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Lipschitz Type Characterizations for Bergman Spaces

Hasi Wulan and Kehe Zhu

Abstract. We obtain new characterizations for Bergman spaces with standard weights in terms of Lipschitz type conditions in the Euclidean, hyperbolic, and pseudo-hyperbolic metrics. As a consequence, we prove optimal embedding theorems when an analytic function on the unit disk is symmetrically lifted to the bidisk.

1 Introduction

Let $\mathbb{D}$ be the open unit disk in the complex plane $\mathbb{C}$. For any $\alpha > -1$ we consider the weighted area measure

$$dA_\alpha(z) = (\alpha + 1)(1 - |z|^2)^\alpha \, dA(z),$$

where dA is the normalized area measure on $\mathbb{D}$. It is easy to see that each dA_α is a probability measure on $\mathbb{D}$.

For $p > 0$ and $\alpha > -1$ we denote by A^p_α the space of analytic functions f in $\mathbb{D}$ such that

$$\int_{\mathbb{D}} |f(z)|^p \, dA_\alpha(z) < \infty.$$

These are called *weighted Bergman spaces* with standard weights. See [1] and [6] for the modern theory of Bergman spaces.

Three different metrics on the unit disk will be used in the paper. First, the usual Euclidean metric is of course written as $|z-w|$. Second, the pseudo-hyperbolic metric on $\mathbb{D}$ is given by

$$\rho(z, w) = \left| \frac{z - w}{1 - \overline{z}w} \right|.$$

And finally, the hyperbolic metric on $\mathbb{D}$ is given by

$$\beta(z, w) = \frac{1}{2} \log \frac{1 + \rho(z, w)}{1 - \rho(z, w)}.$$

We mention that the hyperbolic metric is also called the Bergman metric, and sometimes the Poincare metric as well.

The main result of the paper is the following.

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Theorem 1.1 *Suppose $p > 0$, $\alpha > -1$, and f is analytic in $\mathbb{D}$. Then the following conditions are equivalent.*

(a) *f belongs to A^p_α.*

(b) *There exists a continuous function g in $L^p(\mathbb{D}, dA_\alpha)$ such that*

$$|f(z) - f(w)| \le \rho(z, w)(g(z) + g(w))$$

for all z and w in $\mathbb{D}$.

(c) *There exists a continuous function g in $L^p(\mathbb{D}, dA_\alpha)$ such that*

$$|f(z) - f(w)| \le \beta(z, w)(g(z) + g(w))$$

for all z and w in $\mathbb{D}$.

(d) *There exists a continuous function g in $L^p(\mathbb{D}, dA_{p+\alpha})$ such that*

$$|f(z) - f(w)| \le |z - w|(g(z) + g(w))$$

for all z and w in $\mathbb{D}$.

Note that the same measure dA_α appears in conditions (a), (b), and (c), but condition (d) involves a different measure, $dA_{p+\alpha}$.

Similar characterizations for Hardy–Sobolev spaces have appeared in the literature before. See [4, 5] for example. The present paper was motivated by [10]. As another motivation for our result, we mention that the classical Bloch space $\mathcal{B}$, consisting of analytic functions f in $\mathbb{D}$ such that

$$\sup\{(1 - |z|^2)|f'(z)| : z \in \mathbb{D}\} < \infty,$$

also admits a Lipschitz type characterization. More specifically, an analytic function f in $\mathbb{D}$ belongs to $\mathcal{B}$ if and only if there exists a positive constant C such that

$$|f(z) - f(w)| \le C\beta(z, w)$$

for all z and w in $\mathbb{D}$; see [6] or [14] for example. It is then clear that an analytic function f in $\mathbb{D}$ belongs to $\mathcal{B}$ if and only if there exists a continuous function g in $L^\infty(\mathbb{D})$ such that

$$|f(z) - f(w)| \le \beta(z, w)(g(z) + g(w))$$

for all z and w in $\mathbb{D}$.

Since the Bergman metric is based on the reproducing kernel of the Bergman space, it is no surprise that our Lipschitz type characterizations for weighted Bergman spaces appear more natural when the Bergman metric (and its bounded counterpart, the pseudo-hyperbolic metric) is used. Although a characterization in terms of the Euclidean metric (condition (d)) is possible, it gives one the impression of something artificial.

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2 Preliminaries

We collect some preliminary results in this section that involve the hyperbolic and pseudo-hyperbolic metrics.

For any $0 < r < 1$ and $z \in \mathbb{D}$ we let $D(z,r) = \{w \in \mathbb{D} : \rho(w,z) < r\}$ denote the pseudo-hyperbolic disk centered at z with radius r. It is well known that $D(z,r)$ is actually a Euclidean disk with Euclidean center and Euclidean radius given by

$$\frac{1-r^2}{1-r^2|z|^2}\,z, \qquad \frac{1-|z|^2}{1-r^2|z|^2}\,r,$$

respectively; see [3] for example. In particular, if r is fixed, then the area of $D(z,r)$ is comparable to $(1-|z|^2)^2$.

The pseudo-hyperbolic metric is bounded by 1. But the hyperbolic metric is unbounded. For any $R > 0$ and $z \in \mathbb{D}$ we let $E(z,R) = \{w \in \mathbb{D} : \beta(w,z) < R\}$ denote the hyperbolic disk centered at z with radius R. If $0 < r < 1$ and

$$R = \frac{1}{2}\log\frac{1+r}{1-r}, \tag{2.1}$$

then it is clear that $E(z,R) = D(z,r)$. Consequently, if R is fixed, then the area of $E(z,R)$ is comparable to $(1-|z|^2)^2$ as well. By the same token, any estimate in terms of the pseudo-hyperbolic metric can be translated to one in terms of the hyperbolic metric, and vice versa.

Lemma 2.1 *For any fixed $r \in (0,1)$ there exists a positive constant C such that*

$$C^{-1} \le \frac{1-|z|^2}{|1-\bar{z}w|} \le C$$

whenever $\rho(z,w) \le r$. Consequently, there exists a positive constant C such that

$$C^{-1} \le \frac{1-|z|^2}{1-|w|^2} \le C$$

whenever $\rho(z,w) \le r$.

Proof This is well known. See [6] or [14] for example. ∎

Lemma 2.2 *For any fixed $r \in (0,1)$ there exists a positive constant C such that*

$$|f(z)|^p \le \frac{C}{(1-|z|^2)^2}\int_{D(z,r)} |f(w)|^p\, dA(w)$$

for all $z \in \mathbb{D}$, all $p > 0$, and all analytic f in $\mathbb{D}$.

Proof This is well known as well. See [6] or [14] for example. ∎

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Lemma 2.3 *For any fixed $z \in \mathbb{D}$ we have*

$$\lim_{w \to z} \frac{\beta(z,w)}{|z-w|} = \lim_{w \to z} \frac{\rho(z,w)}{|z-w|} = \frac{1}{1-|z|^2}.$$

Proof This follows from elementary calculations. ∎

Lemma 2.4 *For any $\alpha > -1$ and $p > 0$ there exists a constant $C > 0$ such that*

$$\int_{\mathbb{D}} |f(z)|^p \, dA_\alpha(z) \le C\Big[\,|f(0)|^p + \int_{\mathbb{D}} |g(z)|^p \, dA_\alpha(z)\Big]$$

and

$$|f(0)|^p + \int_{\mathbb{D}} |g(z)|^p \, dA_\alpha(z) \le C \int_{\mathbb{D}} |f(z)|^p \, dA_\alpha(z)$$

for all analytic functions f in $\mathbb{D}$, where $g(z) = (1-|z|^2)f'(z)$ for $z \in \mathbb{D}$.

Proof See [6] or [14] for example. ∎

3 The Main Result

We now prove the main result of the paper. The proof is constructive in the sense that we will actually produce a formula for the function g that appears in various conditions of Theorem 1.1. We prove Theorem 1.1 as three separate results.

Theorem 3.1 *Suppose $p > 0$, $\alpha > -1$, and f is analytic in $\mathbb{D}$. Then $f \in A^p_\alpha$ if and only if there exists a continuous function $g \in L^p(\mathbb{D}, dA_\alpha)$ such that*

$$|f(z) - f(w)| \le \rho(z,w)(g(z) + g(w)) \tag{3.1}$$

for all z and w in $\mathbb{D}$.

Proof First assume that condition (3.1) holds. Then

$$\Big|\frac{f(z)-f(w)}{z-w}\Big| \le \frac{\rho(z,w)}{|z-w|}(g(z)+g(w))$$

for all $z \ne w$ in $\mathbb{D}$. Fix any $z \in \mathbb{D}$, let $w \to z$, and use Lemma 2.3. We obtain

$$(1-|z|^2)|f'(z)| \le 2g(z), \qquad z \in \mathbb{D}.$$

Since $g \in L^p(\mathbb{D}, dA_\alpha)$, an application of Lemma 2.4 shows that $f \in A^p_\alpha$.

Next assume that $f \in A^p_\alpha$. We are going to produce a continuous function g that satisfies condition (3.1). To this end we fix a radius $r \in (0,1)$ and consider any two points z and w in $\mathbb{D}$ with $\rho(z,w) < r$. By the fundamental theorem of calculus,

$$f(z) - f(w) = (z-w)\int_0^1 f'\big(tz + (1-t)w\big)\, dt.$$

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Since z and w are points in the convex set $D(z, r)$, we see that $tz + (1-t)w \in D(z, r)$ for all $t \in [0, 1]$. It follows that

$$|f(z) - f(w)| \le |z - w| \sup\{|f'(u)| : u \in D(z, r)\}.$$

By Lemma 2.1, there exists a positive constant C that only depends on r such that $|f(z) - f(w)| \le \rho(z, w)h(z)$, where

$$h(z) = C \sup\{(1 - |u|^2)|f'(u)| : u \in D(z, r)\}, \qquad z \in \mathbb{D}.$$

Obviously, we also have $|f(z) - f(w)| \le \rho(z, w)(h(z) + h(w))$, if $\rho(z, w) < r$.

If $\rho(z, w) \ge r$, then clearly $|f(z) - f(w)| \le \frac{\rho(z,w)}{r}(|f(z)| + |f(w)|)$. If we define

$$g(z) = \frac{|f(z)|}{r} + h(z), \qquad z \in \mathbb{D},$$

then we have $|f(z) - f(w)| \le \rho(z, w)(g(z) + g(w))$ for all z and w in $\mathbb{D}$.

It is clear that the function g above is continuous on $\mathbb{D}$. It remains for us to show that $g \in L^p(\mathbb{D}, dA_\alpha)$. Since f is already in $L^p(\mathbb{D}, dA_\alpha)$, it suffices for us to show that $h \in L^p(\mathbb{D}, dA_\alpha)$.

Recall that if r and R are related as in (2.1), then $D(z, r) = E(z, R)$. So we can choose $r' \in (0, 1)$ so that $D(z, r') = E(z, 2R)$ for all $z \in \mathbb{D}$. By the triangle inequality for the Bergman metric β we have $E(u, R) \subset E(z, 2R)$ whenever $u \in E(z, R)$. Equivalently, we have $D(u, r) \subset D(z, r')$ whenever $u \in D(z, r)$. It follows from this and Lemma 2.2 that there exists a constant $C > 0$ that depends only on r such that

$$h(z)^p \le \frac{C}{(1 - |z|^2)^{2-p}} \int_{D(z,r')} |f'(w)|^p \, dA(w)$$

for all $z \in \mathbb{D}$. If we use χ_z to denote the characteristic function of the set $D(z, r')$, then clearly $\chi_z(w) = \chi_w(z)$, and

$$h(z)^p \le \frac{C}{(1 - |z|^2)^{2-p}} \int_{\mathbb{D}} |f'(w)|^p \chi_z(w) \, dA(w)$$

for all $z \in \mathbb{D}$. Writing $C_1 = (\alpha + 1)C/\pi$ and using Fubini's theorem, we obtain

$$\begin{aligned}
\int_{\mathbb{D}} h(z)^p \, dA_\alpha(z) &\le C_1 \int_{\mathbb{D}} (1 - |z|^2)^{p+\alpha-2} \, dA(z) \int_{\mathbb{D}} |f'(w)|^p \chi_z(w) \, dA(w) \\
&= C_1 \int_{\mathbb{D}} |f'(w)|^p \, dA(w) \int_{\mathbb{D}} (1 - |z|^2)^{p+\alpha-2} \chi_w(z) \, dA(z) \\
&= C_1 \int_{\mathbb{D}} |f'(w)|^p \, dA(w) \int_{D(w,r')} (1 - |z|^2)^{p+\alpha-2} \, dA(z).
\end{aligned}$$

Combining this with Lemma 2.1 and the fact that the area of $D(w, r')$ is comparable to $(1 - |w|^2)^2$, we obtain another positive constant C_2, which depends only on α and r, such that

$$\int_{\mathbb{D}} h(z)^p \, dA_\alpha(z) \le C_2 \int_{\mathbb{D}} |f'(w)|^p (1 - |w|^2)^{p+\alpha} \, dA(w).$$

In view of Lemma 2.4, this shows that $h \in L^p(\mathbb{D}, dA_\alpha)$ and completes the proof of Theorem 3.1. ■

Theorem 3.2 *Suppose $p > 0$, $\alpha > -1$, and f is analytic in $\mathbb{D}$. Then $f \in A^p_\alpha$ if and only if there exists a continuous function $g \in L^p(\mathbb{D}, dA_\alpha)$ such that*

$$|f(z) - f(w)| \leq \beta(z, w)(g(z) + g(w)) \tag{3.2}$$

for all z and w in $\mathbb{D}$.

Proof If condition (3.2) is satisfied, we divide both sides of (3.2) by $|z-w|$ and, with the help of Lemma 2.3, take the limit as $w \to z$. The result is

$$(1 - |z|^2)|f'(z)| \leq 2g(z), \qquad z \in \mathbb{D}.$$

This along with Lemma 2.4 shows that $f \in A^p_\alpha$.

If $f \in A^p_\alpha$, then by Theorem 3.1 there exists a continuous function $g \in L^p(\mathbb{D}, dA_\alpha)$ such that condition (3.1) holds. Since $\rho(z, w) \leq \beta(z, w)$ for all z and w in $\mathbb{D}$, the same function g also satisfies condition (3.2). This completes the proof of Theorem 3.2. ■

Theorem 3.3 *Suppose $p > 0$, $\alpha > -1$, and f is analytic in $\mathbb{D}$. Then $f \in A^p_\alpha$ if and only if there exists a continuous function $g \in L^p(\mathbb{D}, dA_{p+\alpha})$ such that*

$$|f(z) - f(w)| \leq |z - w|(g(z) + g(w)) \tag{3.3}$$

for all z and w in $\mathbb{D}$.

Proof If condition (3.3) holds, we divide both sides of (3.3) by $|z - w|$ and take the limit as $w \to z$. The result is $|f'(z)| \leq 2g(z)$, so that

$$(1 - |z|^2)|f'(z)| \leq 2(1 - |z|^2)g(z)$$

for all $z \in \mathbb{D}$. Since $g \in L^p(\mathbb{D}, dA_{p+\alpha})$, we see that the function $(1-|z|^2)f'(z)$ belongs to $L^p(\mathbb{D}, dA_\alpha)$, which, according to Lemma 2.4, means that $f \in A^p_\alpha$.

If $f \in A^p_\alpha$, then by Theorem 3.1 there exists a continuous function h in $L^p(\mathbb{D}, dA_\alpha)$ such that

$$|f(z) - f(w)| \leq \rho(z, w)(h(z) + h(w))$$

for all z and w in $\mathbb{D}$. Rewrite this as

$$|f(z) - f(w)| \leq |z - w| \left[\frac{h(z)}{|1 - \bar{z}w|} + \frac{h(w)}{|1 - \bar{z}w|} \right],$$

and apply the triangle inequalities

$$|1 - \bar{z}w| \geq 1 - |z|, \quad |1 - \bar{z}w| \geq 1 - |w|.$$

We obtain

$$|f(z) - f(w)| \leq |z - w|(g(z) + g(w)), \qquad z, w \in \mathbb{D},$$

where

$$g(z) = \frac{h(z)}{1 - |z|} \leq \frac{2h(z)}{1 - |z|^2}.$$

Since $h \in L^p(\mathbb{D}, dA_\alpha)$, we have $g \in L^p(\mathbb{D}, dA_{p+\alpha})$. This completes the proof of Theorem 3.3. ■

4 Lifting Functions from the Disk to the Bidisk

Let $\mathbb{D}^2 = \mathbb{D} \times \mathbb{D}$ denote the bidisk in $\mathbb{C}^2$ and let $H(\mathbb{D}^2)$ denote the space of all holomorphic functions in $\mathbb{D}^2$. Similarly, $H(\mathbb{D})$ is the space of all analytic functions in $\mathbb{D}$. For $p > 0$ and $\alpha > -1$ we define $A^p_\alpha(\mathbb{D}^2)$ as the space of functions $f \in H(\mathbb{D}^2)$ such that

$$\int_{\mathbb{D}} \int_{\mathbb{D}} |f(z,w)|^p \, dA_\alpha(z) \, dA_\alpha(w) < \infty.$$

These are also called weighted Bergman spaces.

In this section we present an application of our main theorem to the problem of lifting analytic functions from the unit disk to the bidisk. Thus we consider the symmetric lifting operator

$$L\colon H(\mathbb{D}) \to H(\mathbb{D}^2)$$

defined by

$$L(f)(z,w) = \frac{f(z) - f(w)}{z - w}.$$

We will also need the associated diagonal operator $\Delta\colon H(\mathbb{D}^2) \to H(\mathbb{D})$ which is defined by $\Delta(f)(z) = f(z,z)$. The action of the diagonal operator on Hardy and Bergman spaces of the polydisk has been studied by several authors, (see [2,9,12,13]). The diagonal operator was also used in [7] and [8] to study the reproducing kernel for certain weighted Bergman spaces in the bidisk. We begin with the following property of the diagonal operator.

Lemma 4.1 *Suppose $p > 0$ and $\alpha > -1$. Then the operator Δ maps $A^p_\alpha(\mathbb{D}^2)$ boundedly onto $A^p_{2(\alpha+1)}(\mathbb{D})$.*

Proof See [12] or [13]. ■

The following standard estimate will be needed in the proof of our lifting theorems.

Lemma 4.2 *Suppose $s > -1$, t is real, and*

$$I(z) = \int_{\mathbb{D}} \frac{(1 - |w|^2)^s \, dA(z)}{|1 - \overline{z}w|^{2+s+t}}, \qquad z \in \mathbb{D}.$$

Then $I(z)$ is bounded in $\mathbb{D}$ whenever $t < 0$, and $I(z)$ is bounded by $(1-|z|^2)^{-t}$ whenever $t > 0$.

Proof See [6] or [14]. ■

We now obtain the first lifting theorem.

Theorem 4.3 *Suppose $\alpha > -1$ and $0 < p < \alpha + 2$. Then the symmetric lifting operator L maps $A^p_\alpha(\mathbb{D})$ boundedly into $A^p_\alpha(\mathbb{D}^2)$. Moreover, this is no longer true when $p > \alpha + 2$.*

Proof Given $f \in A^p_\alpha(\mathbb{D})$, we apply Theorem 3.1 to find a function $g \in L^p(\mathbb{D}, dA_\alpha)$ such that condition (3.1) holds. Then there exists a constant $C = C_p$ such that

$$\big|L(f)(z,w)\big|^p \le C\Big[\frac{g(z)^p}{|1-\bar{z}w|^p} + \frac{g(w)^p}{|1-\bar{z}w|^p}\Big]. \tag{4.1}$$

It follows that

$$\int_{\mathbb{D}}\int_{\mathbb{D}} |L(f)(z,w)|^p \; dA_\alpha(z)\, dA_\alpha(w) \le 2C \int_{\mathbb{D}} g(z)^p \, dA_\alpha(z) \int_{\mathbb{D}} \frac{dA_\alpha(w)}{|1-\bar{z}w|^p}.$$

When $p < 2+\alpha$, an application of Lemma 4.2 shows that there exists another constant $C > 0$ such that

$$\int_{\mathbb{D}}\int_{\mathbb{D}} |L(f)(z,w)|^p \; dA_\alpha(z)\, dA_\alpha(w) \le C \int_{\mathbb{D}} g(z)^p \, dA_\alpha(z).$$

This shows that L maps $A^p_\alpha(\mathbb{D})$ into $A^p_\alpha(\mathbb{D}^2)$. A standard argument based on the closed graph theorem then shows that the operator $L\colon A^p_\alpha(\mathbb{D}) \to A^p_\alpha(\mathbb{D}^2)$ must be bounded.

On the other hand, let us suppose that $L\colon A^p_\alpha(\mathbb{D}) \to A^p_\alpha(\mathbb{D}^2)$ is bounded. Then by Lemma 4.1, the operator $D = \Delta \circ L$ maps $A^p_\alpha(\mathbb{D})$ boundedly into $A^p_{2(\alpha+1)}(\mathbb{D})$. It is easy to see that $Df = f'$. Therefore, $f \in A^p_\alpha(\mathbb{D})$ would imply that

$$\int_{\mathbb{D}} \big| (1-|z|^2) f'(z) \big|^p \, dA_{2(\alpha+1)-p}(z) < \infty,$$

which, according to Lemma 2.4, is a condition that is strictly stronger than $f \in A^p_\alpha(\mathbb{D})$ when $p > 2+\alpha$. So our lifting theorem cannot possibly hold for $p > 2+\alpha$. ■

We mention that the case $p = 2+\alpha$ is not covered by the above result. When $\alpha = 0$, we can show by Taylor expansion that the operator L does not map $A^2(\mathbb{D})$ into $A^2(\mathbb{D}^2)$ (these are the unweighted Bergman spaces). In fact, if $f(z) = \sum_{k=0}^{\infty} a_k z^k$ is a function in $A^2(\mathbb{D})$, then

$$\int_{\mathbb{D}} |f(z)|^2 \, dA(z) = \sum_{k=0}^{\infty} \frac{|a_k|^2}{k+1}.$$

On the other hand,

$$L(f)(z,w) = \sum_{k=0}^{\infty} a_k \frac{z^k - w^k}{z-w} = \sum_{k=1}^{\infty} a_k \sum_{i+j=k-1} z^i w^j,$$

and for $k \neq m$, the homogeneous polynomials

$$\sum_{i+j=k-1} z^i w^j, \qquad \sum_{i+j=m-1} z^i w^j,$$

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are orthogonal with respect to the measure $dA(z)dA(w)$ on $\mathbb{D}^2$. So the integral

$$I = \int_{\mathbb{D}} \int_{\mathbb{D}} |L(f)(z,w)|^2 \, dA(z)\, dA(w)$$

can be computed as follows.

$$\begin{aligned} I &= \sum_{k=1}^{\infty} |a_k|^2 \int_{\mathbb{D}} \int_{\mathbb{D}} \Big| \sum_{i+j=k-1} z^i w^j \Big|^2 \, dA(z)\, dA(w) \\ &= \sum_{k=1}^{\infty} |a_k|^2 \sum_{i+j=k-1} \frac{1}{(i+1)(j+1)} = \sum_{k=1}^{\infty} \frac{|a_k|^2}{k+1} \sum_{i+j=k-1} \Big[\frac{1}{i+1} + \frac{1}{j+1} \Big] \\ &= 2 \sum_{k=1}^{\infty} \frac{|a_k|^2}{k+1} \sum_{j=0}^{k-1} \frac{1}{j+1} \sim \sum_{k=1}^{\infty} \frac{|a_k|^2}{k+1} \log(k+1) \sim \int_{\mathbb{D}} |f(z)|^2 \log \frac{1}{1-|z|^2} \, dA(z). \end{aligned}$$

This shows that the integral I is not necessarily finite, so the symmetric lifting operator L does not map $A^2(\mathbb{D})$ into $A^2(\mathbb{D}^2)$.

The following result tells us what happens when $p > \alpha + 2$.

Theorem 4.4 *Suppose $\alpha > -1$, $p > \alpha + 2$, and β is determined by $2(\beta + 1) = p + \alpha$. Then the symmetric lifting operator L maps $A^p_\alpha(\mathbb{D})$ boundedly into $A^p_\beta(\mathbb{D}^2)$.*

Proof Given $f \in A^p_\alpha(\mathbb{D})$, we once again appeal to Theorem 3.1 to obtain a function $g \in L^p(\mathbb{D}, dA_\alpha)$ such that condition (4.1) holds. We then have

$$\int_{\mathbb{D}} \int_{\mathbb{D}} \big| L(f)(z,w) \big|^p \, dA_\beta(z)\, dA_\beta(w) \le 2C \int_{\mathbb{D}} g(z)^p \, dA_\beta(z) \int_{\mathbb{D}} \frac{dA_\beta(w)}{|1-\bar{z}w|^p}.$$

Since $p + \alpha = 2(\beta + 1)$ and $p > \alpha + 2$, we must have $\beta > -1$ and $\beta > \alpha$. We write the inner integral above as

$$(\beta+1) \int_{\mathbb{D}} \frac{(1-|w|^2)^\beta \, dA(w)}{|1-\bar{z}w|^{2+\beta+(\beta-\alpha)}}$$

and apply Lemma 4.2 to find another positive constant C such that

$$\int_{\mathbb{D}} \int_{\mathbb{D}} \big| L(f)(z,w) \big|^p \, dA_\beta(z)\, dA_\beta(w) \le C \int_{\mathbb{D}} g(z)^p \, dA_\alpha(z).$$

This along with the closed graph theorem proves the desired result. ∎

Once again, with the help of Lemmas 4.1 and 2.4, we can check that the lifting effect of L guaranteed by Theorem 4.4 is best possible.

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5 Generalization to the Unit Ball

In this section we explain how to generalize our main result to the context of the unit ball in $\mathbb{C}^n$. Thus we let $\mathbb{B}_n = \{z \in \mathbb{C}^n : |z| < 1\}$ denote the open unit ball in $\mathbb{C}^n$. For $\alpha > -1$ let $dv_\alpha(z) = c_\alpha(1-|z|^2)^\alpha\, dv(z)$, where dv is normalized volume measure on $\mathbb{B}_n$ and c_α is a positive normalizing constant so that $v_\alpha(\mathbb{B}_n) = 1$.

For $p > 0$ and $\alpha > -1$ let

$$A^p_\alpha(\mathbb{B}_n) = L^p(\mathbb{B}_n, dv_\alpha) \cap H(\mathbb{B}_n)$$

denote the weighted Bergman spaces, where $H(\mathbb{B}_n)$ is the space of all holomorphic functions in $\mathbb{B}_n$. See [14] for basic properties of these spaces.

For any $a \in \mathbb{B}_n$ there exists a biholomorphic map φ_a on $\mathbb{B}_n$ such that $\varphi_a(0) = a$ and $\varphi_a^{-1} = \varphi_a$. These are sometimes called symmetrices (or involutive automorphisms) of $\mathbb{B}_n$. Explicit formulas are available for φ_a, (see [11] or [14]).

It is well known that the Bergman metric on $\mathbb{B}_n$ induces the following distance:

$$\beta(z,w) = \frac{1}{2}\log\frac{1+|\varphi_z(w)|}{1-|\varphi_z(w)|}.$$

It follows that $\rho(z,w) = |\varphi_z(w)|$ is also a distance function on $\mathbb{B}_n$. We shall also call ρ the pseudo-hyperbolic metric on $\mathbb{B}_n$. The Euclidean metric on $\mathbb{B}_n$ is still denoted by $|z-w|$.

Theorem 5.1 *Suppose $p > 0$, $\alpha > -1$, and $f \in H(\mathbb{B}_n)$. Then the following conditions are equivalent.*

(a) *f belongs to $A^p_\alpha(\mathbb{B}_n)$.*

(b) *There exists a continuous function $g \in L^p(\mathbb{B}_n, dv_\alpha)$ such that*

$$|f(z) - f(w)| \le \rho(z,w)(g(z)+g(w))$$

for all z and w in $\mathbb{B}_n$.

(c) *There exists a continuous function $g \in L^p(\mathbb{B}_n, dv_\alpha)$ such that*

$$|f(z) - f(w)| \le \beta(z,w)(g(z)+g(w))$$

for all z and w in $\mathbb{B}_n$.

(d) *There exists a continuous function $g \in L^p(\mathbb{B}_n, dv_{p+\alpha})$ such that*

$$|f(z) - f(w)| \le |z-w|(g(z)+g(w))$$

for all z and w in $\mathbb{B}_n$.

The proof follows the same ideas used in the proofs of Theorems 3.1, 3.2, and 3.3. Lemma 2.1 holds for the ball without any change, (see [14]).

The only change needed in Lemma 2.2 is the exponent of $1-|z|^2$. In the context of $\mathbb{B}_n$, it should be $(1-|z|^2)^{n+1}$ instead of $(1-|z|^2)^2$, (see [14]). Similarly, the volume of $D(z,r)$ (or $E(z,R)$) is comparable to $(1-|z|^2)^{n+1}$ whenever r (or R) is fixed.

Lemmas 2.3 and 2.4 need to be modified substantially before they can be used. These are the contents of the next two lemmas. But first, we recall three useful notations of differentiation in $\mathbb{B}_n$.

Given $f \in H(\mathbb{B}_n)$, we write

$$Rf(z) = \sum_{k=1}^{n} z_k \frac{\partial f}{\partial z_k}(z),$$

and call it the radial derivative of f at z. In fact, $Rf(z)$ is the directional derivative of f at z in the radial direction (that is, the direction in z):

$$Rf(z) = \lim_{t \to 1} \frac{f(tz) - f(z)}{t},$$

where t is a scalar.

The complex gradient of f at z is defined by

$$|\nabla f(z)| = \Big[\sum_{k=1}^{n} \Big| \frac{\partial f}{\partial z_k}(z) \Big|^2 \Big]^{1/2}.$$

And the invariant complex gradient of f at z is given by

$$|\widetilde{\nabla} f(z)| = |\nabla (f \circ \varphi_z)(0)|, \qquad z \in \mathbb{B}_n.$$

We can now state the analogs of Lemmas 2.3 and 2.4 that will suit our needs.

Lemma 5.2 *Suppose $z \in \mathbb{B}_n$ and $w = tz$, where t is a scalar. Then*

$$\lim_{w \to z} \frac{\rho(w, z)}{|z - w|} = \lim_{w \to z} \frac{\beta(z, w)}{|z - w|} = \frac{1}{1 - |z|^2}.$$

Proof This follows from the explicit formulas for φ_a given in [11] and [14]. ■

Lemma 5.3 *Suppose $p > 0$, $\alpha > -1$, and $f \in H(\mathbb{B}_n)$. Then the following conditions are equivalent.*

(a) *The function f is in $A^p_\alpha(\mathbb{B}_n)$.*
(b) *The function $(1 - |z|^2) Rf(z)$ is in $L^p(\mathbb{B}_n, dv_\alpha)$.*
(c) *The function $(1 - |z|^2)|\nabla f(z)|$ is in $L^p(\mathbb{B}_n, dv_\alpha)$.*
(d) *The function $|\widetilde{\nabla} f(z)|$ is in $L^p(\mathbb{B}_n, dv_\alpha)$.*

Proof See [14]. ■

We can now outline the proof of Theorem 5.1.

First assume that there exists a continuous function $g \in L^p(\mathbb{B}_n, dv_\alpha)$ such that

$$|f(z) - f(w)| \le \rho(z, w)(g(z) + g(w)) \tag{5.1}$$

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for all z and w in $\mathbb{B}_n$. We then fix z in $\mathbb{B}_n$ and let $w = tz$, where t is a scalar. Then

$$\frac{|f(z) - f(w)|}{|z - w|} \leq \frac{\rho(z, w)}{|z - w|}(g(z) + g(w))$$

for all $z \neq w$ in $\mathbb{B}_n$. Let w approach z in the radial direction and apply Lemma 5.2. We obtain $(1 - |z|^2)|Rf(z)| \leq 2g(z)$ for all $z \in \mathbb{B}_n$. According to Lemma 5.3, this is the same as $f \in A^p_\alpha(\mathbb{B}_n)$.

On the other hand, for any holomorphic function f in $\mathbb{B}_n$ and any $z \in \mathbb{B}_n$, we have

$$f(z) - f(0) = \int_0^1 \Big[\sum_{k=1}^n z_k \frac{\partial f}{\partial z_k}(tz)\Big]\, dt.$$

It follows that for $\rho(z, 0) < r$, where $r \in (0, 1)$ is any fixed radius in the pseudo-hyperbolic metric, we have

$$|f(z) - f(0)| \leq |z|\,|\sup\{|\nabla f(w)| : w \in D(0, r)\}.$$

It is easy to see that in the relatively compact set $D(0, r)$ the Euclidean metric is comparable to the pseudo-hyperbolic metric (as well as the Bergman metric β). It is also easy to see that $|\nabla f(w)|$ is comparable to $|\widetilde{\nabla} f(w)|$ in the relatively compact set $D(0, r)$. So we can find a constant $C > 0$, that depends on r but not on f, such that

$$|f(z) - f(0)| \leq C\rho(z, 0) \sup\{|\widetilde{\nabla} f(w)| : w \in D(0, r)\}$$

for all $z \in D(0, r)$. Replace f by $f \circ \varphi_w$, then replace z by $\varphi_w(z)$, and use the Möbius invariance of the pseudo-hyperbolic metric and the invariant gradient. We obtain

$$|f(z) - f(w)| \leq C\rho(z, w) \sup\{|\widetilde{\nabla} f(u)| : u \in D(z, r)\}$$

for all z and w in $\mathbb{B}_n$ with $\rho(z, w) < r$. Let

$$g(z) = \frac{|f(z)|}{r} + C \sup\{|\widetilde{\nabla} f(u)| : u \in D(z, r)\}.$$

Then it is clear that condition (5.1) is satisfied, and as in the proof of Theorem 3.1, $g \in L^p(\mathbb{B}_n, dv_\alpha)$.

So conditions (a) and (b) are equivalent in Theorem 5.1.

If condition (b) holds in Theorem 5.1, then condition (c) holds for the same function g, because $\rho \leq \beta$. If condition (c) holds, then an application of Lemma 5.2 shows that $(1 - |z|^2)|Rf(z)| \leq 2g(z)$ for all $z \in \mathbb{B}_n$, which, according to Lemma 5.3, implies condition (a). Therefore, conditions (a), (b), and (c) are all equivalent.

Now let us assume that condition (d) holds, so there exists a continuous function g in $L^p(\mathbb{B}_n, dv_{p+\alpha})$ such that

$$|f(z) - f(w)| \leq |z - w|(g(z) + g(w)) \tag{5.2}$$

for all z and w in $\mathbb{B}_n$. Rewrite this as

$$\frac{|f(z)-f(w)|}{|z-w|} \le g(z)+g(w)$$

and let w approach z in the direction of a complex coordinate axis. We obtain

$$\Big|\frac{\partial f}{\partial z_k}(z)\Big| \le 2g(z), \qquad 1 \le k \le n,$$

so that $|\nabla f(z)| \le 2\sqrt{n}\, g(z)$ for all $z \in \mathbb{B}_n$. This together with the assumption that $g \in L^p(\mathbb{B}_n, dv_{p+\alpha})$ shows that the function $(1-|z|^2)|\nabla f(z)|$ is in $L^p(\mathbb{B}_n, dv_\alpha)$. In view of Lemma 5.3, this is the same as $f \in A^p_\alpha(\mathbb{B}_n)$. So condition (d) implies (a) in Theorem 5.1.

Finally let us assume that condition (b) holds in Theorem 5.1. It follows from the well-known identity (see [11] or [14])

$$1-|\varphi_z(w)|^2 = \frac{(1-|z|^2)(1-|w|^2)}{|1-\langle z,w\rangle|^2}$$

that

$$\frac{\rho(z,w)^2}{|z-w|^2} = \frac{|z-w|^2+|\langle z,w\rangle|^2-|z|^2|w|^2}{|z-w|^2|1-\langle z,w\rangle|^2}.$$

By the triangle inequality for the natural inner product in $\mathbb{C}^n$, we always have $|\langle z,w\rangle|^2 \le |z|^2|w|^2$. We deduce that

$$\rho(z,w) \le \frac{|z-w|}{|1-\langle z,w\rangle|}$$

for all z and w in $\mathbb{B}_n$. As in the proof of Theorem 3.3, this along with condition (b) implies the existence of a continuous function g in the space $L^p(\mathbb{B}_n, dv_{p+\alpha})$ such that condition (5.2) holds. So condition (b) implies (d), and the proof of Theorem 5.1 is complete.

Although we have not checked whether the function

$$d(z,w) = \frac{|z-w|}{|1-\langle z,w\rangle|}, \qquad z,w \in \mathbb{B}_n,$$

defines a metric on $\mathbb{B}_n$, it is clear by now that the pseudo-hyperbolic metric ρ used in Theorem 5.1 can be replaced by d. In other words, a holomorphic function f in $\mathbb{B}_n$ belongs to the Bergman space $A^p_\alpha(\mathbb{B}_n)$ if and only if there exists a continuous function $g \in L^p(\mathbb{B}_n, dv_\alpha)$ such that $|f(z)-f(w)| \le d(z,w)(g(z)+g(w))$ for all z and w in $\mathbb{B}_n$.

References

[1] P. Duren and A. Schuster, *Bergman spaces.* Mathematical Surveys and Monographs 100, American Mathematical Society, Providence, RI, 2004.

[2] P. Duren and A. Shields, *Restriction of H^p functions to the diagonal of the polydisk.* Duke Math. J. **42**(1975), no. 4, 751–753.

[3] J. Garnett, *Bounded analytic functions.* Pure and Applied Mathematics 96, Academic Press, New York, 1981.

[4] P. Hajlasz, *Sobolev spaces on an arbitrary metric space.* Potential Anal. **5**(1996), no. 4, 403–415.

[5] ———, *Sobolev spaces on metric-measure spaces.* In: Heat kernels and analysis on manifolds, graphs, and metric spaces, Contemp. Math. 338, American Mathematical Society, Providence, RI, 2003, pp. 173–218.

[6] H. Hedenmalm, B. Korenblum, and K. Zhu, *Theory of Bergman spaces.* Graduate Texts in Mathematics 199, Springer-Verlag, New York, 2000.

[7] H. Hedenmalm and S. Shimorin, *Weighted Bergman spaces and the integral means spectrum of conformal mappings.* Duke Math. J. **127**(2005), no. 2, 341–393.

[8] H. Hedenmalm, S. Shimorin, and A. Sola, *Norm expansion along a zero variety.* J. Funct. Anal. **254**(2008), no. 6, 1601–1625.

[9] C. Horowitz and D. M. Oberlin, *Restriction of $H^p(\mathbb{D}^n)$ functions to the diagonal of $\mathbb{D}^n$.* Indiana Univ. Math. J. **24**(1974/75), 767–772.

[10] P. Koskela and E. Saksman, *Pointwise characterizations of Hardy-Sobolev functions.* Math. Res. Lett. **15**(2008), no. 4, 727–744.

[11] W. Rudin, *Function theory in the Unit Ball of $\mathbb{C}^n$.* Fundamental Principles in Mathematical Science 241, Springer-Verlag, New York-Berlin, 1980.

[12] F. A. Shamoyan, *Applications of Dzhrbashyan integral representations to some problems of analysis.* Dokl. Akad. Nauk SSSR **261**(1981), no. 3, 557–561.

[13] M. Stessin and K. Zhu, *Composition operators on embedded disks.* J. Operator Theory **56**(2006), no. 2, 423–449.

[14] K. Zhu, *Spaces of holomorphic functions in the unit ball.* Graduate Texts in Mathematics 226, Springer-Verlag, New York, 2005.

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