29–36.
17. J. R. Hunton, Complete cohomology theories and the homology of their omega spectra, *Topology* **41** (2002), 931–943.
18. J. R. Hunton and N. Ray, A rational approach to Hopf rings, *J. Pure Appl. Alg.* **101** (1995), 313–333.
19. J. R. Hunton and B. Schuster, Subalgebras of group cohomology defined by infinite loop spaces, preprint math.AT/0112169 (revised and extended version of University of Leicester Department of Mathematics Technical Report 2004/30).
20. J. R. Hunton and P. R. Turner, Coalgebraic algebra, *J. Pure Appl. Alg.* **129** (1998), 297–313.
21. J. R. Hunton and P. R. Turner, The homology of spaces representing exact pairs of homotopy functors, *Topology* **38** (1999), 621–634.
22. D. C. Johnson and W. S. Wilson, Projective dimension and Brown–Peterson homology, *Topology* **12** (1973), 327–353.
23. P. S. Landweber, Homological properties of comodules over $MU_*(MU)$ and $BP_*(BP)$, *Am. J. Math.* **98** (1976), 591–610.
24. P. S. Landweber, D. C. Ravenel and R. Stong, Periodic cohomology theories defined by elliptic curves, *Contemp. Math.* **181** (1995), 317–337.
25. D. C. Ravenel and W. S. Wilson, The Hopf ring for complex cobordism, *J. Pure Appl. Alg.* **9** (1977), 241–280.
26. W. S. Wilson, The Ω-spectrum for Brown–Peterson cohomology, Part I, *Comment. Math. Helv.* **48** (1973), 45–55.
27. W. S. Wilson, *A BP introduction and sampler*, CBMS Regional Conference Series in Mathematics, vol. 48 (American Mathematical Society, Providence, RI, 1980).