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AN ALGORITHM FOR STEM PRODUCTS AND ONE-RELATOR GROUPS

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We give an algorithm which for a given set of generators can decide whether a stem product of infinite cyclic groups is a one-relator group. We also generalize this to the case of one-relator products of groups.

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1. Introduction

A significant amount of work has been done concerning one-relator groups with centre. Baumslag and Taylor in [2] give an algorithm with which one can calculate the centre of a one-relator group. Pietrowski in 1974 [**10**] proved that every one-relator group G with centre, such that G/G' is not free abelian, has a presentation of the form

$$G = \langle x_1, \ldots, x_{n+1} \mid x_1^{p_1} = x_2^{q_1}, \ldots, x_n^{p_n} = x_{n+1}^{q_n} \rangle \tag{1}$$

with $p_i, q_i \geq 2$ and $(p_i, q_j) = 1$ for $i > j$.

Meskin, Pietrowski and Steinberg in 1973 (see [7]) proposed the following problem: which of the groups with the presentation (1) are one-relator groups. Although it is not explicitly mentioned in [7], the problem Meskin, Pietrowski and Steinberg deal with is when the groups (1) are one-relator groups in the generating set $\{x_1, x_{n+1}\}$. In [7] a necessary condition and a sufficient condition are given on the ordered set of integers $(p_1, q_1, \ldots, p_n, q_n)$ for (1) to be a one-relator group but not a necessary and sufficient condition. Collins in [4] shows that any generating pair of G is Nielsen equivalent to a pair $\{x_1^r, x_{n+1}^s\}$ with some extra conditions for r and s. Finally, McCool in [**6**] proves that if G is of the form (1) with $p_i = q_i$ for every $i = 1, \ldots, n+1$ then G is a one-relator group if H can be obtained from a suitable group $\langle \alpha, \beta \mid \alpha^p = \beta^p \rangle$ by repeated applications of a procedure consisting of applying central Nielsen transformations followed by adjoining a root of a generator.

In this paper we demonstrate an algorithm by which one can decide whether a group G is a one-relator group in some generating pair $\{x_1^r, x_{n+1}^s\}$ with $r, s \geq 1$.

We are also able to generalise the above results to the case of one-relator products. It has been shown recently (see [**8**]) that a one-relator product G of non-cyclic locally indicable groups A and B, with non-trivial centre, can be uniquely presented as a stem product

$$G = \langle A, B, t_1, \ldots, t_n \mid \alpha = t_1^{p_1}, t_1^{q_1} = t_2^{p_2}, \ldots, t_{n-1}^{q_{n-1}} = t_n^{p_n}, t_n^{q_n} = \beta \rangle$$

where $\alpha \in A, \beta \in B$ and $(p_i, q_j) = 1$ for every $1 \leq j \leq i \leq n$ (except possibly in the case where one of A, B is cyclic and the generator of this cyclic group has zero exponent sum in the relator). We prove here that our algorithm works also for the above stem products, if we know that some power of α and some power of β belong to the centre of A and B respectively.

2. The algorithm

Theorem 1. *There is an algorithm to decide of any group G given by a presentation*

$$G = \langle x_1, \ldots, x_{n+1} \mid x_1^{p_1} = x_2^{q_1}, \ldots, x_n^{p_n} = x_{n+1}^{q_n} \rangle,$$

where $p_i, q_i \in \mathbb{Z}$ for every $i = 1, \ldots, n$ and $(p_i, q_j) = 1$ for every $i > j$, whether G is a one-relator group in some generating set $\{x, y\}$.

Proof. From [7], G has a presentation of the form $\langle x_1, x_{n+1} \mid x_1^{p_1 \cdots p_n} = x_{n+1}^{q_1 \cdots q_n}, [x_1^{p_1}, x_{n+1}^{q_2 \cdots q_n}], \ldots, [x_1^{p_1 \cdots p_{n-1}}, x_{n+1}^{q_n}] \rangle$. Hence, there is a unique epimorhism $\phi : G \twoheadrightarrow \mathbb{Z}$ (modulo $\mathrm{Aut}(\mathbb{Z})$) given by the mapping $x_1 \rightarrow \frac{q_1 \cdots q_n}{g}$ and $x_{n+1} \rightarrow \frac{p_1 \cdots p_n}{g}$, where $g = gcd(p_1 \ldots p_n, q_1 \ldots q_n)$, such that the kernel H of ϕ is free of finite rank (see also [**5**, **9**, **10**]). Using the presentation that is given for G we can easily calculate the image of each x_i in $\mathbb{Z}$ by solving the system of equations $p_i \alpha_i = q_i \alpha_{i+1}$ with $i = 1, \ldots, n$ for α_i. Then $\alpha_i = \frac{p_1 \cdots p_{i-1} q_i \cdots q_n}{g}$, $i = 1, \ldots, n+1$.

On the other hand, by Bass-Serre Theory (see for example [3]), H is the fundamental group of a graph of groups, say $(\mathcal{H}, X)$, with trivial vertex and edge groups. More specifically, by applying the Structure Theorem (see [3]) we find that H is the fundamental group of a graph of groups $((\mathcal{H}, X))$ whose vertices (with trivial vertex and edge groups, since $H \cap v\langle x_i \rangle v^{-1}$ is trivial for every $i = 1, \ldots, n$, where v runs over a suitable double coset representative system for $H \backslash G / \langle x_i \rangle$) correspond to each x_i of G and whose edges connect vertices corresponding to x_i to vertices corresponding to x_{i+1}. The number of vertices that correspond to the generator x_i is $|\frac{G}{H} : \langle x_i H \rangle| = \alpha_i$. The number of edges that emanate from the vertices that correspond to x_i and are connected to the vertices of x_{i+1} is $|\frac{G}{H} : \langle x_i^{p_i} H \rangle| = p_i \alpha_i$. It is obvious that for any presentation of G as in the statement of the theorem, we can easily create the graph of groups $(\mathcal{H}, X)$. If T is a maximal tree of X, then the edges of X that do not belong to T form a generating set for H. From X we can always derive such a generating set, which we shall call $\mathcal{N}$. A formula for calculating the rank of H can be found using the solution of the system of equations $p_i \alpha_i = q_i \alpha_{i+1}$, with $i = 1, \ldots, n$, for α_i and the formula in [3]. Specifically, H is free with rank

$$r = \left(\sum_{i=1}^{n} \alpha_i(p_i - 1)\right) - \alpha_{n+1} + 1.$$

Since $\{x, y\}$ is a generating set of G, then by the results in [4], $\{x, y\}$ is Nielsen equivalent to $\{x_1^s, x_{n+1}^t\}$ for some $s, t \in \mathbb{Z}$. So after a finite number of Nielsen transformations we can change our generating set to $\{x_1^s, x_{n+1}^t\}$ for some $s, t \in \mathbb{Z}$, and we have that $\phi(x_1^s) = s\alpha_1$ and $\phi(x_{n+1}^t) = t\alpha_{n+1}$. By applying the Euclidean algorithm, we can effectively transform $\{x_1^s, x_{n+1}^t\}$ into a generating set $\{z(x_1^s, x_{n+1}^t),\ w(x_1^s, x_{n+1}^t)\}$ such that $\phi(z) = 1$ and $\phi(w) = 0$.

We claim that G is one-relator in terms of $\{x_1^s, x_{n+1}^t\}$ if and only if $\{z^i wz^{-i}, i = 0, \ldots, r-1\}$ is a basis for H. Given the claim, the proof of the theorem is immediate, since it is routine to express the elements $z^i wz^{-i}$, $i = 0, \ldots, r-1$, in terms of the elements of $\mathcal{N}$ and then use Nielsen transformations to determine whether $\{z^i wz^{-i}, i = 0, \ldots, r-1\}$ is a generating set for the whole of H.

If G is one-relator in terms of $\{x_1^s, x_{n+1}^t\}$ then it is also one-relator in terms of $\{z, w\}$ and using $\{z^m, m \in \mathbb{Z}\}$ as a Schreier transversal yields a presentation

$$H = \langle \ldots, w_{-1}, w_0, w_1, \ldots \mid \ldots, R_{-1}, R_0, R_1, \ldots \rangle$$

where $w_i = z^i wz^{-i}$ and $R_i = z^i R_0 z^{-i}$. By standard theory of one-relator groups, the fact that H is finitely generated means that if $R_0 = R_0(w_k, \ldots, w_l)$ then both w_k and w_l appear exactly once in R_0. Moreover, H is free with $\{w_i, i = 0, \ldots, l-k-1\}$ as basis; in particular $l - k = r$.

Conversely, suppose that $\{z^i wz^{-i}, i = 0, \ldots, r-1\}$ is a basis for H. Since G is the semidirect product of H by the infinite cyclic group on z, G has a presentation of the form

$$\begin{aligned}\langle w_0, \ldots, w_{r-1}, z \mid zw_0z^{-1} = w_1, \ldots, zw_{r-1}z^{-1} &= W_1(w_0, \ldots, w_{r-1}),\\ z^{-1}w_0z &= W_2(w_0, \ldots, w_{r-1})\rangle.\end{aligned}$$

However, the generator w_0 can appear only once in W_1; for otherwise $W_1 = Aw_0^{\varepsilon}Bw_0^{\eta}C$ where $\varepsilon, \eta = \pm 1$ and A, C are words in $\{w_1, \ldots, w_{r-1}\}$. But then the set

$$\{w_0^{\varepsilon}Bw_0^{\eta}, w_1, \ldots, w_{r-1}\}$$

is Nielsen equivalent to the set $\{W_1, w_1, \ldots, w_{r-1}\}$ and since it is also Nielsen reduced and clearly generates a proper subgroup of H, we have a contradiction.

It follows that the relation $z^{-1}w_0z = W_2(w_0, \ldots, w_{r-1})$ is a consequence of the other relations and hence G is the one-relator group

$$\begin{aligned}\langle w_0, \ldots, w_{r-1}, z \mid zw_0z^{-1} = w_1, \ldots, z^r w_{r-1}z^{-r} &= W_1(w_0, \ldots, w_{r-1})\rangle\\ &= \langle w, z \mid z^r wz^{-r} = W(z, w)\rangle.\end{aligned}$$

It follows immediately that G is one-relator in terms of $\{x_1^s, x_{n+1}^t\}$ and so in $\{x, y\}$. $\square$

3. On one-relator products

It has been shown in [8] that every one-relator product of two locally indicable groups A and B, with non-trivial centre, is a stem product with presentation

$$\langle A, B, t_1, \ldots, t_n \mid \alpha = t_1^{p_1}, t_1^{q_1} = t_2^{p_2}, \ldots, t_{n-1}^{q_{n-1}} = t_n^{p_n}, t_n^{q_n} = \beta \rangle$$

where $\alpha \in A, \beta \in B$ and $(p_i, q_j) = 1$ for every $1 \le j \le i \le n$ (except possibly in the case where one of A, B is cyclic, and the generator of that cyclic group has zero exponent sum in the relator). Our algorithm in Section 2 can be easily generalised to decide whether stem products of the above form are one-relator products of A and B or not. We prove the following.

Theorem 2. *There is an algorithm to decide of any group G with presentation*

$$\langle A, B, t_1, \ldots, t_n \mid \alpha = t_1^{p_1}, t_1^{q_1} = t_2^{p_2}, \ldots, t_{n-1}^{q_{n-1}} = t_n^{p_n}, t_n^{q_n} = \beta \rangle,$$

where $\alpha \in A$, $\beta \in B$ and $(p_i, q_j) = 1$ for every $1 \le j \le i \le n$ with A and B finitely generated locally indicable groups with non-trivial centre, soluble generalised word problem relative to A and B and such that some power of α is in the centre of A and some power of β is in the centre of B, whether G is a one-relator product of groups A and B or not.

Proof. From the results in [5], G is locally indicable and there is an epimorphism $\phi : G \twoheadrightarrow \mathbb{Z}$ with kernel H. Since the centre of G is non-trivial, it is therefore not contained in H, but is contained in both A and B (see [1]). More specifically, the centre of G is infinite cyclic, generated by some power of α which is equal to some power of β. So we have that $\phi(A) = p\mathbb{Z}$ and $\phi(B) = q\mathbb{Z}$ for some $p, q \in \mathbb{Z}$ with $(p, q) = 1$ and $A = A^\circ \rtimes \langle \gamma \rangle$ and $B = B^\circ \rtimes \langle \delta \rangle$, $\phi(\gamma) = p$, $\phi(\delta) = q$ with $\phi(\alpha) = kp$ and $\phi(\beta) = \lambda q$ for some $k, \lambda \in \mathbb{Z}$ (see also [1]).

On the other hand, from Bass-Serre Theory and an application of the Structure Theorem, H is the fundamental group of a graph of groups $(\mathcal{H}, X)$, whose vertices that correspond to $\langle t_i \rangle$ have trivial vertex groups and whose vertices that correspond to A and B have vertex groups $A_i = \delta^i A^\circ \delta^{-i}$, $i = 1, \ldots, p$, and $B_j = \gamma^j B^\circ \gamma^{-j}$, $j = 1, \ldots, q$, respectively, since $\{\delta, \ldots, \delta^p\}$ and $\{\gamma, \ldots, \gamma^q\}$ are double coset representative systems for $H \backslash G / A$ and $H \backslash G / B$ respectively. Since the edge groups are also trivial, H is the group

$$H = F_r * (*_{i=1}^{p} \delta^i A^\circ \delta^{-i}) * (*_{j=1}^{q} \gamma^j B^\circ \gamma^{-j}). \tag{2}$$

Now, if we "kill" A° and B° in G we get a group

$$G' = \langle \gamma, \delta, t_1, \ldots, t_n \mid \gamma^k = t_1^{p_1}, t_1^{q_1} = t_2^{p_2}, \ldots, t_{n-1}^{q_{n-1}} = t_n^{p_n}, t_n^{q_n} = \delta^\lambda \rangle.$$

If G is a one-relator product with non-trivial centre then G' is a two-generator one-relator group. But we can decide if this is true. If the coprimeness conditions of the algorithm in Section 2 do not apply, then G' is not a one-relator group and so G is not a one-relator product. If on the other hand the conditions apply, then we can apply to G' the algorithm described in Section 2. If G' is not a one-relator group then G is not a one-relator product and we are done.

If G' is a one-relator group, then by Theorem 1 in [7] we have that $(p_i, q_j) = 1$ for every $1 \leq j \leq i \leq n$. Then a generating set $\mathcal{N}_1$ for H in (2) can be easily obtained by choosing a generating set for each $\delta^i A^\circ \delta^{-i}, i = 1, \dots, p$, a generating set for each $\gamma^i B^\circ \gamma^{-i}, i = 1, \dots, q$, and a generating set for F_r. This latter one can be easily found by the same method as in Theorem 1.

Now by applying the Euclidean algorithm to the pair $\{\gamma, \delta\}$, we can find words $w_1(\gamma, \delta)$ and $w_2(\gamma, \delta)$ such that $\phi(w_1) = 1$ and $\phi(w_2) = 0$.

Let $\mathcal{N}_2$ be the set consisting of one generating set for each $w_1^i A^\circ w_1^{-i}$, $i = 1, \dots, p$, one generating set for each $w_1^i B^\circ w_1^{-i}$, $i = 1, \dots, q$, and the set $\{w_2, \dots, w_1^{r-1} w_2 w_1^{-r+1}\}$.

We claim that G is a one-relator product of A and B if and only if $\mathcal{N}_1$ and $\mathcal{N}_2$ generate the same subgroup of G.

Given the claim, the proof of the theorem is immediate, since by the Grushko-Neuman Theorem the generating set $\mathcal{N}_2$ can be carried to $\mathcal{N}_1$ by a sequence of elementary Nielsen transformations. Notice that such a calculation requires a solution to the generalised word problem for G relative to both A and B.

If G is a one-relator product of A and B, then $\{w_1^r\}$ as a Schreier transversal for H in G yields a presentation

$$\langle w_1^i A^\circ w_1^{-i}, w_1^j B^\circ w_1^{-j}, w_1^s w_2 w_1^{-s} \mid \mathcal{R} \rangle$$

where $i = 1, \dots, p$, $j = 1, \dots, q$, $s = \dots, -1, 0, 1, \dots$, and $\mathcal{R}$ consists of all the relations of A° conjugated by w_1^i, all the relations of B° conjugated by w_1^j and the relator R conjugated by w_1^s for every $i = 1, \dots, p$, $j = 1, \dots, q$, $s = \dots - 1, 0, 1, \dots$.

Then, by the results in [**8**], H is a free product of the form $F_r * (*_i A_i) * (*_j B_j)$ where F_r is a free group of finite rank and the groups A_i and B_j are subgroups of the form $H \cap gAg^{-1}$ and $H \cap gBg^{-1}$, as g ranges over a certain set of double coset representatives in $H \backslash G / A$ and $H \backslash G / B$ respectively. Hence, [**8**] yields $\mathcal{N}_1$ as a generating set for H. But since G is a one-relator product, $\mathcal{N}_2$ is necessarily a generating set for H. So $\mathcal{N}_1$ and $\mathcal{N}_2$ generate H.

Conversely, if $\mathcal{N}_1$ and $\mathcal{N}_2$ generate the same subgroup H of G then $w_1^r w_2 w_1^{-r} = W(\mathcal{N}_2)$ and by a similar argument to that in Theorem 2.1 we have that $w_1 w_2 w_1^{-1}$ occurs only once in $W(\mathcal{N}_2)$ and so G is the one-relator product $\langle A * B \mid w_1^r w_2 w_1^{-r} = W \rangle$. □

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