AN INTEGRAL FORMULA FOR COMPACT HYPERSURFACES IN A EUCLIDEAN SPACE AND ITS APPLICATIONS

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1. Introduction. Let M be a compact hypersurface in a Euclidena space $\mathbb{R}^{n+1}$. The support function ρ of M is the component of the position vector field of M in $\mathbb{R}^{n+1}$ along the unit normal vector field to M, which is a smooth function defined on M. Let S be the scalar curvature of M. The object of the present paper is to prove the following theorems.

THEOREM 1. *Let M be a compact hypersurface of $\mathbb{R}^{n+1}$ with non-negative Ricci curvature. Then*

$$\mathrm{Av}(S) \geq n(n-1)/\mathrm{diam}^2(M),$$

where $\mathrm{Av}(S)$ *is the average scalar curvature of M given by the Einstein functional* $\mathrm{Av}(S) = 1/\mathrm{vol}(M) \int_M S\, dv$, *and* $\mathrm{diam}(M)$ *is the diameter of M.*

THEOREM 2. *Let M be a compact hypersurface of $\mathbb{R}^{n+1}$ with non-negative Ricci curvature. If M is centrally symmetric and R is the radius of the escribed sphere, then* $R^2 \geq n(n-1)/\mathrm{Av}(S)$.

THEOREM 3. *Let M be a compact and connected hypersurface of positive Ricci curvature in $\mathbb{R}^{n+1}$. If the support function ρ of M satisfies $\rho^2 \leq n(n-1)/S$, then ρ is a constant and M is a sphere of radius ρ.*

THEOREM 4. *Let M be a compact and connected hypersurface of non-negative Ricci curvature in $\mathbb{R}^{n+1}$. If M is contained in a closed ball of radius R centered at origin in $\mathbb{R}^{n+1}$ and the scalar curvature S of M satisfies* $\sup S = n(n-1)R^{-2}$, *then M is the sphere of radius R.*

All above theorems are consequences of an integral formula which we prove in Section 2. We observe that Theorem 4 generalizes Theorem 1 in **[1]** for hypersurfaces of non-negative Ricci curvature in a Euclidean space. We also get the following corollary to Theorem 1 which generalizes the result of Jacobwitz **[2]** for non-immersibility of a compact Riemannian manifold of non-negative Ricci curvature into a closed ball in a Euclidean space.

COROLLARY. *Let M be a compact n-dimensional Riemannian manifold of non-negative Ricci curvature whose average scalar curvature satisfies* $\mathrm{Av}(S) < n(n-1)R^{-2}$. *Then no isometric immersion of M into $\mathbb{R}^{n+1}$ is contained in a closed ball of radius R in $\mathbb{R}^{n+1}$.*

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2. Integral formula. Let M be a compact hypersurface in $\mathbb{R}^{n+1}$ and N be the globally defined unit normal vector field on M. We denote by g, ∇ and A, the induced metric, the covariant derivative operator with respect to the induced Riemannian

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connection and the shape operator on M. Then we have

$$\bar{\nabla}_X Y = \nabla_X Y + g(AX, Y)N, \qquad \bar{\nabla}_X N = -AX, \qquad X, Y \in \mathscr{X}(M), \tag{2.1}$$

where $\bar{\nabla}$ is the covariant derivative operator with respect to the Euclidean connection on $\mathbb{R}^{n+1}$ and $\mathscr{X}(M)$ is the Lie algebra of vector fields on M. Let T be the position vector field on $\mathbb{R}^{n+1}$. Then the smooth function $\rho: M \to R$ defined by $\rho = \langle T|_M, N\rangle$ is called the support function of the hypersurface M, where $\langle\, ,\rangle$ is the Euclidean metric on $\mathbb{R}^{n+1}$. We have $T|_M = \xi + \rho N$, $\xi \in \mathscr{X}(M)$. Then since $\bar{\nabla}_X T = X$ holds for any $X \in \mathscr{X}(M)$, using (2.1), we have

$$\nabla_X \xi = X + \rho AX, \qquad d\rho(X) = -g(AX, \xi), \qquad X \in \mathscr{X}(M). \tag{2.2}$$

From the equation of Gauss for hypersurface M in $\mathbb{R}^{n+1}$, we get the following expressions for Ricci curvature and scalar curvature S of M (cf. [3])

$$\operatorname{Ric}(X, Y) = n\alpha g(AX, Y) - g(AX, AY), \qquad X, Y \in \mathscr{X}(M), \tag{2.3}$$

$$S = n^2\alpha^2 - \operatorname{tr} A^2, \tag{2.4}$$

where $\alpha = 1/n \sum g(Ae_i, e_i)$ is the mean curvature of M and $\{e_1, \ldots, e_n\}$ is a local orthonormal frame on M.

LEMMA 2.1.

$$\int_M \{\operatorname{Ric}(\xi, \xi) + n(n-1) - \rho^2 S\}\, dv = 0.$$

Proof. We use equation (2.2) to compute the Laplacian of the support function ρ and the divergence of the vector field ξ, and obtain

$$\Delta\rho = -n\, d\alpha(\xi) - n\alpha - \rho \operatorname{tr} A^2, \qquad \operatorname{div} \xi = n(1 + \rho\alpha). \tag{2.5}$$

Integrating second equation over M we get

$$\int_M (1 + \rho\alpha)\, dv = 0. \tag{2.6}$$

Also we have

$$-n\rho\, d\alpha(\xi) = n\alpha \operatorname{div}(\rho\xi) - \operatorname{div}(n\alpha\rho\xi) = n\alpha\, d\rho(\xi) + n\rho\alpha \operatorname{div} \xi - \operatorname{div}(n\alpha\rho\xi).$$

Using second equation in (2.2) and above equation in (2.5), we find

$$\rho\, \Delta\rho = -n\alpha g(A\xi, \xi) + n^2\rho\alpha + n^2\rho^2\alpha^2 - n\rho\alpha - p^2 \operatorname{tr} A^2 - \operatorname{div}(n\alpha\rho\xi). \tag{2.7}$$

We find $\operatorname{grad} \rho = -A\xi$ from equation (2.2) and use it together with equation (2.7) in $\frac{1}{2}\Delta\rho^2 = \rho\, \Delta\rho + \|\operatorname{grad} \rho\|^2$ to obtain

$$\tfrac{1}{2}\Delta\rho^2 = -\operatorname{Ric}(\xi, \xi) - n(n-1) + n(n-1)(1 + \rho\alpha) + \rho^2 S - \operatorname{div}(n\alpha\rho\xi),$$

where we have also used equation (2.3) and (2.4). Integrating above equation over M and using integral formula (2.6), we get the desired result.

3. Proofs of theorems. Theorems 1 and 2 follow directly from Lemma 2.1, as their hypotheses give

$$\operatorname{diam}^2(M) \int_M S\, dv \geq \int_M \rho^2 S\, dv \geq n(n-1)\operatorname{vol}(M).$$

For Theorem 3, we observe that Lemma 2.1 gives $\xi = 0$, which when combined with equation (2.2) gives ρ is a constant. That ρ is a non-zero constant is guarnateed by integral formula (2.6). Thus in this case from first equation in (2.2), we have that M is a totally umbilic and hence a sphere of radius ρ (cf. [**3**], p. 30).

The hypothesis of Theorem 4 confirms that $R^2 S \le n(n-1)$. Now using a unit vector field $t = \xi/\|\xi\|$, defined on the open subset of M where ξ is non-zero, and $\|T|_M\|^2 = \|\xi\|^2 + \rho^2$ in Lemma 2.1, we obtain

$$\int_M \{\|\xi\|^2 (\mathrm{Ric}(t,t) + S) + (n(n-1) - \|T|_M\|^2 S)\}\, dv = 0.$$

Since M lies in a closed ball of radius R in $\mathbb{R}^{n+1}$, we may assume that $\|T|_M\|^2 \le R^2$ and consequently we have

$$n(n-1) - \|T|_M\|^2 S \ge n(n-1) - R^2 S \ge 0,$$

where we have used $S \ge 0$ which follows from the hypothesis. Thus the above integral together with the inequality above gives $\xi = 0$ and $\|T|_M\|^2 = R^2 = \rho^2$. This proves the theorem.

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