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VOICULESCU'S THEOREM FOR NONSEPARABLE C*-ALGEBRAS

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Abstract. We prove that Voiculescu's noncommutative version of the Weyl-von Neumann Theorem can be extended to all unital, separably representable C*-algebras whose density character is strictly smaller than the (uncountable) cardinal invariant $\mathfrak{p}$. We show moreover that Voiculescu's Theorem consistently fails for C*-algebras of larger density character.

§1. Introduction. Voiculescu's work in [17], a far-reaching extension of the results by Weyl [18] and von Neumann [16] on unitary equivalence up to compact perturbation of self-adjoint operators, is one of the cornerstones of the theory of extensions of separable C*-algebras (see [9] for a nice exposition of these results).

The label 'Voiculescu's Theorem' often refers to a collection of results and corollaries from [17], rather than a specific Theorem. Through this paper, it always refers to the following statement, where for a complex Hilbert space H, $\mathcal{B}(H)$ is the algebra of linear bounded operators from H into itself, and $\mathcal{K}(H)$ is the algebra of compact operators.

VOICULESCU'S THEOREM. *Let H,L be two separable Hilbert spaces, $\mathcal{A}\subseteq\mathcal{B}(H)$ a separable unital C^*-algebra and $\sigma:\mathcal{A}\to\mathcal{B}(L)$ a unital completely positive map such that $\sigma(a)=0$ for all $a\in\mathcal{A}\cap\mathcal{K}(H)$. Then there is a sequence of isometries $V_n:L\to H$ such that $\sigma(a)-V_n^*aV_n\in\mathcal{K}(L)$ and $\lim_{n\to\infty}\|\sigma(a)-V_n^*aV_n\|=0$ for all $a\in\mathcal{A}$.*

See Section 2 for a definition of completely positive map. A prototypical example of application of Voiculescu's Theorem is the case where $H=L$ is infinite-dimensional, $\mathcal{A}\cap\mathcal{K}(H)=\{0\}$ and σ is a $*$-homomorphism such that $\sigma(a)\in\mathcal{K}(H)$ if and only if $a=0$. Let $q:\mathcal{B}(H)\to\mathcal{Q}(H)$ be the quotient map onto the Calkin algebra $\mathcal{Q}(H)=\mathcal{B}(H)/\mathcal{K}(H)$. In this setting, both the restriction of q to $\mathcal{A}$ and the composition $q\circ\sigma$ are unital embeddings of $\mathcal{A}$ inside $\mathcal{Q}(H)$. Voiculescu's Theorem implies that these two maps are unitarily equivalent,[1] and more in general it entails that any two unital embeddings of $\mathcal{A}$ inside $\mathcal{Q}(H)$ which are liftable to unital $*$-homomorphisms to $\mathcal{B}(H)$, are unitarily equivalent.

Another specific instance of Voiculescu's Theorem is the case where $L=\mathbb{C}$ and σ is a state (i.e., a functional on $\mathcal{A}$ such that $\sigma(1)=1$; states are completely positive

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[1]Two maps $\phi_1,\phi_2:\mathcal{A}\to\mathcal{B}$ between unital C*-algebras are *unitarily equivalent* if there is a unitary $u\in\mathcal{B}$ such that $u\phi_1(a)u^*=\phi_2(a)$ for all $a\in\mathcal{A}$.

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maps, see [7, Example 1.5.2]). This statement has been known for a longer time, and it is usually referred to as Glimm's Lemma [15, Lemma 3.6.1].

GLIMM'S LEMMA. *Let H be a separable Hilbert space, $\mathcal{A} \subseteq \mathcal{B}(H)$ a separable unital C^*-algebra and $\sigma : \mathcal{A} \to \mathbb{C}$ a state such that $\sigma(a) = 0$ for all $a \in \mathcal{A} \cap \mathcal{K}(H)$. There exists a sequence of orthonormal vectors $\{\xi_n\}_{n\in\mathbb{N}}$ such that $\sigma(a) = \lim_{n\to\infty}\langle a\xi_n, \xi_n\rangle$ for every $a \in \mathcal{A}$.*

We refer the reader to [1] and [15, Sections 3.4–3.6] for a proof of these classical results.

One of the main fields of applications of Voiculescu's Theorem has been (and still is) the study of the extensions of a C*-algebra and of the invariant Ext, which flourished after the seminal work in [6] (see [15] or [5] for an introduction to this subject). Given a separable and unital C*-algebra $\mathcal{A}$, Ext$(\mathcal{A})$ is the set of all unital embeddings of $\mathcal{A}$ into the Calkin algebra $\mathcal{Q}(H)$ (modulo the relation of unitary equivalence). The set Ext$(\mathcal{A})$ can be endowed with a semigroup structure, and one of the main consequences of Voiculescu's Theorem in this framework is that Ext$(\mathcal{A})$ has an identity, namely the class of all embeddings which are liftable to unital $*$-homomorphisms to $\mathcal{B}(H)$ (see [15, Sections 2–3]). This, along with the results in [8], implies for instance that Ext$(\mathcal{A})$ is a group for every nuclear, separable, unital C*-algebra $\mathcal{A}$.

Voiculescu's Theorem has also been recently employed in combination with set theory in the study of which nonseparable C*-algebras embed into the Calkin algebra, in [11] and [12].

The current paper pushes further the interaction of the results in [17] with set theory, extending Voiculescu's Theorem to certain 'small' nonseparable C*-algebras. Here, the notion of 'smallness' is determined by the cardinal invariant $\mathfrak{p}$, of which we quickly recall the definition (a reference for a general introduction to cardinal invariants is [2]). Let $[\mathbb{N}]^{\mathbb{N}}$ the set of all infinite subsets of $\mathbb{N}$. We define an order relation $\subseteq^*$ on $[\mathbb{N}]^{\mathbb{N}}$ by saying that, for $X, Y \in [\mathbb{N}]^{\mathbb{N}}$, $X \subseteq^* Y$ iff Y contains X up to a finite number of elements, namely iff $X \setminus Y$ is finite. A family $\mathcal{F} \subseteq [\mathbb{N}]^{\mathbb{N}}$ is *centered* iff for every $X_1, \dots, X_n \in \mathcal{F}$ there exists $X \in [\mathbb{N}]^{\mathbb{N}}$ such that $X \subseteq^* X_j$ for all $j \leq n$. The value $\mathfrak{p}$ is defined as the smallest possible cardinality of a centered family $\mathcal{F}$ for which there is no $Y \in [\mathbb{N}]^{\mathbb{N}}$ such that $Y \subseteq^* X$ for all $X \in \mathcal{F}$. It is a simple exercise to check that $\mathfrak{p}$ is uncountable. Indeed, let $\{X_n\}_{n\in\mathbb{N}} \subseteq [\mathbb{N}]^{\mathbb{N}}$ be a centered family. For each $n \in \mathbb{N}$, let $Y_n \in [\mathbb{N}]^{\mathbb{N}}$ be such that $Y_n \subseteq^* X_j$ for all $j \leq n$. This allows to find, for every $n \in \mathbb{N}$, some $z_n \in Y_n$ so that $z_n \in \cap_{j\leq n} X_j$ and $z_n \neq z_j$ for all $j < n$. It follows that $Z := \{z_n\}_{n\in\mathbb{N}} \in [\mathbb{N}]^{\mathbb{N}}$ and $Z \subseteq^* X_j$ for all $j \in \mathbb{N}$. It is consistent with ZFC that $\mathfrak{p} = \mathfrak{c}$ (where $\mathfrak{c}$ is the cardinality of the continuum), for instance this happens when the continuum hypothesis or Martin's Axiom are assumed (see [3]).

Before stating the main result of this note, one last observation. Let H, L be two separable Hilbert spaces, $\mathcal{A} \subseteq \mathcal{B}(H)$ a nonseparable unital C*-algebra and $\sigma : \mathcal{A} \to \mathcal{B}(L)$ a unital completely positive map such that $\sigma(a) = 0$ for all $a \in \mathcal{A} \cap \mathcal{K}(H)$. If we do not assume anything else, all that Voiculescu's Theorem can guarantee is the existence of a net of isometries $\{V_\lambda\}_{\lambda\in\Lambda}$ from L into H such that, for any separable subalgebra $\mathcal{B} \subset \mathcal{A}$ and $\varepsilon > 0$, there is $\mu \in \Lambda$ such that $\lambda \geq \mu$ implies $\sigma(a) - V_\lambda^* a V_\lambda \in \mathcal{K}(L)$ and $\|\sigma(a) - V_\lambda^* a V_\lambda\| < \varepsilon$ for all $a \in \mathcal{B}$. In this note we prove the following Theorem.

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THEOREM 1. *Let H be a separable Hilbert space.*

(1) *Let L be a separable Hilbert space, $\mathcal{A} \subseteq \mathcal{B}(H)$ a unital C^*-algebra of density character strictly less than $\mathfrak{p}$ and $\sigma : \mathcal{A} \to \mathcal{B}(L)$ a unital completely positive map such that $\sigma(a) = 0$ for all $a \in \mathcal{A} \cap \mathcal{K}(H)$. Then there is a countable sequence of isometries $V_n : L \to H$ such that $\sigma(a) - V_n^* a V_n$ is compact and $\lim_{n\to\infty} \|\sigma(a) - V_n^* a V_n\| = 0$ for all $a \in \mathcal{A}$.*

(2) *Given a cardinal λ, it is consistent with $\mathsf{ZFC} + \mathfrak{c} \geq \lambda$ that there exist a unital C^*-algebra $\mathcal{A} \subseteq \mathcal{B}(H)$ of density character less than $\mathfrak{c}$, and σ, a state of $\mathcal{A}$ annihilating $\mathcal{A} \cap \mathcal{K}(H)$, for which Glimm's Lemma fails.*

Theorem 1 gives the following corollary.

COROLLARY 2. *The statement 'Voiculescu's Theorem holds for all separably representable C^*-algebras of density character less than $\mathfrak{c}$' is independent from* ZFC. *Moreover, it is independent from* $\mathsf{ZFC} + \mathfrak{c} \geq \lambda$ *for any cardinal* λ.

PROOF. Under Martin's Axiom, which implies $\mathfrak{p} = \mathfrak{c}$, the statement holds by Item 1 of Theorem 1. The consistent failure of the statement follows by Item 2 of Theorem 1. ⊣

The argument used to obtain the first part of Theorem 1 is inspired to the proof of Voiculescu's Theorem as given by Arveson in [1] (see also [15, Sections 3.4–3.6]). We show that such proof essentially consists of a sequence of diagonalization arguments which are equivalent to applications of the Baire category Theorem to certain σ-centered partial orders (see Section 2 for a definition). Item 1 of Theorem 1 then follows by the results in [3], where it is shown that Martin's Axiom (see Section 2 for a definition) holds for κ-sized families of dense subsets of σ-centered partial orders if and only if $\kappa < \mathfrak{p}$. This equivalence is the reason why $\mathfrak{p}$ appears in the statement of Theorem 1.

The second item of Theorem 1 is obtained via an application of Cohen's forcing and a simple cardinality argument. Starting from a C*-algebra $\mathcal{A}$ of density character $\mathfrak{c}$ for which Glimm's Lemma fails, we show that the lemma still fails for $\mathcal{A}$ also after adding enough (but not too many) Cohen reals.

We remark that Voiculescu's Theorem is false in general for subalgebras of $\mathcal{B}(H)$ of denisty character $\mathfrak{c}$, as witnessed by $\ell^\infty(\mathbb{N})$, $L^\infty([0,1])$ and $\mathcal{B}(H)$ itself (see Section 4). We do not know if the notion of smallness given by $\mathfrak{p}$ in this context is optimal, or whether it is consistent that there are C*-algebras of density character greater than or equal to $\mathfrak{p}$ for which the conclusion of Voiculescu's Theorem holds.

The paper is organized as follows. Section 2 is devoted to definitions and preliminaries. In Section 3 we prove Item 1 of Theorem 1 and list some standard corollaries of Voiculescu's Theorem which generalize to C*-algebras of density character smaller than $\mathfrak{p}$. Finally in Section 4 we give a proof of Item 2.

§2. Preliminaries. Through this paper, given a complex Hilbert space H, $\mathcal{B}(H)$ is the algebra of linear bounded operators from H into itself, and $\mathcal{K}(H)$ is the algebra of compact operators. The Calkin algebra $\mathcal{Q}(H)$ is the quotient $\mathcal{B}(H)/\mathcal{K}(H)$.

An *approximate unit* of a C*-algebra $\mathcal{A}$ is a net $\{h_\lambda\}_{\lambda\in\Lambda}$ of positive contractions of $\mathcal{A}$ such that $\lim_\lambda \|h_\lambda a - a\| = \lim_\lambda \|a h_\lambda - a\| = 0$ for all $a \in \mathcal{A}$. Given a C*-algebra

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$\mathcal{A} \subseteq \mathcal{B}(H)$, an approximate unit of $\mathcal{K}(H)$ is *quasicentral* for $\mathcal{A}$ if $\lim_\lambda \|[h_\lambda, a]\| = 0$ for all $a \in \mathcal{A}$, where $[b,c]$ denotes, for two operators $b, c \in \mathcal{B}(H)$, the commutant $bc - cb$.

For a C*-algebra $\mathcal{A}$, let $M_n(\mathcal{A})$ be the C*-algebra of $n \times n$ matrices with entries in $\mathcal{A}$. Given two C*-algebras $\mathcal{A}$ and $\mathcal{B}$, a bounded linear map $\sigma : \mathcal{A} \to \mathcal{B}$ is *completely positive* if for all $n \in \mathbb{N}$ all the maps $\phi_n : M_n(\mathcal{A}) \to M_n(\mathcal{B})$, defined as

$$\phi_n([a_{ij}]) = [\phi(a_{ij})],$$

are positive, i.e., they send positive elements into positive elements.

The notation $F \Subset G$ stands for 'F is a finite subset of G'.

A *partially ordered set* (or simply *poset*) $(\mathbb{P}, \leq)$ is a set equipped with a binary, transitive, antisymmetric, reflexive relation $\leq$. The poset $(\mathbb{P}, \leq)$ is *centered* if for any $F \Subset \mathbb{P}$ there is $q \in \mathbb{P}$ such that $q \leq p$ for all $p \in F$, and it is σ*-centered* if it is the union of countably many centered sets.

A subset $D \subseteq \mathbb{P}$ is *dense* if for every $p \in \mathbb{P}$ there is $q \in D$ such that $q \leq p$. A set $G \subseteq \mathbb{P}$ is a *filter* if $q \in G$ and $q \leq p$ implies $p \in G$, and if for any $p, q \in G$ there is $r \in G$ such that $r \leq p$, $r \leq q$. Given $\mathcal{D}$ a collection of dense subsets of $\mathbb{P}$, a filter G is $\mathcal{D}$*-generic* if $G \cap D \neq \emptyset$ for all $D \in \mathcal{D}$. By the results in [3], $\kappa < \mathfrak{p}$ is equivalent to the following weak form of Martin's Axiom.

MA_κ (σ-centered). Given a σ-centered poset $(\mathbb{P}, \leq)$ and $\mathcal{D}$ a collection of size κ of dense subsets of $\mathbb{P}$, there exists a $\mathcal{D}$-generic filter on $\mathbb{P}$.

Before moving to the proof of Theorem 1, we prove a simple preliminary fact. It is known that for every C*-algebra $\mathcal{A} \subseteq \mathcal{B}(H)$ there is an approximate unit of the compact operators which is quasicentral for $\mathcal{A}$ (see [1, Theorem 1, p. 330]). Moreover, if $\mathcal{A}$ is separable, the quasicentral approximate unit can be chosen to be countable, hence sequential. This property can be generalized to all C*-algebras of density character κ for which $\mathsf{MA}_\kappa(\sigma\text{-centered})$ holds. This is a simple fact, nevertheless its proof gives a fairly clear idea, at least to the reader familiar with the proof of Voiculescu's Theorem given in [1], of how to prove Item 1 of Theorem 1.

PROPOSITION 2.1. *Let H be a separable Hilbert space and $\mathcal{A} \subseteq \mathcal{B}(H)$ a C*-algebra of density character less than $\mathfrak{p}$. Then there exists a sequential approximate unit $\{h_n\}_{n \in \mathbb{N}}$ of $\mathcal{K}(H)$ which is quasicentral for $\mathcal{A}$.*

PROOF. Fix a countable dense K in the set of all positive norm one elements of $\mathcal{K}(H)$, and B dense in $\mathcal{A}$ of size smaller than $\mathfrak{p}$. Let $\mathbb{P}$ be the set of tuples

$$p = (F_p, J_p, n_p, (h_j^p)_{j \leq n_p}),$$

where $F_p \Subset B$, $J_p \Subset K$, $n_p \in \mathbb{N}$ and $h_j^p \in K$ for all $j \leq n_p$. For $p, q \in \mathbb{P}$ we say $p \leq q$ if and only if

(1) $F_q \subseteq F_p$,
(2) $J_q \subseteq J_p$,
(3) $n_q \leq n_p$,
(4) $h_j^p = h_j^q$ for all $j \leq n_q$,
(5) if $n_q < n_p$ then, for all $n_q < j \leq n_p$, all $k \in J_q$ and all $a \in F_q$, the following holds

$$\max\{\|[h_j^p, a]\|, \|h_j^p k - k\|, \|k h_j^p - k\|\} < 1/j.$$

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The poset $(\mathbb{P}, \leq)$ is σ-centred since, for any finite $X \Subset \mathbb{P}$ such that there is $n \in \mathbb{N}$ and $(h_j)_{j \leq n} \in K^n$ satisfying $n_p = n$ and $(h_j^p)_{j \leq n} = (h_j)_{j \leq n}$ for all $p \in X$, the condition

$$r = \left(\bigcup_{p \in X} F_p, \bigcup_{p \in X} J_p, n, (h_j)_{j \leq n} \right)$$

is a lower bound for X. Let $\mathcal{D}$ be the collection of the sets

$$\Delta_{F,J,n} = \{p \in \mathbb{P} : F_p \supseteq F, J_p \supseteq J, n_p \geq n\}$$

for $F \Subset B$, $J \Subset K$ and $n \in \mathbb{N}$. The sets $\Delta_{F,J,n}$ are dense since for every separable subalgebra of $\mathcal{B}(H)$ there is a sequential approximate unit of $\mathcal{K}(H)$ which is quasicentral for it. A $\mathcal{D}$-generic filter produces a sequential approximate unit of $\mathcal{K}(H)$ which is quasicentral for $\mathcal{A}$. Such filter exists by $\mathsf{MA}_{|\mathcal{D}|}(\sigma\text{-centered})$, which holds since $\mathcal{D}$ has size smaller than $\mathfrak{p}$. ⊣

§3. Voiculescu's Theorem and Martin's Axiom. Similarly to what happens in [1], we split the proof of Item 1 of Theorem 1 in two steps. First we prove the statement assuming that the completely positive map σ is block-diagonal (see Lemma 3.1), then in Lemma 3.2 we show that the general case can be reduced to the block-diagonal case.

3.1. Block-diagonal maps. A completely positive map $\sigma : \mathcal{A} \to \mathcal{B}(L)$ is *block-diagonal* if there is a decomposition $L = \bigoplus_{n \in \mathbb{N}} L_n$, where L_n is finite-dimensional for all $n \in \mathbb{N}$, which in turn induces a decomposition $\sigma = \bigoplus_{n \in \mathbb{N}} \sigma_n$ where the maps $\sigma_n : \mathcal{A} \to \mathcal{B}(L_n)$ are completely positive.

LEMMA 3.1. *Let H, L be two separable Hilbert spaces, $\mathcal{A} \subseteq \mathcal{B}(H)$ a unital C^*-algebra of density character less than $\mathfrak{p}$ and $\sigma : \mathcal{A} \to B(L)$ a block-diagonal, unital, completely positive map such that $\sigma(a) = 0$ for all $a \in \mathcal{A} \cap \mathcal{K}(H)$. Then there is a sequence of isometries $V_n : L \to H$ such that $\sigma(a) - V_n^* a V_n \in \mathcal{K}(L)$ and $\lim_{n \to \infty} \|\sigma(a) - V_n^* a V_n\| = 0$ for all $a \in \mathcal{A}$.*

PROOF. Fix $\varepsilon > 0$. By hypothesis $L = \bigoplus_{n \in \mathbb{N}} L_n$, with L_n finite-dimensional for all $n \in \mathbb{N}$, and σ decomposes as $\bigoplus_{n \in \mathbb{N}} \sigma_n$, where $\sigma_n(a) = 0$ whenever $a \in \mathcal{A} \cap \mathcal{K}(H)$ for all $n \in \mathbb{N}$. Let K be a countable dense subset of the unit ball of H such that, for every $\xi \in K$ the set $\{\eta \in K : \eta \perp \xi\}$ is dense in $\{\eta \in H : \|\eta\| = 1,\ \eta \perp \xi\}$. Fix an orthonormal basis $\{\xi_j^n\}_{j \leq k_n}$ for each L_n. Consider the set $\mathbb{P}$ of the tuples

$$p = (F_p, n_p, (W_i^p)_{i \leq n_p}),$$

where $F_p \Subset \mathcal{A}$, $n_p \in \mathbb{N}$ and W_i^p is an isometry of L_i into H such that $W_i^p \xi_j^i \in K$ for every $j \leq k_i$ and $i \leq n_p$. We say $p \leq q$ for two elements in $\mathbb{P}$ if and only if

(1) $F_q \subseteq F_p$,
(2) $n_q \leq n_p$,
(3) $W_i^p = W_i^q$ for all $i \leq n_q$,
(4) for $n_q < i \leq n_p$ (if any) we require $W_i L_i$ to be orthogonal to $\{W_j L_j, a W_j L_j, a^* W_j L_j : j \leq i,\ a \in F_q\}$ and

$$\|\sigma_i(a) - W_i^* a W_i\| < \varepsilon / 2^i$$

for all $a \in F_q$.

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For any finite set of conditions $X \Subset \mathbb{P}$ such that there is $n \in \mathbb{N}$ and $(W_i)_{i \leq n}$ satisfying $n_p = n$ and $(W^p)_{i \leq n_p} = (W_i)_{i \leq n}$ for all $p \in X$, the condition

$$r = \left(\bigcup_{p \in X} F_p, n, (W_i)_{i \leq n} \right)$$

is a lower bound for X. Thus the poset $(\mathbb{P}, \leq)$ is σ-centered. Let $\mathcal{D}$ be the collection of the sets

$$\Delta_{F,n} = \{p \in \mathbb{P} : F_p \supseteq F, n_p \geq n\}$$

for $n \in \mathbb{N}$ and $F \Subset B$, where B is a fixed dense subset of $\mathcal{A}$ of size smaller than $\mathfrak{p}$. By Theorem 1 every $\Delta_{F,n}$ is dense in $\mathbb{P}$ (the orthogonality condition in Item 3 of the definition of the order relation can be obtained using Proposition 3.6.7 in [15]). Let G be a $\mathcal{D}$-generic filter, which exists since $|\mathcal{D}| < \mathfrak{p}$, and thus $\mathsf{MA}_{|\mathcal{D}|}(\sigma\text{-centered})$, holds. Let V be the isometry from $\bigoplus_{n \in \mathbb{N}} L_n$ into H defined as $\bigoplus_{n \in \mathbb{N}} W_n$ where $W_n = W_n^p$ for some $p \in G$ such that $n_p \geq n$. The isometry is well defined since G is a filter. The proof that $\sigma(a) - V^* a V \in \mathcal{K}(L)$ and that $\|\sigma(a) - V^* a V\| < \varepsilon$ for all $a \in \mathcal{A}$ is the same as in Lemma 3.5.2 in [15]. ⊣

3.2. The general case. The following lemma generalizes Theorem 3.5.5 of [15] to all C*-algebras of density character smaller than $\mathfrak{p}$.

Lemma 3.2. *Let H, L, L' be separable Hilbert spaces, $\mathcal{A} \subseteq \mathcal{B}(H)$ a unital C*-algebra of density character less than $\mathfrak{p}$ and $\sigma : \mathcal{A} \to B(L)$ a unital completely positive map such that $\sigma(a) = 0$ for all $a \in \mathcal{A} \cap \mathcal{K}(H)$. Then there is a block-diagonal, unital completely positive map $\sigma' : \mathcal{A} \to B(L')$, such that $\sigma'(a) = 0$ for all $a \in \mathcal{A} \cap \mathcal{K}(H)$, and a sequence of isometries $V_n : H \to L$ such that $\sigma(a) - V_n^* \sigma'(a) V_n \in \mathcal{K}(H)$ and $\lim_{n \to \infty} \|\sigma(a) - V_n^* \sigma'(a) V_n\| = 0$ for all $a \in \mathcal{A}$.*

Proof. Fix $\varepsilon > 0$. We use the same poset (and notation) defined in Proposition 2.1 to generate an approximate unit of $\mathcal{K}(H)$ which is quasicentral for $\sigma[\mathcal{A}]$. Adjusting suitably the inequality in Item 2 of the definition of such poset (see [1, lemma, p. 332]), by $\mathsf{MA}_{|\mathcal{D}|}(\sigma\text{-centered})$ there is a filter of $\mathbb{P}$ which generates an approximate unit $(h_n)_{n \in \mathbb{N}}$ such that if $a \in F_p$ for some $p \in G$, then for all $n > n_p$ we have

$$\|[(h_{n+1} - h_n)^{1/2}, \sigma(a)]\| < \varepsilon / 2^n$$

From this point the proof goes verbatim as in Theorem 3.5.5 of [15]. ⊣

The thesis of Item 1 of Theorem 1 follows composing the isometries obtained from Lemmas 3.1 and 3.2.

3.3. Corollaries and remarks. Voiculescu's Theorem allows to infer several corollaries if the completely positive map σ is assumed to be a $*$-homomorphism. Using Item 1 of Theorem 1, these results generalize to separably representable C*-algebras of density character less than $\mathfrak{p}$. We omit the proofs in this part as they can be obtained following verbatim the arguments used in the separable case.

We introduce some definitions to ease the notation in the following statements. Given a representation $\phi : \mathcal{A} \to \mathcal{B}(H)$, let H_e be the Hilbert space spanned by $(\phi[\mathcal{A}] \cap \mathcal{K}(H))H$. Since $\phi[\mathcal{A}] \cap \mathcal{K}(H)$ is an ideal of $\phi[A]$, the space H_e is invariant for $\phi[A]$. The *essential part of* ϕ, denoted ϕ_e, is the restriction of ϕ to H_e.

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Two representations $\phi : \mathcal{A} \to \mathcal{B}(H_1)$ and $\psi : \mathcal{A} \to \mathcal{B}(H_2)$ are *equivalent* if there is a unitary map $U : H_1 \to H_2$ such that $U^*\psi(a)U = \phi(a)$ for all $a \in \mathcal{A}$. They are *approximately equivalent* if there is a sequence of unitary maps $U_n : H_1 \to H_2$ such that $U_n^*\psi(a)U_n - \phi(a) \in \mathcal{K}(H_1)$ and $\lim_{n\to\infty} \| U_n^*\psi(a)U_n - \phi(a)\| = 0$ for all $a \in \mathcal{A}$. Finally, they are *weakly approximately equivalent* if there are two sequences of unitary maps $U_n : H_1 \to H_2$ and $V_n : H_2 \to H_1$ such that $U_n^*\psi(a)U_n \to \phi(a)$ and $V_n^*\phi(a)V_n \to \psi(a)$ in the weak operator topology.

Corollaries 3.3 and 3.4 can be proved using the proofs of [1, Corollary 2, p. 339] and [1, Theorem 5] plus [1, Corollary 1, p. 343] respectively, after substituting all the instances of Voiculescu's Theorem with Item 1 of Theorem 1.

Corollary 3.3. *Let H, L be two separable Hilbert spaces, $\mathcal{A} \subseteq \mathcal{B}(H)$ a unital C^*-algebra of density character less than $\mathfrak{p}$ and $\phi : \mathcal{A} \to B(L)$ a unital representation such that $\phi(a) = 0$ for all $a \in \mathcal{A} \cap \mathcal{K}(H)$. Then the direct sum representation $Id \oplus \phi$ on $H \oplus L$ is approximately equivalent to ϕ.*

Corollary 3.4. *Let $\mathcal{A}$ be a separably representable unital C^*-algebra of density less than $\mathfrak{p}$ and ϕ, ψ two unital representations on some separable, infinite-dimensional Hilbert space H. The following are equivalent.*

(1) *ϕ and ψ are approximately equivalent,*
(2) *ϕ and ψ are weakly approximately equivalent,*
(3) *$ker(\phi) = ker(\psi)$, $ker(\pi \circ \phi) = ker(\pi \circ \psi)$ (here $\pi : \mathcal{B}(H) \to \mathcal{Q}(H)$ is the quotient map) and ϕ_e is equivalent to ψ_e.*

In particular, if $ker(\phi) = ker(\psi)$ and $\phi[A] \cap \mathcal{K}(H) = \psi[A] \cap \mathcal{K}(H) = \{0\}$ then ϕ and ψ are approximately equivalent.

A further consequence of Voiculescu's Theorem is that every separable unital subalgebra of the Calkin algebra is equal to its double commutant in the Calkin algebra (see [1, p. 345]; see also [10] for a version of this statement in the context of ultrapowers). This result is known to fail for C*-algebras of density character $\mathfrak{c}$ (see [4]). Nevertheless, it is not clear whether $\mathsf{MA}_\kappa(\sigma$-centered) could be used to generalize it to C*-algebras of density character $\kappa < \mathfrak{p}$, even assuming they are separably representable.

§4. Independence.

In this section we prove Item 2 of Theorem 1.

Proof of Item 2 of Theorem 1. Let H be a separable Hilbert space and $\mathcal{A} \subseteq \mathcal{B}(H)$ a maximal abelian atomic subalgebra, hence isomorphic to $\ell^\infty(\mathbb{N})$. Since the pure states of $\mathcal{A}$ annihilating $\mathcal{A} \cap \mathcal{K}(H)$ are in bijection with the nonprincipal ultrafilters on $\mathbb{N}$ (see [13, Example 6.2]), there are $2^\mathfrak{c}$ of them. Since there are only $\mathfrak{c}$ (countable) sequences of vectors in H, there are $2^\mathfrak{c}$ states of $\mathcal{A}$ for which Glimm's Lemma fails (it actually fails for all pure states annihilating $\mathcal{A} \cap \mathcal{K}(H)$, as shown in Proposition 2.7 of [14]). We prove the statement of Item 2 of Theorem 1 for $\lambda = \aleph_2$, as the proof in the general case is analogous. Consider a model of ZFC where $\mathfrak{c} = \aleph_1$ and $2^{\aleph_1} = \aleph_3$ and add to it $\aleph_2$ Cohen reals. In the generic extension we have $\mathfrak{c} = \aleph_2$, thus (the closure of) $\mathcal{A}$ has density character strictly smaller than $\mathfrak{c}$. Glimm's Lemma fails for $\mathcal{A}$ also in the generic extension. There are in fact at most $\aleph_2$ new sequences of vectors of H, which are still not enough to cover all the $\aleph_3$ states of $\mathcal{A}$ for which Glimm's Lemma failed in the ground model. ⊣

The argument we just exposed can be generalized verbatim to other C*-algebras of density character $\mathfrak{c}$ such as $\mathcal{B}(H)$ or $L^\infty([0,1])$, which all have more than $2^{\mathfrak{c}}$ different states.[2]

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[2]Notice that, if V is the ground model of ZFC and $V[G]$ a generic extension, the closure of $\mathcal{B}(H)^V$ in $V[G]$ is generally strictly contained in $\mathcal{B}(H)^{V[G]}$. The same happens for $\ell^\infty(\mathbb{N})$ and $L^\infty([0,1])$.