Eighth Meeting, June 10, 1892.

Professor J. E. A. STEGGALL, M.A., President, in the Chair.

Integrals of the form
$$\int_0^{2\pi}\log\left\{(x-a\cos\theta)^2+(y-b\sin\theta)^2\right\}{\cos \atop \sin}m\theta d\theta$$
and allied Integrals.

By GEORGE A. GIBSON, M.A.

1. If θ be the eccentric angle of a point P on an ellipse whose semi-axes are a and b and if x, y be the rectangular co-ordinates of a point Q, then $PQ^2=(x-a\cos\theta)^2+(y-b\sin\theta)^2$. When $a=b$ and $x^2+y^2=f^2$, $PQ^2=f^2-2af\cos\theta+a^2$, if the line from which θ is measured passes through Q. The integral

$$\int_0^{2\pi}\log(f^2-2af\cos\theta+a^2)\cos m\theta d\theta$$

is a well-known one. I propose in the present paper to consider the more general form which the integral takes when P lies on an ellipse. Some of the results I establish are proved in Roberts's *Integral Calculus*, pp. 199-201, by a totally different method; others seem to me to be new.

I use the following notation throughout.

$$a=c\cosh u \qquad x=\alpha\cos\phi=c\cosh v\cos\phi$$
$$b=c\sinh u \qquad y=\beta\sin\phi=c\sinh v\sin\phi$$

so that
$$c^2=a^2-b^2=\alpha^2-\beta^2.$$

P and Q thus lie on two confocal ellipses. Putting $PQ=R$, then R^2 may be thrown into the form

$$R^2=c^2\{\cosh(u+v)-\cos(\theta+\phi)\}\{\cosh(u-v)-\cos(\theta-\phi)\}$$
$$=c^2\Delta \text{ suppose.}$$

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This expression for R^2 is given in Greenhill's *Calculus* (2nd ed., p. 372). Before noticing this form I had used complex factors, which are rather more troublesome to work with. By means of the factors the functions to be integrated are readily expressed by a series.

The usual methods of expansion give

$$\log(A-\cos x)=\log\frac{A+\sqrt{(A^2-1)}}{2}-2\sum_1^\infty\{A-\sqrt{(A^2-1)}\}^n\frac{\cos nx}{n}$$

A being greater than unity, and the root sign being so chosen that $\mathrm{mod}\{A-\sqrt{(A^2-1)}\}<1$. For example, if $A=\cosh(u-v)$ and $u<v$, $A-\sqrt{(A^2-1)}=e^{u-v}$, $A+\sqrt{(A^2-1)}=e^{v-u}$.

2. $$\int_0^{2\pi}\log(c^2\Delta)d\theta.$$

The integral is equal to

$$\int_0^{2\pi}\log c^2 d\theta+\int_0^{2\pi}\log\{\cosh(u+v)-\cos(\theta+\phi)\}d\theta+\int_0^{2\pi}\log\{\cosh(u-v)-\cos(\theta-\phi)\}d\theta$$

$$=4\pi\log c+\int_0^{2\pi}\log\{\cosh(u+v)-\cos\theta\}d\theta+\int_0^{2\pi}\log\{\cosh(u-v)-\cos\theta\}d\theta$$

since the function to be integrated has the period 2π and ϕ is real.

(i.) Suppose $u>v$, that is, suppose Q inside the ellipse on which P lies.

$$\therefore\quad\int_0^{2\pi}\log(c^2\Delta)d\theta=4\pi\log c+2\pi(u+v-\log 2)+2\pi(u-v-\log 2)$$

$$=4\pi\left(u+\log\frac{c}{2}\right)$$

$$=4\pi\log\frac{a+b}{2}$$

since $u=\log e^u=\log(\cosh u+\sinh u)$. The integrals are of course obtained by expanding the logarithms, noting that $\int_0^{2\pi}\cos n\theta d\theta=0$.

It will be observed that the value is independent of v, that is of the position of Q within the ellipse.

(ii.) Suppose $u<v$, that is, Q outside the ellipse ; then

$$\int_0^{2\pi}\log(c^2\Delta)d\theta = 4\pi\log c + 2\pi(u+v-\log 2) + 2\pi(v-u-\log 2)$$

$$= 4\pi\left(v + \log\frac{c}{2}\right)$$

$$= 4\pi\log\frac{\alpha+\beta}{2}\ .$$

In this case the integral is independent of u.

(iii.) Suppose $u=v$. The integral is still determinate, and we get

$$\int_0^{2\pi}\log(c^2\Delta)d\theta = 4\pi\log\frac{a+b}{2}\ .$$

By supposing the ellipse on which P lies to become a circle of radius a, we get the well-known values

$$\int_0^{2\pi}\log(\mathrm{PQ}^2)d\theta = 4\pi\log a \text{ or } 4\pi\log f$$

according as Q lies inside or outside the circle, f being the distance of Q from the centre.

3. $$\int_0^{2\pi}\log(c^2\Delta)\cos m\theta d\theta$$

where m is a positive integer.

If $x=\theta\pm\phi$, $\cos nx = \cos n\theta\cos n\phi \mp \sin n\theta\sin n\phi$. Hence $\log(c^2\Delta)$ may be expanded in a series involving the terms $\mathrm{C}_n\cos n\theta$ and $\mathrm{S}_n\sin n\theta$. But

$$\int_0^{2\pi}\cos n\theta\cos m\theta d\theta = 0 \text{ if } m \neq n, \text{ but} = \pi \text{ if } m=n$$

and $$\int_0^{2\pi}\sin n\theta\cos m\theta d\theta = 0.$$

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Therefore expanding the logarithms we get

$$\int_0^{2\pi}\log\{\cosh(u+v)-\cos(\theta+\phi)\}\cos m\theta d\theta$$

$$= -\frac{2\pi}{m}\cos m\phi . e^{-m(u+v)}$$

$$\int_0^{2\pi}\log\{\cosh(u-v)-\cos(\theta-\phi)\}\cos m\theta d\theta$$

$$= -\frac{2\pi}{m}\cos m\phi . e^{-m(u-v)} \text{ if } u>v$$

$$= -\frac{2\pi}{m}\cos m\phi e^{+m(u-v)} \text{ if } u<v.$$

Hence $$\int_0^{2\pi}\log(c^2\Delta)\cos m\theta d\theta$$

$$= -\frac{4\pi}{m}\cos m\phi\cosh mv(\cosh mu - \sinh mu) \text{ if } u>v$$

$$= -\frac{4\pi}{m}\cos m\phi\cosh mu(\cosh mv - \sinh mv) \text{ if } u<v$$

If $u=v$, the integral is still determinate, and has for value

$$-\frac{4\pi}{m}\cos m\phi\cosh mu(\cosh mu - \sinh mu).$$

The values of the integral in this case depend on both the co-ordinates of Q.

In order to deduce the values when $a=b$, it may be noticed that $\cosh mv(\cosh mu - \sinh mu)$

$$= \tfrac{1}{2}\{(\cosh v+\sinh v)^m + (\cosh v - \sinh v)^m\}\{\cosh u - \sinh u\}^m$$

$$= \tfrac{1}{2}\frac{\{(\alpha+\beta)^m+(\alpha-\beta)^m\}(a-b)^m}{c^{2m}}$$

$$= \tfrac{1}{2}\,\frac{(\alpha+\beta)^m+(\alpha-\beta)^m}{(a+b)^m}$$

Hence if Q be at distance f from the centre of the circle, we get

$$\int_0^{2\pi}\log\{f^2-2af\cos(\theta-\phi)+a^2\}\cos m\theta d\theta = -\frac{2\pi}{m}\cos m\phi\frac{f^m}{a^m}$$

$$\text{or} \qquad -\frac{2\pi}{m}\cos m\phi\frac{a^m}{f^m}$$

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according as f is less or greater than a, and is equal to either value when f and a are equal.

4. $$\int_0^{2\pi}\log(c^2\Delta)\sin m\theta d\theta,$$

where m is a positive integer.

As in the preceding example, it may be shown that

$$\begin{aligned}\int_0^{2\pi}\log(c^2\Delta)\sin m\theta d\theta &= -\frac{4\pi}{m}\sin m\phi\sinh mv(\cosh mu - \sinh mu) \text{ if } u>v \\ &= -\frac{4\pi}{m}\sin m\phi\sinh mu(\cosh mv - \sinh mv) \text{ if } u<v \\ &= -\frac{4\pi}{m}\sin m\phi\sinh mu(\cosh mu - \sinh mu) \text{ if } u=v\end{aligned}$$

When $a=b$, the values are as in the corresponding cases of § 3 with $\sin m\phi$ instead of $\cos m\phi$.

5. $$\int_0^{\pi}\log(c^2\Delta)\begin{matrix}\sin\\ \cos\end{matrix} m\theta d\theta$$

where m is a positive integer.

In general these integrals have to be expressed in series; the particular case

$$\int_0^{\pi}\log\{(x-a\cos\theta)^2+b^2\sin^2\theta\}\cos m\theta.d\theta \ (i.e., \ \phi=0)$$

is, however, simply half the integral of § 3 with ϕ put equal to zero. As the series can be readily expressed, it is needless to write them down.

6. By differentiation of the integrals, others may be deduced. Thus

$$\begin{aligned}&\int_0^{\pi}\log\{(x-a\cos\theta)^2+b^2\sin^2\theta\}\cos m\theta d\theta \\ &\qquad = -\frac{2\pi}{m}\cosh mv(\cosh mu - \sinh mu) \quad u>v \\ &\qquad = -\frac{\pi}{m}\cdot\frac{\{x+\sqrt{(x^2-a^2+b^2)}\}^m+\{x-\sqrt{(x^2-a^2+b^2)}\}^m}{(a+b)^m}\end{aligned}$$

since in this case $\alpha=x$, $\beta=\sqrt{a^2-c^2}=\sqrt{(x^2-a^2+b^2)}$.

Now differentiate with respect to b.

$$\therefore \int_0^\pi \frac{\sin^2\theta\cos m\theta d\theta}{(x-a\cos\theta)^2+b^2\sin^2\theta}$$

$$= \frac{\pi}{2}\cdot\frac{\{x+\sqrt{(x^2-a^2+b^2)}\}^m+\{x-\sqrt{(x^2-a^2+b^2)}\}^m}{b(a+b)^{m+1}}$$

$$-\frac{\pi}{2}\cdot\frac{\{x+\sqrt{(x^2-a^2+b^2)}\}^{m-1}-\{x-\sqrt{(x^2-a^2+b^2)}\}^{m-1}}{(a+b)^m.\sqrt{(x^2-a^2+b^2)}}$$

If $a=b$, this becomes

$$\int_0^\pi \frac{\sin^2\theta\cos m\theta d\theta}{x^2-2ax\cos\theta+a^2} = -\frac{\pi}{4}(a^2-x^2)\frac{x^{m-2}}{a^{m+2}} \quad x<a$$

If $x>a$, we get

$$\int_0^\pi \frac{\sin^2\theta\cos m\theta d\theta}{(x-a\cos\theta)^2+b^2\sin^2\theta} = \frac{\pi}{2}\,\frac{(a+b)^m+(a-b)^m}{\sqrt{(x^2-a^2+b^2)}\{x+\sqrt{(x^2-a^2+b^2)}\}^{m+1}}$$

$$-\frac{\pi}{2}\,\frac{(a+b)^{m-1}-(a-b)^{m-1}}{b\{x+\sqrt{(x^2-a^2+b^2)}\}^m}$$

and if $a=b$

$$\int_0^\pi \frac{\sin^2\theta\cos m\theta d\theta}{x^2-2ax\cos\theta+a^2} = -\frac{\pi}{4}(x^2-a^2)\frac{a^{m-2}}{x^{m+2}} \quad x>a.$$

Similarly by differentiating the integrals of §§ 3 and 4 with respect to u or v, various integrals may be obtained.

7. $$\int_0^{2\pi}\frac{d\theta}{c^2\Delta.}$$

In this case $u\neq v$. We may write

$$\cosh(u+v)-\cos(\theta+\phi) = -\frac{\cos\phi+i\sin\phi}{2x}(x-x_1)(x-x_2)$$

where $x=e^{i\theta}$, mod$x_1<1$, mod$x_2>1$, x_1, x_2 being the roots of

$$(\cos\phi+i\sin\phi)x^2-2\cosh(u+v).x+\cos\phi-i\sin\phi=0.$$

Similarly

$$\cosh(u-v)-\cos(\theta-\phi) = -\frac{\cos\phi-i\sin\phi}{2x}(x-x_3)(x-x_4)$$

where mod$x_3<1$, mod$x_4>1$, and x_3, x_4 are roots of

$$(\cos\phi-i\sin\phi)x^2-2\cosh(u-v)x+\cos\phi+i\sin\phi=0.$$

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Hence
$$\frac{1}{\Delta}=\frac{4x^2}{(x-x_1)(x-x_2)(x-x_3)(x-x_4)}$$
$$=\frac{A}{x-x_1}+\frac{B}{x-x_2}+\frac{C}{x-x_3}+\frac{D}{x-x_4}\,.$$

It is obvious on expanding $A/(x-x_1)$, etc., that

$$\int_0^{2\pi}\frac{d\theta}{x-x_1}=0=\int_0^{2\pi}\frac{d\theta}{x-x_3}\;;\;\int_0^{2\pi}\frac{d\theta}{x-x_2}=-\frac{2\pi}{x_2}\,,\;\int_0^{2\pi}\frac{d\theta}{x-x_4}=-\frac{2\pi}{x_4}\,.$$

But $$B=\frac{4x_2^{\,2}}{(x_2-x_1)(x_2-x_3)(x_2-x_4)}\,,\;D=\frac{4x_4^{\,2}}{(x_4-x_1)(x_4-x_2)(x_4-x_3)}$$

$$\therefore\qquad \frac{B}{x_2}+\frac{D}{x_4}=\frac{4(x_1x_3-x_2x_4)}{(x_2-x_1)(x_2-x_3)(x_1-x_4)(x_3-x_4)}$$
$$=-\frac{2\sinh 2u}{(\cosh 2u-\cosh 2v)(\cosh 2u-\cos 2\phi)}\qquad u>v$$
$$=-\frac{2\sinh 2v}{(\cosh 2v-\cosh 2u)(\cosh 2v-\cos 2\phi)}\qquad u<v$$

$$\therefore\qquad \int_0^{2\pi}\frac{d\theta}{c^2\Delta}=-\frac{2\pi}{c^2}\left(\frac{B}{x_2}+\frac{D}{x_4}\right)$$
$$=\frac{4\pi\sinh 2u}{c^2(\cosh 2u-\cosh 2v)(\cosh 2u-\cos 2\phi)}\qquad u>v$$
$$=\frac{4\pi\sinh 2v}{c^2(\cosh 2v-\cosh 2u)(\cosh 2v-\cos 2\phi)}\qquad u<v$$

If these values be expressed in terms of x, y, α, β, a, b, we get

$$2\pi ab/(a^2b^2-b^2x^2-a^2y^2)\ \text{and}\ 2\pi\alpha\beta/(\alpha^2-a^2)(\alpha^2+\beta^2-x^2-y^2)$$
respectively.

8. $$\int_0^{2\pi}\frac{\cos m\theta\, d\theta}{c^2\Delta}$$

where m is a positive integer.

Writing $1/\Delta$ as in § 7, expanding and integrating after multiplication by $\cos m\theta$, we get

$$\int_0^{2\pi}\frac{\cos m\theta\, d\theta}{c^2\Delta}=\frac{\pi}{c^2}\left\{Ax_1^{\,m-1}-\frac{B}{x_2^{\,m+1}}+Cx_3^{\,m-1}-\frac{D}{x_4^{\,m+1}}\right\}$$

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The reduction after substituting for A, B, C, D is somewhat tedious, but we get, if $u > v$, for the value of the integral

$$4\pi e^{-mu}\frac{\cosh mv \sin m\phi \sin 2\phi(\cosh 2u - \cosh 2v) + \sinh mv \sinh 2v \cos m\phi}{c^2(\cosh 2u - \cosh 2v)(\cosh 2u - \cos 2\phi)}$$
$$\frac{(\cosh 2u - \cos 2\phi) + \cosh mv \sinh 2u \cos m\phi(\cosh 2v - \cos 2\phi)}{(\cosh 2v - \cos 2\phi)}$$

If $u < v$, then u and v must be interchanged.

If we suppose $a = b$, and $\alpha = \beta = \sqrt{x^2 + y^2} = f$ as before, then

$$\int_0^{2\pi}\frac{\cos m\theta d\theta}{f^2 - 2af\cos(\theta - \phi) + a^2} = \frac{2\pi f^m \cos m\phi}{a^m(a^2 - f^2)} \text{ or } \frac{2\pi a^m \cos m\phi}{f^m(f^2 - a^2)}$$

according as $f <$ or $> a$.

If we had taken $\phi = 0$, the integral would have been much more easily evaluated, since

$$\int_0^{2\pi}\cos m\theta . d\theta/(\mathrm{A} - \cos\theta)$$

is very easily evaluated.

Matthew Stewart's Theorem.

By J. S. Mackay, M.A., LL.D.

In 1746, when he was a candidate for the Chair of Mathematics in the University of Edinburgh rendered vacant by the death of Maclaurin, Matthew Stewart published his first work, *Some General Theorems of considerable use in the higher parts of Mathematics.* In the preface to it he states that "the theorems contained in the following sheets . . . . are entirely new, save one or two at most," but he does not specify the two. They are*

Theorem 1.

If in triangle ABC any straight line AD be drawn to BC, and DE, DF be drawn parallel to AC, AB, and meeting AB, AC in E, F, then

$$AD^2 + BD \cdot CD = AB \cdot AE + AC \cdot AF.$$

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* The enunciations of these theorems have been modernised.