Prologue

Abstract mathematics brings me great joy. It is also enlightening, illuminating, applicable, and indeed “useful”, but for me that is not its driving force. For me its driving force is joy. The joy it gives me moves me to want to pursue it further and further, immerse myself in it more and more deeply, and engage with the discipline it seems to involve.

This is how I lead all of my life, where I can. I might look disciplined: in the way I do research, write books, practice the piano, follow complex recipes. But I firmly believe that you only need discipline to accomplish something unpleasant. I prefer to find a way to enjoy it. This approach also works better: the enjoyment takes me much further than discipline ever could.

Abstract mathematics is different from what is usually seen in school mathematics. Mathematics in school is typically about numbers, equations and solving problems. Abstract mathematics is not. Mathematics in school has a focus on getting the right answer. Abstract mathematics does not. Mathematics in school unfortunately puts many people off the subject. Abstract mathematics need not.

The aim of this book is to introduce abstract mathematics not usually seen by non-specialists, and to change attitudes about what mathematics is, what it is for, and how it works. The purpose might either be general interest or further study. The specific subject of the book is Category Theory, but along the way we will get a taste of various important mathematical objects including different types of number, shape, surface and space, types of abstract structure, the worlds they form, and some open research questions about them.

This Prologue will provide some background about the book’s motivation, style and contents, and guidance about a range of intended audiences. In summary: if you’re interested in learning some advanced mathematics that is very different from school math, but find traditional textbooks too dry or requiring too much background, read on.

The status of mathematics

Math has an image problem. Many people are put off it at school and end up as adults either hating it, being afraid of it, or defensively boasting about how bad they are at it or how irrelevant it is anyway. Complaints about math that I hear most commonly from my art students include that it is rigid, uncreative, and requires too much memorization; that the questions have nothing to do with real life and that the answers involve too many rules to be interesting; that it's useful for scientists and engineers but pointless for anyone else.

On the other hand, as an abstract mathematician I revel in how flexible and creative the field is, and how *little* memorization it requires. I am invigorated and continually re-awakened by how the way of thinking is pertinent to all aspects of life. I adore how its richness and insight come from not having to follow anyone else's rules but instead creating different worlds from different rules and seeing what is possible. And I believe that while certain parts are useful for science and engineering, my favorite parts are powerful and illuminating for everyone.

I think there are broadly three reasons math education is important.

1. As a foundation for further study in mathematical fields.
2. For direct usefulness in life.
3. To develop a particular way of thinking.

The first point, further study, is the one that is obviously not relevant to everyone: it doesn't apply if you have absolutely decided you are not going into further study in mathematical (and by extension scientific) fields.

The second point, usefulness, is often emphasized as a reason that math is compulsory for so long at school, but there seems to be a wide range of views about what this actually means. It certainly doesn't seem to justify the endless study of triangles, graph sketching, solving quadratic equations, trigonometric identities and so on. Some people focus on arithmetic and are convinced that math is important so that we don't have to rely on calculators to add up our grocery bill, calculate a tip at a restaurant, or work out how much we'll pay when something is on sale at 20% off. Others argue that the math we teach is not relevant enough and we should teach things like mortgages, interest rates, and how to do your taxes. All of these views are much more utilitarian than the view this book will take.

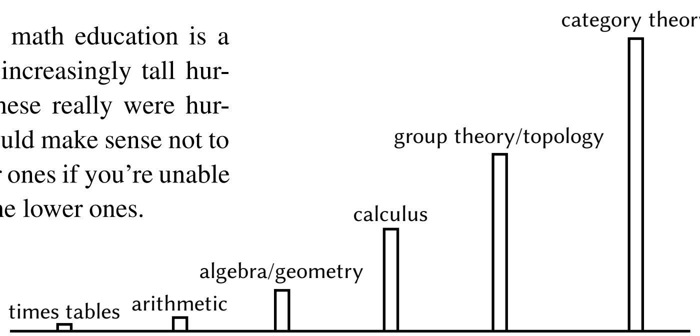
The third point is about math as a way of thinking, and is the one that drives both my research and my teaching. Abstract mathematics is not just a topic of study. It is a way of thinking that makes connections between diverse situations to help us unify them and think about them more efficiently. It focuses our attention on what is relevant for a particular point of view and temporarily disregards the rest so that we can get to the heart of a structure or an argument. In making these connections and finding these deep structures we package up intractably complex situations into succinct units, enabling us to address yet more complicated situations and use our limited brain power to greater effect. This starts with numbers, where instead of saying "1 + 1" all the time we can call it 2, or we fit squares together and call the result a cube, and then build up to more complex mathematical structures as we'll see throughout this book.

This is what I think the power and importance of abstract mathematics are. The idea that it is relevant to the whole of life and thus illuminating for everyone may be surprising, but is demonstrated by the wide range of examples that I have found where category theory helps, despite the field being considered perhaps the "most abstract" of all mathematics. This includes examples such as privilege, sexism, racism, sexual harassment. These are not the sort of contrived real life examples involving the purchase of 17 watermelons, but are *real* real life questions, things we actually do (or should) think about in our daily lives.

If people are put off math then they are put off these ways of thinking that could really intrigue and help them. The sad part is that they are put off an entirely different kind of math usually involving algorithms, formulae, memorization and rigid rules, which is not what this abstract math is about at all. Math is misunderstood, and the first impression many people get of it is enough to put them off, forever, something that they might have been able to appreciate and benefit from if they saw it in its true light.

Traditional mathematics: subjects

A typical math education is a series of increasingly tall hurdles. If these really were hurdles it would make sense not to try higher ones if you're unable to clear the lower ones.



"Hurdles" model of math learning.

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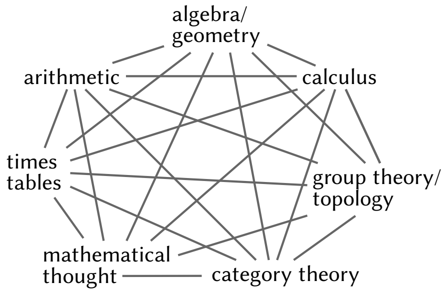
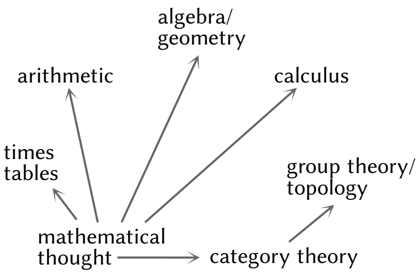
However, math is really more like an interconnected web of ideas, perhaps like this; everything is connected to everything else, and thus there are many possible routes around this web depending on what sort of brain you have.



"Interconnectedness" in math learning.

Some people do need to build up gradually through concrete examples towards abstract ideas. But not everyone is like that. For some people, the concrete examples don't make sense until they've grasped the abstract ideas or, worse, the concrete examples are so offputting that they will give up if presented with those first. When I was first introduced to single malt whisky I thought I didn't like it, but I later discovered it was because people were trying to introduce me "gently" via single malts they considered "good for beginners". It turns out I only like the extremely smoky single malts of Islay, not the sweeter, richer ones you might be expected to acclimatize with.

I am somewhat like that with math as well. My route through the web of mathematics was something like this diagram.



My progress to higher level mathematics did not use my knowledge of mathematical subjects I was taught earlier. In fact after learning category theory I went back and understood everything again and much better.

I have confirmed from several years of teaching abstract mathematics to art students that I am not the only one who prefers to use abstract ideas to illuminate concrete examples rather than the other way round. Many of these art students consider that they're bad at math because they were bad at memorizing times tables, because they're bad at mental arithmetic, and they can't solve equations. But this doesn't mean they're bad at math — it just means they're not very good at times tables, mental arithmetic and equations, an absolutely tiny part of mathematics that hardly counts as abstract at all. It turns out that they do not struggle nearly as much when we get to abstract things such as

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higher-dimensional spaces, subtle notions of equivalence, and category theory structures. Their blockage on mental arithmetic becomes irrelevant.

It seems to me that we are denying students entry into abstract mathematics when they struggle with non-abstract mathematics, and that this approach is counter-productive. Or perhaps some students self-select out of abstract mathematics if they did not enjoy non-abstract mathematics. This is as if we didn't let people try swimming because they are a slow runner, or if we didn't let them sing until they're good at the piano.

One aim of this book is to present abstract mathematics directly, in a way that does not depend on proficiency with other parts of mathematics. It doesn't have to matter if you didn't make it over some of those earlier hurdles.

Traditional mathematics: methods

When I studied modern languages at school there were four facets tested with different exams: reading, writing, speaking and listening. Of those, writing and speaking are "productive" where reading and listening are "receptive". For full mastery of the language all four are needed of course, but if complete fluency is beyond you it can still be rewarding to be able to do only some of these things. I later studied German for the purposes of understanding the songs of Schubert (and Brahms, Strauss, Schumann, and so on). My productive German is almost non-existent, but I can understand Romantic German poetry at a level including some nuance, and this is rewarding for me and helps me in my life as a collaborative pianist.

I think there is a notion of "productive" and "receptive" mathematics as well. Productive mathematics is about being able to answer questions, say, homework questions or exam questions, and, later on, produce original research. There is a fairly widely held view that the only way to understand math is to work through problems. There is a further view that this is the only way of doing math that is worthwhile. I would like to change that.

I view "receptive" mathematics as being about appreciating math even if you can't solve unseen problems. It's being able to follow an argument even if you wouldn't be able to build it yourself.

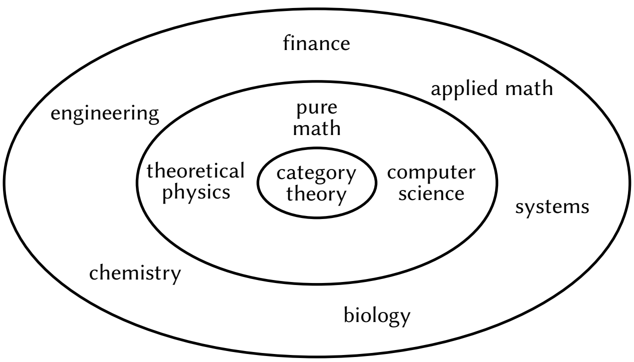
I can appreciate German poetry, restaurant food, a violin concerto, a Caravaggio, a tennis match. Imagine if appreciation were only taught by doing. I can even read a medical research paper although I can't practice medicine. The former is still valuable. In math some authors call this "mathematical tourism" with undertones of disdain. But I think tourism is fine — it would be a shame if the only options for traveling were to move somewhere to live there or else stay at home. I actually once spoke to a representative from a health insurance company who thought this was the case, and did not comprehend the concept that I might visit a different state and ask about coverage there.

One particular feature of this book is that I will not demand that the reader does any exercises in order to follow the book. It is standard in math books to exhort the reader to work through exercises, but I believe this is offputting to many non-mathematicians, as well as some mathematicians (including me). I will provide "Things to think about" from time to time, but these will really be questions to ponder rather than exercises of any sort. And one of the main purposes of those questions will be to develop our instincts for the sorts of questions that mathematicians ask. The hope is that as we progress, the reader will think of those questions spontaneously, before I have made them explicit. Thinking of "natural" next questions is one important aspect of mathematical thinking. Where working through them is beneficial to understanding what follows I will include that discussion afterwards.

The content in this book

Category theory was introduced by Eilenberg and Mac Lane in the 1940s and has since become more or less ubiquitous in pure mathematics. In some fields it is at the level of a language, in others it is a framework, in others a tool, in others it is the foundations, in others it is what the whole structure depends on.

Category theory quickly found uses beyond pure math, in theoretical physics and computer science. The view of things at the end of the 20th century might be regarded like this, with the diagram showing applications moving outwards from category theory:



However, since then category theory has become increasingly pervasive,

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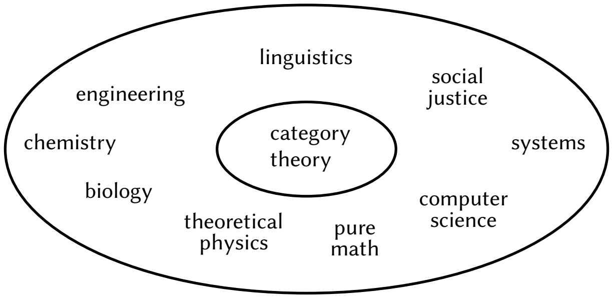
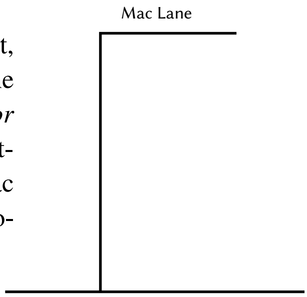
finding direct applications in a much wider range of subjects further from pure mathematics, such as ecological diversity, chemistry, systems control, engineering, air traffic control, linguistics, social justice. The picture now might be thought of as more like this:



For some time the only textbook on the subject, from which everyone had to try and learn, was the classic graduate text by Mac Lane, *Categories for the Working Mathematician* (from 1971). The situation was this: there was a huge step up to Mac Lane, which many people, even those highly motivated, failed to be able to reach.



As is the way with these things, what started as a research field had become something that graduate students (tried to) study, and eventually it trickled down into a few undergraduate courses at individual universities around the world that happened to have an expert there who wanted to teach it at that level. This spawned several much more approachable textbooks at the turn of the 21st century, notably those by Lawvere and Schanuel (1997), followed by a sort of second wave with Awodey (2006), Leinster (2014) and Riehl (2016). There was still a gap up to those books, and the gap was still insurmountable for many people who didn't have the background of an undergraduate mathematician, either in terms of experience with the formality of mathematics or background for the examples being used.[†]

In 2015 I wrote *How to Bake π*, a book about category theory for an entirely

[†] Lawvere and Schanuel include high school students in their stated target audience but I think they have in mind quite advanced ones. There are also some recent books aimed at specific types of audience, which are less in the vein of standard textbooks; see Further reading.

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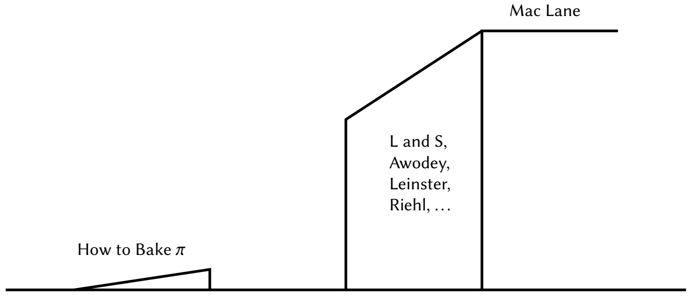
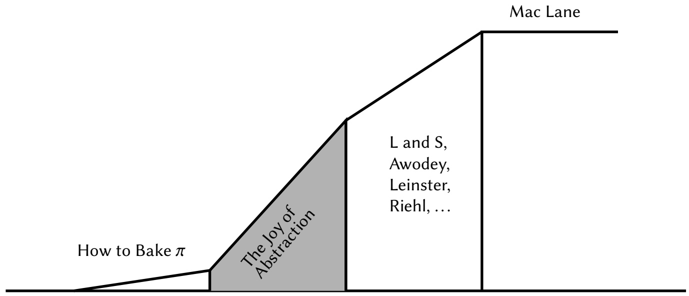
general audience with no math background whatever. The situation became like this:



How to Bake π provides a ramp from more or less nothing but does not get very far into the subject and remains mostly informal throughout. The role of the present work is to fill the remaining gap:



This book will not assume undergraduate level mathematics training, nor even a disposition for it, but it will become technical. The aim, after all, is to bridge the gap up to the undergraduate textbooks. We will build up very gradually towards the rigorous formality of category theory. While there is no particular technical prerequisite for learning category theory, the formality of any mathematical field can be offputting to those who are not used to it. The intention is that you can read this book *and turn the pages*, in contrast with so many math books where you have to sit and think for an hour, week or month about a single paragraph (although those books have their place too). It will be informal and personal in style, including descriptions of my personal feelings about various aspects of the subject. This is not strictly part of mathematics but I believe that building an emotional engagement is an important part of learning anything. There will also be many diagrams to help with visualizing, partly to engage visual people and partly because the subject is very visual. There will be both formal mathematical diagrams, and informal "schematic" diagrams. All these things mean that the book will in some sense be the op-

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posite of terse — long, perhaps, for the technical content contained in it. But I believe this is the key to reaching a wide audience not inclined (or not yet prepared) for reading terse technical mathematics.

Audience

I am aiming this book at anyone who wants to delve into some contemporary mathematics that is not done in school, and is very different from the kind of math done in school. This is regardless of your previous math achievement and your future math goals. I will not assume any knowledge or any recollection of school math, and I will gradually build up the level of formality in case it is the symbols and notation that have previously put you off. Here are some different types of reader I imagine:

- Adults who regret being put off math in the past and think, deep down, that they should be able to understand math if it's presented differently.
- Adults who always liked math and miss it, and feel like having some further mathematical stimulation.
- Anyone who wishes to learn some contemporary mathematics not covered in the standard curriculum, though I hope it might be one day.
- Math teachers who want to extend their knowledge and/or get ideas for how to teach abstract math without much (or any) prerequisite.
- School students who want extra math to stretch themselves, and/or an introduction into higher level math different from what students are usually stretched with at school.
- School students who are unhappy with their math classes and who might benefit from seeing a more profound, less routine type of math.
- Non-mathematicians who want to learn category theory but find the existing textbooks beyond them. Judging from correspondence since *How to Bake π* this might include programmers, engineers, business people, psychologists, linguists, writers, artists and more.
- Undergraduate mathematicians who have heard that category theory is important but are not sure why such abstraction is necessary, or are not sure how to approach it.
- Those with a math degree who would still like to have a gentle companion book to the existing texts, that contains more of the spirit of category theory alongside the technicalities.
- Home-schools and summer camps.

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How to Bake π is not exactly a prerequisite but having read it will almost certainly help.

This material is developed from my teaching art students at the School of the Art Institute of Chicago. Most of the students had bad experiences of school math and many of them either can't remember any of it or have deliberately forgotten all of it as they found it so traumatic. This book seeks to be different from all of those types of experiences. It might seem long in terms of pages, but I hope you will quickly find that you can get through the pages much faster than you can for a standard math textbook. If the content here were written in a standard way it might only take 100 pages. I didn't want to make it shorter by explaining things less, so I have made it longer by explaining things more fully. I will gradually introduce formal mathematical language, and have included a glossary at the end for quick reference. I occasionally include the names of related concepts that are beyond the scope of this book, not because I think you need to know them, but in case you are interested and would like to look them up.

One obstacle to non-mathematicians trying to learn category theory is that the examples used are often taken from other parts of pure mathematics. In this book I will be sure to use examples that do not do that, including examples from daily experience such as family relationships, train journeys, freezing and thawing food, and more hard-hitting subjects such as racism and privilege. I have found that this helps non-mathematicians connect with abstract mathematics in ways that mathematical examples do not. Where I do include mathematical examples I will either introduce them fully as new concepts, or point out where they are not essential for continuing but are included for interest for those readers who have heard of them.

In particular, if you think you're bad at mental arithmetic, terrible at algebra, can't solve equations, and shudder at the thought of sketching graphs, that need not be an obstacle for you to read this book. I am not saying that you will find the book easy: abstraction is a way of thinking that takes some building of ability. We will build up through Part One of the book, and definitely take off in Part Two. It should be intellectually stretching, otherwise we wouldn't have achieved anything. But your previous experiences with math need not bar your way in here as they might previously have seemed to do. Most of all, aside from the technicalities of category theory I want to convey the joy I feel in the subject: in learning it, researching it, using it, applying it, thinking about it. More than technical prowess or a huge litany of theorems, I want to share the joy of abstraction.