THE BETTI NUMBERS OF THE SIMPLE LIE GROUPS

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1. Introduction. The purpose of the present paper[1] is to simplify the calculation of the Betti numbers of the simple compact Lie groups.

For the unimodular group and the orthogonal group on a space of odd dimension the form of the Poincaré polynomial was correctly guessed by E. Cartan in 1929 (**5**, p. 183). The proof of his conjecture and its extension to the four classes of classical groups was given by L. Pontrjagin (**13**) using topological arguments and then by R. Brauer (**2**) using algebraic methods. However, the case of the exceptional Lie groups proved more recalcitrant and was finally settled only in 1949 by C. T. Yen (**21**). Borel and Chevalley (**1**) have recently simplified the calculations for the exceptional groups. Even so, they make use of a large number of disparate algebraic and topological results including the known facts for the classical groups. Much of their paper was already covered by results of Coxeter (**9**) and Racah (**14**). Their method entails a tedious discussion of special cases.

Hopf (**10**) and Samelson (**15**) showed that for a compact Lie group the Poincaré polynomial (the coefficients of which are the Betti numbers) is of the form

$$P(t) = \prod_{i=1}^{n} (1 + t^{p_i})$$

where n is the rank of the group and p_i are odd integers. Chevalley (**6**) proved that $p_i = 2k_i - 1$ where k_i is the degree of a minimal homogeneous invariant of the group. We shall show how the k_i may be easily obtained.

As is well known (**18**), the classification of simple compact Lie groups of rank n is closely related to the classification of finite orthogonal groups generated by reflections in a space of dimension n. Since this finite group associated with the Lie group was first introduced by Killing, we shall call it the Killing group and denote it by $\mathfrak{K}$. This group has also been called the "kaleidoscopic group" and the "Weyl group." The latter name has little justification since, though Weyl made use of it in 1926, it was used by Killing and, following him, Cartan, to effect the classification of simple Lie groups. All particular cases of the Killing group were described in some detail in Cartan's paper (**4**) in 1896. It has been studied and used by many authors including Cartan (**3**, p. 58; **4**), Coxeter (**8**, chap. 11), Stiefel (**18**), Weyl

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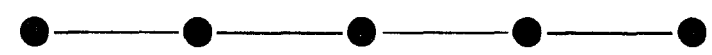
(19, § 4 et seq.**)**, and Witt **(20)**. The Killing group of a compact Lie group is isomorphic to the quotient group relative to $\mathfrak{T}$ of the normalizer of a maximal abelian subgroup $\mathfrak{T}$. Chevalley **(6)** has shown that by restricting the minimal invariants to $\mathfrak{T}$, the problem of finding k_i reduces to that of finding the degrees of the minimal homogeneous invariants of the Killing group represented by orthogonal transformations in a space of n dimensions. Here and throughout the paper n is the rank of the Lie group.

2. The product of reflections. For a simple Lie group, $\mathfrak{G}$, the Killing group, $\mathfrak{K}$, represented as a group of congruent transformations in Euclidean n-space, E_n, is generated by n reflections which we shall denote by R_i, $1 \leqslant i \leqslant n$. The relations among R_i are conveniently indicated by a Coxeter graph **(8**, § 11.3, and p. 297**)**. The Coxeter graph of $\mathfrak{G}$ consists of n nodes joined by branches. The ith node corresponds to R_i. Two nodes are joined if the corresponding two reflections do not commute. The (i, j) branch is marked to indicate the period of R_iR_j, but we can neglect this for present purposes. We shall, however, distinguish two types of Coxeter graph.

In *Case* I the graph consists of a single chain such as



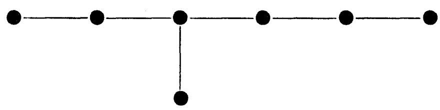
which is the graph for A_5. Such graphs occur for the following simple Lie groups:

A_n (unimodular), B_n (orthogonal group on $2n + 1$ variables),

C_n (symplectic), G_2 (the exceptional group of rank 2 and dimension 14),

F_4 (the exceptional group of rank 4 and dimension 52).

In *Case* II the graph consists of a principal chain with $n - 1$ nodes, with a second chain containing one node emanating from the principal chain. For example,



is the graph for E_7. *Case* II includes:

D_n (the orthogonal group on $2n$ variables),

E_6, E_7, E_8 (the exceptional groups of rank 6, 7, 8 and dimension 78, 133, 248 respectively).

The product of the n generating reflections, $R = R_1R_2 \ldots R_n$ and its order, h, play a fundamental role in what follows. It is of historical interest to note that Killing **(12**, pp. 18-23; **3**, p. 58**)** made use of this same product. Coxeter has made a careful study of R **(8**, § 12.3; **9)** and we depend heavily on his work. In particular he noticed that if ζ is a primitive hth root of unity and

$$\zeta^{m_i}, \quad \text{where} \quad 0 < m_i < h,$$

are the eigenvalues of R, then $k_i = m_i + 1$. We shall prove this. Hence Coxeter's calculation of m_i determines k_i. We shall call the positive integers m_i *the exponents of* $\mathfrak{K}$.

Definition. By a *regular vector* we shall mean one which does not lie on a reflecting hyperplane of $\mathfrak{R}$. Thus $\mathbf{x}$ is regular if and only if $\mathbf{r}_\alpha \cdot \mathbf{x} \neq 0$ for every positive root vector $\mathbf{r}_\alpha$, since the root vectors are orthogonal to the reflecting hyperplanes of $\mathfrak{R}$.

LEMMA 1. *The operation R, of order h, has a primitive hth root of unity, ζ, as an eigenvalue which corresponds to a regular eigenvector. With a_{ij} as defined below and α the minimum eigenvalue of (a_{ij}), $\zeta = e^{i\theta}$, where $\theta = 2\pi/h =$* $4 \arcsin (\frac{1}{2}\alpha)^{\frac{1}{2}}$.

Proof. As was shown by Cartan **(3**, p. 58**)** and by Coxeter **(9**, p. 767**)** the equation

$$R\mathbf{x} = \lambda \mathbf{x} \tag{1}$$

is equivalent to the equations

$$b_i^j x_j = 0 \tag{2}$$

where the co-ordinates x_j are distances from the sides of a fundamental simplex $\mathbf{F}$ of $\mathfrak{R}$. If $\mathbf{e}_i$ are unit vectors orthogonal to sides of $\mathbf{F}$, pointing inwards, and $\mathbf{e}^i$ is the reciprocal basis of E_n such that $\mathbf{e}_i \cdot \mathbf{e}^j = \delta_i{}^j$, then $\mathbf{x} = x_i \mathbf{e}^i$. The matrix $b_i{}^j$ has the form:

$$\begin{pmatrix} \frac{1}{2}(\lambda+1) & a_{12}\lambda & \cdots & a_{1n}\lambda \\ a_{21} & \frac{1}{2}(\lambda+1) & \cdots & a_{2n}\lambda \\ \cdots & & \cdots & \\ a_{n1} & a_{n2} & \cdots & \frac{1}{2}(\lambda+1) \end{pmatrix}$$

where $a_{ij} = a_{ji} = -\cos(\pi/p_{ij})$, and p_{ij} is the period of $R_i R_j$. Thus $a_{ii} = 1$, $a_{ij} < 0$ if the distinct i and j nodes are connected by a branch, and $a_{ij} = 0$ otherwise. The matrix (a_{ij}) therefore corresponds to what Coxeter calls an a-form **(8**, § 10.2**)**. We order the co-ordinates $x_1, x_2, \ldots, x_{n-1}$ to correspond in succession to the nodes of the principal chain from left to right, with x_n corresponding to the end node in Case I, and to the node on the second chain in Case II. In Case II we let $q = 2$ for D_n and $q = 3$ for E_6, E_7, E_8.

With the notation fixed in this way the only non-vanishing elements in the ith row of (a_{ij}) are:

(a) *in Case* I: for $i = 1$, a_{11}, a_{12}; for $i = n$, a_{nn-1}, a_{nn}; otherwise a_{ii-1}, a_{ii}, a_{ii+1};

(b) *in Case* II: for $i = 1$, a_{11}, a_{12}; for $i = q$, a_{qq-1}, a_{qq}, a_{qq+1}, a_{qn}; for $i = n-1$, $a_{n-1\,n-2}$, $a_{n-1\,n-1}$; for $i = n$, a_{nq}, a_{nn}; otherwise, a_{ii-1}, a_{ii}, a_{ii+1}.

The equations (2) are now transformed in *Case* I by setting

$$x_j = \lambda^{-\frac{1}{2}j} y^j \tag{3a}$$

and multiplying the ith row by $\lambda^{\frac{1}{2}(i-1)}$; and in *Case* II by setting

$$x_j = \lambda^{-\frac{1}{2}j} y^j, \qquad 1 \leqslant j \leqslant n-1; \qquad x_n = \lambda^{-\frac{1}{2}(q+1)} y^n \tag{3b}$$

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and multiplying the ith row by $\lambda^{\frac{1}{2}(i-1)}$ for $1 \leqslant i \leqslant n-1$, and the nth row by $\lambda^{\frac{1}{2}q}$. Equations (2) then take the form

$$(4) \qquad (a_{ij} - (1 - \Lambda)\delta_{ij})y^j = 0$$

where

$$\Lambda = \tfrac{1}{2}(\lambda^{\frac{1}{2}} + \lambda^{-\frac{1}{2}}).$$

For $\lambda = 1$, (2) reduces to $a_{ij}y^j = 0$ which has no non-trivial solution for the simple groups. Thus no vector is fixed under R, and $-1 < \Lambda < 1$.

Let α be the smallest eigenvalue of (a_{ij}), then $(a_{ij} - \alpha\delta_{ij})y^iy^j$ is a positive semi-definite connected a-form, so that $(a_{ij} - \alpha\delta_{ij})$ is of nullity 1 and the equations (4), with $\alpha = 1 - \Lambda$ have a solution $y_0{}^j$ with $y_0{}^j > 0$ for all j. Let $\zeta = e^{i\theta}$ be the corresponding value of λ where $0 < \theta < 2\pi$. Since α is the minimum eigenvalue of (a_{ij}), θ is the smallest positive angle ϕ for which $e^{i\phi}$ is an eigenvalue of R. The angle $\theta = 2\pi/p$ where p is an integer. For let p be the smallest integer such that $p\theta$ is a multiple of 2π. Then $e^{i\theta}$ is a primitive pth root of unity. Since the characteristic equation of R has rational coefficients it has as roots all primitive pth roots of unity and in particular $e^{i2\pi/p}$. The above minimum property of θ implies that $\theta = 2\pi/p$.

Equations ($3a$) or ($3b$), with λ replaced by ζ and y^j by $y_0{}^j$, give an eigenvector $\mathbf{x}_0$ of R. We must show that $\mathbf{r}_\alpha \cdot \mathbf{x}_0 \neq 0$ for all positive root vectors $\mathbf{r}_\alpha$. To this end, note that $\theta \leqslant 2\pi/(n+1)$. We prove this by induction on n. For $n = 1$, $R = R_1$, $\lambda = -1$, $\theta = 2\pi/2$. In general, the minimum eigenvalue α of (a_{ij}), is less than or equal to the minimum eigenvalue α' of the sub-matrix $(a_{ij})'$ with $i, j > 1$, corresponding to the Coxeter graph obtained by removing the first node of the given graph. Now $y_0{}^1$, which does not vanish, is proportional by **(8**, 10.27**)** to the square-root of the (1, 1) principal minor of $(a_{ij} - \alpha\delta_{ij})$. Thus $(a_{ij} - \alpha\delta_{ij})'$ is regular and α is not an eigenvalue of $(a_{ij})'$. Thus α is in fact less than α' and if $\theta' = 2\pi/t$, $\theta \leqslant 2\pi/(t+1)$. This enables us to complete the induction.

The fact that $\mathbf{x}_0$ is regular now follows easily. Each positive root vector $\mathbf{r}_\alpha = r_\alpha{}^j\mathbf{e}_j$ where $r_\alpha{}^j$ are positive rational numbers. Thus

$$\mathbf{r}_\alpha \cdot \mathbf{x}_0 = r_\alpha{}^j x_j = \sum_j (r_\alpha{}^j \cos \phi_j y_0{}^j - i r_\alpha{}^j \sin \phi_j y_0{}^j)$$

where the ϕ_j are obtained from ζ by equations (3): in *Case* I,

$$0 < \phi_j < \tfrac{1}{2}j\theta \leqslant \pi \frac{j}{n+1} < \pi$$

and similarly in *Case* II, $0 < \phi_j < \pi$. Thus $\sin \phi_j > 0$ and $\Sigma r_\alpha{}^j \sin \phi_j y_0{}^j > 0$ which implies $\mathbf{r}_\alpha \cdot \mathbf{x}_0 \neq 0$.

To show that ζ is a primitive hth root, assume that $\zeta^u = 1$, for $u \leqslant h$. Then $R^u\mathbf{x}_0 = \mathbf{x}_0$. But $\mathbf{x}_0$ does not lie on a reflecting plane, so it is fixed only under the identity **(19)**. Hence $R^u = I$, and $u = h$.

Since $\Lambda = 1 - \alpha = \cos \frac{1}{2}\theta = 1 - 2\sin^2 \frac{1}{4}\theta$, we have $\theta = 4 \arcsin (\frac{1}{2}\alpha)^{\frac{1}{2}}$.

COROLLARY 1. *All the primitive* hth-*roots of unity occur as eigenvalues of* R *and the corresponding eigenvectors are regular.*

Proof. Since the characteristic equation of R has rational coefficients, the first part of the statement follows immediately.

The mapping of ζ onto ζ', another primitive h-root of unity, while keeping the rationals $\mathscr{R}$ fixed, determines an automorphism of the field $\mathscr{R}(\zeta)$, which sends the co-ordinates x_{0i} of $\mathbf{x}_0$ into the co-ordinates of the eigenvector $\mathbf{x}_0'$ corresponding to ζ'. Under this mapping $\mathbf{r}_\alpha \cdot \mathbf{x}_0$, which is different from zero will not be mapped onto zero. Therefore $\mathbf{x}_0'$ is regular.

It follows from this that $\phi(h) \leqslant n$, where ϕ is Euler's function. From the proof of the theorem we also have the limitation $h \geqslant n + 1$.

COROLLARY 2. *The number of reflections in* $\mathfrak{K}$ *is an integral multiple of* $\frac{1}{2}h$.

Proof. $R^u \mathbf{r}_\alpha \cdot \mathbf{x}_0 = \mathbf{r}_\alpha \cdot R^{-u} \mathbf{x}_0 = \zeta^{-u}(\mathbf{r}_\alpha \cdot \mathbf{x}_0)$. These are distinct and different from zero for $1 \leqslant u \leqslant h$. Thus $R^u \mathbf{r}_\alpha$ are distinct. The desired result follows, if we note that $\mathbf{r}_\alpha$ and $-\mathbf{r}_\alpha$ give rise to the same reflection, by partitioning the reflections into equivalent classes under the cyclic group generated by R.

It is in fact easy to verify **(8**, 12.61**)** that *the number of reflections is equal to*

$$\frac{1}{2}nh. \tag{5}$$

3. The Jacobian of a basic set of invariants. Chevalley **(7)** has given an elegant proof of the fact that any polynomial in x which is invariant under $\mathfrak{K}$ belongs to the ring generated by n minimal invariants I_i. If I_i has degree k_i then by a theorem of Molien **(16)**,

$$g \prod_{i=1}^{n} (1 - t^{k_i})^{-1} = \sum_{k} \prod_{i=1}^{n} (1 - \omega_i^k t)^{-1} \tag{6}$$

where g is the order of $\mathfrak{K}$ and ω_i^k are the eigenvalues of the operator $k \in \mathfrak{K}$. Following Shephard and Todd **(16**, p. 289**)** we multiply (6) by $(1 - t)^n$ and set $t = 1$, whence $g = \Pi k_i$. Subtract $(1 - t)^{-n}$ from both sides of (6), multiply by $(1 - t)^{n-1}$, set $t = 1$ and we deduce that *the number of reflections in* $\mathfrak{K}$ is

$$\Sigma(k_i - 1). \tag{7}$$

Consider the equations $I_i(x) = w_i$, where I_i are any n algebraically independent polynomial invariants of $\mathfrak{K}$. For a point $\mathbf{x}$ at which the Jacobian $J = |\partial_j I_i| \neq 0$, there will be open neighbourhoods of $\mathbf{x}$ and $\mathbf{w}$ in one-to-one correspondence. However, if $\mathbf{x}$ lies on one of the reflecting hyperplanes of $\mathfrak{K}$, any open neighbourhood of $\mathbf{x}$ contains points which are equivalent under $\mathfrak{K}$ and correspond to the same point $\mathbf{w}$. Thus $J = 0$ on the reflecting hyperplanes of $\mathfrak{K}$.

In particular, if I_i are a set of minimal invariants the degree $\Sigma(k_i - 1)$ of J is equal to the number of reflecting hyperplanes of $\mathfrak{K}$. Hence, **(17)**.

LEMMA 2. *The Jacobian of* n *minimal polynomial invariants of* $\mathfrak{K}$ *is equal, within a multiplicative constant, to the product of the linear forms whose vanishing gives the reflecting hyperplanes of* $\mathfrak{K}$.

With the above preparation, we may now easily reach the main result of the paper by evaluating J for a set of n minimal invariants, in a system of co-ordinates in which R is diagonal. Let $\mathbf{u}_i$ be an eigenvector of R such that

$$R\,\mathbf{u}_i = \zeta^{m_i}\,\mathbf{u}_i$$

where m_i are the exponents of $\mathfrak{K}$, with $m_1 = 1$. Thus if $\mathbf{x} = x^i\mathbf{u}_i$ goes into

$$\bar{\mathbf{x}} = R\mathbf{x} = x^i\zeta^{m_i}\mathbf{u}_i = \bar{x}^i\mathbf{u}_i$$

then

$$\bar{x}^i = \zeta^{m_i}\,x^i.$$

Since $\mathbf{u}_1$ is regular by Lemma 1, $J \neq 0$ at $\mathbf{x} = x^1\mathbf{u}_1$. Thus for each i there is a j such that $\partial_j I_i \neq 0$ at $x^1\mathbf{u}_1$. But at this point $\partial_j I_i$ is a multiple of

$$(x^1)^{k_i-1}$$

and this term arises from a term

$$(x^1)^{k_i-1}x^j$$

in I_i. This term must be invariant under R; therefore

$$k_i - 1 + m_j \equiv 0\ (h). \tag{8}$$

Since R is real, together with m_j, $h - m_j$ is an exponent, and since for $J \neq 0$ all j must occur, by reordering m_j we can arrange that $k_i \equiv m_i + 1\ (h)$. But

$$\sum m_j = \sum\ (h - m_j) = \tfrac{1}{2}nh = \sum\ (k_i - 1)$$

by (5). Hence

$$k_i = m_i + 1.$$

This conclues the proof[2] of the

THEOREM. *The degrees k_i of a set of minimal polynomial invariants of the Killing group, $\mathfrak{K}$, are given by $k_i = m_i + 1$, where m_i are the exponents of $\mathfrak{K}$.*

Hence from Coxeter's elegant calculation of m_i **(9)** we obtain the $p_i = 2k_i - 1 = 2m_i + 1$, which define the Poincaré polynomial. For the simple compact Lie groups the p_i are as follows

A_n	:	$3, 5, 7, 9, \ldots, 2n + 1$
B_n, C_n	:	$3, 7, 11, 15, \ldots, 4n - 1$
D_n	:	$3, 7, 11, \ldots, 4n - 5, 2n - 1$
G_2	:	$3, 11$
F_4	:	$3, 11, 15, 23$
E_6	:	$3, 9, 11, 15, 17, 23$
E_7	:	$3, 11, 15, 19, 23, 27, 35$
E_8	:	$3, 15, 23, 27, 35, 39, 47, 59.$

[2]Professor Coxeter has pointed out that the proof of this theorem is valid not only for the groups $\mathfrak{K}$ associated with Lie Groups but for any real finite group generated by reflections.

4. Remarks.

(i) If h is known, Coxeter's calculation of m_i can sometimes be simplified. The primitive h-roots of unity are given by ζ^u where $(u, h) = 1$. For E_8, $h = 30$ and the possible u are 1, 7, 11, 13, 17, 19, 23, 29 giving us the eight m_i. For E_7, $h = 18$, giving 1, 5, 7, 11, 13, 17 for m_i. The seventh root of unity must be real, therefore equal to -1 corresponding to $m_i = 9$. For E_6, $h = 12$ and the above method determines only four of the m_i : 1, 5, 7, 11 and further argument is needed to obtain 4 and 8. For F_4, $h = 12$ giving 1, 5, 7, 11.

From the form of the equations (2) one easily proves they are of nullity one except in Case II for $\lambda = -1$ when they have nullity two. Hence all the eigenvalues are simple except $\lambda = -1$ in Case II which is double. For A_n, $h = n + 1$. We know that $\lambda = 1$ is not an eigenvalue, and that the eigenvalues are all different; so they are completely determined.

(ii) The poles on each side of (6) coincide, therefore if any element of $\mathfrak{R}$ has as eigenvalue a primitive pth root of unity then p divides k_i for some i. Since each subgroup of $\mathfrak{G}$ has associated with it a primitive root of unity by Lemma 1, the k_i provide a limitation on the possible subgroups. Conversely, a knowledge of subgroups partially determines k_i. This, evidently, is the basis of many of the topological arguments for determining the Betti numbers by discussion of subgroups.

(iii) The symmetry in the sequence of first differences of the p_i sometimes referred to as "double duality" is explained by the simple fact that R is a real operator and together with λ, $\bar{\lambda}$ is an eigenvalue.

(iv) Previous methods of obtaining k_i depended on the explicit construction of a set of minimal invariants. These are partly determined by our method. For if $m_s + m_{\bar{s}} = h$ the invariant I_s of degree $k_s = m_s + 1$ contains the term

$$x_1^{m_s} x_{\bar{s}},$$

Indeed, there will be a term in I_s of the form

$$x_i^{m_s} x_j.$$

where $m_s m_i + m_j \equiv 0(h)$, for each m_i relatively prime to h. Probably

$$I_s = \sum_k (kx_1)^{m_s} (kx_{\bar{s}})$$

where k ranges over a set of representatives of the cosets of $\mathfrak{R}$ with respect to the cyclic group R, but this has not been proved.

(v) The above proof still contains the inelegancy of using (5) which has hitherto only been proved by verification. This is unsatisfactory even if one admits that once the h are known the verification is trivial. It would be most desirable to give a general proof[3] of the following three facts, perhaps not

[3]Since this was written, I have learned that R. Steinberg has a paper in the course of publication which deals with (a) and (b).

unrelated, which have been observed: (a) the number of reflections in $\mathfrak{R}$ is $\frac{1}{2}nh$; (b) if the dominant root vector is $z^i\mathbf{t}_i$, where $\mathbf{t}_i$ are a basic set of simple roots, then $h = 1 + \Sigma z^i$; (c) $\mathfrak{R}$ contains a subgroup isomorphic to $\mathfrak{S}_n$, the symmetric group on n objects.

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