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STACKED SUBMODULES OF TORSION MODULES OVER DISCRETE VALUATION DOMAINS

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A submodule W of a torsion module M over a discrete valuation domain is called stacked in M if there exists a basis $\mathcal{B}$ of M such that multiples of elements of $\mathcal{B}$ form a basis of W. We characterise those submodules which are stacked in a pure submodule of M.

1. INTRODUCTION

Let R be a discrete valuation domain and let p be a prime element of R such that Rp is the maximal ideal of R. Let M be a torsion module over R and let W be a submodule of M. In accordance with [7] and [6] we call a set $\{u_\kappa \mid \kappa \in K\}$ a *basis* of M if $M = \bigoplus_{\kappa \in K} Ru_\kappa$. We say that W is *stacked* in M if there exists a basis $\mathcal{X} = \{x_\lambda \mid \lambda \in \Lambda\}$ of W and a basis $\mathcal{U} = \{u_\kappa \mid \kappa \in K\}$ of M such that $\Lambda \subseteq K$ and $x_\lambda = p^{t_\lambda} u_\lambda$ for suitable nonnegative integers t_λ. In that case we call $\mathcal{X}$ a *stacked basis* of W ([4]). If M is of bounded order, that is, if there exists a positive integer m such that $p^m x = 0$ for all $x \in M$, then it is known [7, p. 65] that W is stacked in M if and only if

$$p^n W \cap p^{n+r} M = p^n (W \cap p^r M) \tag{1.1}$$

holds for all $n \geqslant 0$, $r \geqslant 0$. In general however, if M is not of bounded order then condition (1.1) alone need not imply that W is stacked in M (see Exercise 78(b) in [7, p. 65]). In this paper we shall characterise those submodules which are stacked in a pure submodule of M.

Throughout this paper the letters $\mathcal{U}$, $\mathcal{V}$, $\mathcal{X}, \ldots$, will denote subsets of M. We shall use the letters $u, v, x, \ldots$, for elements of the module M, and $\alpha, \beta, \mu, \ldots$, will be elements of the ring R. Using the terminology for Abelian p-groups in [6, p. 4] we say that $x \in M$ has *exponent* k, and we write $\mathrm{e}(x) = k$, if k is the smallest nonnegative integer such that $p^k x = 0$. Clearly, $\mathrm{e}(0) = 0$. An element $x \in M$ is said to have (finite) *height* s if $x \in p^s M$

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and $x \notin p^{s+1}M$. In this case we write $\mathrm{h}(x) = s$. We set $\mathrm{h}(x) = \infty$ if $x \in p^s M$ for all $s \geqslant 0$. Thus $\mathrm{h}(0) = \infty$. Note that the height of all nonzero elements of M is bounded if and only if M is of bounded order.

Let $\langle \mathcal{X} \rangle$ be the submodule spanned by $\mathcal{X}$. When we write

$$\alpha_1 x_1 + \cdots + \alpha_m x_m \in \langle \mathcal{X} \rangle$$

we tacitely assume $x_i \in \mathcal{X}$ and $\alpha_i x_i \neq 0$, $i = 1, \ldots, m$, and $x_i \neq x_j$ if $i \neq j$.

Let R^* be the group of units of R. We set $\alpha \sim \beta$ if $\alpha = \beta\varepsilon$ for some $\varepsilon \in R^*$. It will be convenient to write $\mathrm{h}(\alpha) = s$ if $\alpha \sim p^s$. Let us recall the following properties of the height function on M (see for example, [**6**, p. 154]). For all $x, y \in M$ we have $\mathrm{h}(px) \geqslant \mathrm{h}(x) + 1$, and

$$\mathrm{h}(x + y) \geqslant \min\{\mathrm{h}(x), \mathrm{h}(y)\}. \tag{1.2}$$

Hence

$$\mathrm{h}(\alpha x) \geqslant \mathrm{h}(\alpha) + \mathrm{h}(x) \text{ for all } \alpha \in R. \tag{1.3}$$

We say that an element x is *h-regular* if $\mathrm{h}(x) = \infty$ or if $\mathrm{h}(x)$ is finite and

$$\mathrm{h}(\alpha x) = \mathrm{h}(\alpha) + \mathrm{h}(x) \text{ for all } \alpha \text{ with } \mathrm{h}(\alpha) < \mathrm{e}(x). \tag{1.4}$$

Property (1.4) can be traced back to Baer [**2**]. In [**2**, p. 484] an element x of an Abelian p-group is called *regular* if $\mathrm{h}(x) = \infty$ or if $\mathrm{h}(x) = k < \infty$ and

$$\mathrm{e}(x) + \mathrm{h}(x) = \cdots = \mathrm{e}(p^{k-1}x) + h(p^{k-1}x). \tag{1.5}$$

As usual, a set $\mathcal{X}$ is called *independent* if $0 \notin \mathcal{X}$ and if for any finite subset $\{x_1, \ldots, x_m\}$ of $\mathcal{X}$ a relation $\alpha_1 x_1 + \cdots + \alpha_m x_m = 0$ implies $\alpha_i x_i = 0$, $i = 1, \ldots, m$. We shall employ two stronger concepts of independence. The first one is adapted from Fuchs [**5**]. We call a set $\mathcal{X}$ *p-independent* (or *pure independent*) if it is independent and contains no elements of infinite height, and if

$$\alpha_1 x_1 + \cdots + \alpha_m x_m \in \langle \mathcal{X} \rangle \tag{1.6}$$

implies

$$\mathrm{h}(\alpha_1 x_1 + \cdots + \alpha_m x_m) = \min\{\mathrm{h}(\alpha_i) \mid i = 1, \ldots, m\}.$$

The other definition is motivated by the inequality

$$\mathrm{h}(\alpha_1 x_1 + \cdots + \alpha_m x_m) \geqslant \min\{\mathrm{h}(\alpha_i) + \mathrm{h}(x_i) \mid i = 1, \ldots, m\},$$

which follows from (1.3) and (1.2). We say that $\mathcal{X}$ is *h-independent* if $\mathcal{X}$ is independent and (1.6) implies

$$\mathrm{h}(\alpha_1 x_1 + \cdots + \alpha_m x_m) = \min\{\mathrm{h}(\alpha_i) + \mathrm{h}(x_i) \mid i = 1, \ldots, m\}. \tag{1.7}$$

Our concept of h-independence combines properties used in [3] to describe extendible Jordan bases of marked subspaces. It is obvious that a set $\mathcal{X}$ is p-independent if and only if it is h-independent and all of its elements have height zero.

For the elements x of a submodule S of M we may define $\mathrm{h}_S(x)$ as the height of x in S. We always have $\mathrm{h}_S(x) \leqslant \mathrm{h}(x)$. A submodule S of M is called *pure* in M if $\mathrm{h}_S(x) = \mathrm{h}(x)$ for all $x \in S$, or equivalently if $S \cap p^i M = p^i S$ for all $i \geqslant 0$. The following lemma is due to Fuchs [5].

Lemma 1.1. *For a set $\mathcal{X}$ the following conditions are equivalent.*

(i) *$\mathcal{X}$ is p-independent.*

(ii) *$\mathcal{X}$ is independent and the submodule $\langle \mathcal{X} \rangle$ is pure in M.*

Since M is pure in itself it follows from the preceding lemma that a basis of M is p-independent. It is also obvious that all nonzero elements of M have finite height if M has a basis.

Our main result is the following theorem. It will be proved in Section 3 together with a corollary.

Theorem 1.2. *Let M be a torsion module over a discrete valuation domain and let W be a submodule of M. The following statements are equivalent.*

(i) *There exists a pure submodule S of M such that W is stacked in S.*

(ii) *W has an h-independent basis.*

It will be shown in Proposition 3.3 that condition (1.1) is necessary for the existence of an h-independent basis of W. In the case where M is of bounded order we note the following result.

Corollary 1.3. *Let M of bounded order. For a submodule W of M the following statements are equivalent.*

(i) *W is stacked in M.*

(ii) *W has an h-independent basis.*

(iii) *Condition* (1.1) *holds.*

It is well-known [1] that the Jordan normal form can be studied in the framework of the theory of finitely generated modules over a principal ideal domain. Hence [7, Exercise 79, p. 65], and Theorem 1.2 and its proof provide an alternative access to results in [3] on extensions of Jordan bases for invariant subspaces of a matrix.

2. h-independence

This section contains the results on h-independence which we shall need in the course of the proof of Theorem 1.2. We shall make constant use of the following observations on the height function. Suppose $p^m x \neq 0$ and $\mathrm{h}(p^m x) = m + r$. Then we have $\mathrm{h}(x) \leqslant r$. If $x \neq 0$ is an element with $\mathrm{h}(x) = s$ and $\mathrm{e}(x) = k$ then x is h-regular if and only if

$$\mathrm{h}(p^j x) = j + \mathrm{h}(x),\ j = 1, \ldots, k-1,$$

or equivalently, if and only if

$$\mathrm{h}(p^{k-1} x) = (k-1) + \mathrm{h}(x),$$

or equivalently, $p^j x$ is h-regular for all $j \geqslant 0$.

It is not difficult to see that an independent set $\mathcal{X}$ is h-independent if and only if its elements are h-regular, and if $x = \alpha_1 x_1 + \cdots + \alpha_m x_m \in \langle \mathcal{X} \rangle$, $x \neq 0$, then $\mathrm{h}(x) = \min\{\mathrm{h}(\alpha_i x_i);\ i = 1, \ldots, m\}$ for all $\alpha_i \in R$.

It is obvious that $\mathrm{h}(x) \neq \mathrm{h}(y)$ implies $\mathrm{h}(x+y) = \min\{\mathrm{h}(x), \mathrm{h}(y)\}$. Hence if a strict inequality $\mathrm{h}(x+y) > \min\{\mathrm{h}(x), \mathrm{h}(y)\}$ holds, then $\mathrm{h}(x) = \mathrm{h}(y)$. Therefore, whenever we want to show that an independent set $\{x_1, \ldots, x_m\}$ of h-regular elements is h-independent we have to make sure that $\mathrm{h}(\alpha_i x_i) = r$, $i = 1, \ldots, m$, implies $\mathrm{h}(\alpha_1 x_1 + \cdots + \alpha_m x_m) \leqslant r$.

We shall also make frequent use of the following fact.

Lemma 2.1. *Let $\mathcal{X} = \mathcal{X}_1 \cup \cdots \cup \mathcal{X}_k$ be a disjoint union of h-independent sets. Then $\mathcal{X}$ is h-independent if and only if $x_{i_\tau} \in \langle \mathcal{X}_{i_\tau} \rangle$, $x_{i_\tau} \neq 0$, and $1 \leqslant i_1 < \cdots < i_t \leqslant m$ imply that $\{x_{i_1}, \ldots, x_{i_t}\}$ is h-independent.*

In the following observation we are concerned with a submodule where all elements are h-regular.

Lemma 2.2. *Let $\mathcal{X}$ be h-independent and assume that*

$$\mathrm{h}(x) + \mathrm{e}(x) = t \textit{ for all } x \in \mathcal{X}.$$

Then each nonzero element $y \in \langle \mathcal{X} \rangle$ is h-regular and

$$\mathrm{h}(y) + \mathrm{e}(y) = t. \tag{2.8}$$

Proof: If $x \in \mathcal{X}$ then x is h-regular, and we have

$$\mathrm{h}(\alpha x) + \mathrm{e}(\alpha x) = \mathrm{h}(x) + \mathrm{e}(x) = t \tag{2.9}$$

if $\mathrm{h}(\alpha) < \mathrm{e}(x)$. Let $y = \alpha_1 x_1 + \cdots + \alpha_m x_m \in \langle \mathcal{X} \rangle$ be nonzero with $\mathrm{h}(y) = r$ and $\mathrm{e}(y) = k$. Assume $\mathrm{h}(\alpha_1 x_1) = \min\{\mathrm{h}(\alpha_i x_i) \mid i = 1, \ldots, m\}$. Since $\mathcal{X}$ is h-independent we have $\mathrm{h}(\alpha_1 x_1) = r$. Then (2.9) implies $\mathrm{e}(\alpha_1 x_1) = t - r$. From $r \leqslant \mathrm{h}(\alpha_i x_i)$ we obtain $\mathrm{e}(\alpha_i x_i) \leqslant t - r$. Hence $\mathrm{e}(y) \leqslant t - r$. Since $\mathcal{X}$ is independent it follows from $p^k y = 0$ that $p^k \alpha_i x_i = 0$ for all i. For $i = 1$ we obtain $k \geqslant \mathrm{e}(\alpha_1 x_1) = t - r$, and we deduce $k = \mathrm{e}(y) = t - \mathrm{h}(y)$. Since y was an arbitrary element of $\langle \mathcal{X} \rangle$ it follows that $\mathrm{h}(\alpha y) + \mathrm{e}(\alpha y) = t$ for all $\alpha \neq 0$. Therefore y is h-regular. ∎

The subsequent criterion for h-independence may be of interest in its own right.

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LEMMA 2.3. *If $\mathcal{Y} = \{y_0, y_1, \dots, y_m\} \subseteq M$ is a set of h-regular elements such that*

$$\mathrm{h}(y_0) + \mathrm{e}(y_0) > \cdots > \mathrm{h}(y_m) + \mathrm{e}(y_m), \tag{2.10}$$

then $\mathcal{Y}$ is h-independent.

PROOF: We proceed by induction on $|\mathcal{Y}|$. Set $\mathrm{h}(y_i) = s_i$ and $\mathrm{e}(y_i) = k_i$, $i = 0, 1, \dots, m$. Assume that $\widetilde{\mathcal{Y}} = \{y_1, \dots, y_m\}$ is h-independent. Let y_0 be h-regular satisfying

$$s_0 + k_0 > s_j + k_j,\ j = 1, \dots, m. \tag{2.11}$$

Let us show first that $\mathcal{Y} = \{y_0\} \cup \widetilde{\mathcal{Y}}$ is an independent set. Suppose the contrary such that there exists a nonzero element of the form

$$\alpha_0 y_0 = \alpha_1 y_1 \cdots + \alpha_m y_m. \tag{2.12}$$

Since $\widetilde{\mathcal{Y}}$ is independent we have $\alpha_j y_j \neq 0, j \geqslant 1$, and

$$\mathrm{e}(\alpha_0 y_0) = \max_{j \geqslant 1}\bigl\{\mathrm{e}(\alpha_j y_j)\bigr\}. \tag{2.13}$$

Set $\mathrm{h}(\alpha_0 y_0) = r$. Then $\mathrm{h}(\alpha_0 y_0) + \mathrm{e}(\alpha_0 y_0) = s_0 + k_0$ yields $\mathrm{e}(\alpha_0 y_0) = s_0 + k_0 - r$, and (2.10) implies

$$\mathrm{e}(\alpha_0 y_0) > \mathrm{e}(\alpha_j y_j) + \mathrm{h}(\alpha_j y_j) - r \geqslant \mathrm{e}(\alpha_j y_j) + \Bigl[\min_{j \geqslant 1}\bigl\{\mathrm{h}(\alpha_j y_j)\bigr\} - r\Bigr],\ j \geqslant 1.$$

Since $\widetilde{\mathcal{Y}}$ is h-independent it follows from (2.12) that

$$r = \mathrm{h}(\alpha_0 y_0) = \min_{j \geqslant 1}\bigl\{\mathrm{h}(\alpha_j y_j)\bigr\}.$$

Hence we obtain $\mathrm{e}(\alpha_0 y_0) > \max\bigl\{\mathrm{e}(\alpha_j y_j) \mid j \geqslant 1\bigr\}$, in contradiction to (2.13). Now let us turn to h-independence of $\mathcal{Y}$. Let $y = \alpha_0 y_0 + \alpha_1 y_1 \cdots + \alpha_m y_m$ be nonzero, and $\mathrm{h}(\alpha_i y_i) = r$, $i \geqslant 0$. Then $\mathrm{e}(\alpha_i y_i) = k_i + s_i - r$, $i \geqslant 0$, and by (2.10) we obtain $\mathrm{e}(\alpha_j y_j) < k_0 + s_0 - r$, $j \geqslant 1$. Hence

$$p^{k_0 + s_0 - r - 1} y = p^{k_0 + s_0 - r - 1} \alpha_0 y_0 \neq 0.$$

Since y_0 is h-regular and $\mathrm{h}(\alpha_0 y_0) = r$ it is clear that

$$\mathrm{h}(p^{k_0 + s_0 - r - 1} \alpha_0 y_0) = k_0 + s_0 - 1,$$

and therefore $\mathrm{h}(y) \leqslant r$. ∎

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3. Partitions of bases

For the proof of Theorem 1.2 it will be crucial that h-independence of a set $\mathcal{X}$ can be checked by examining suitably chosen classes of subsets.

Lemma 3.1.

(i) *A set $\mathcal{X}$ is h-independent if the sets*

$$\mathcal{X}^{[t]} = \{x \in \mathcal{X};\, \mathrm{e}(x) + \mathrm{h}(x) = t\},$$

$t \geqslant 1$, are h-independent.

(ii) *Let $\mathcal{U}$ be a set of elements of height zero. Then $\mathcal{U}$ is p-independent if the sets*

$$\mathcal{U}_k = \{u \in \mathcal{U};\, \mathrm{e}(u) = k\}, \tag{3.14}$$

$k \geqslant 1$, are p-independent.

(iii) *Let $\mathcal{Z}$ be a set of elements of exponent 1. Then $\mathcal{Z}$ is h-independent if the sets*

$$\mathcal{Z}^{s-1} = \{z \in \mathcal{Z};\, \mathrm{h}(z) = s-1\},$$

$s \geqslant 1$, are h-independent.

Proof: (i) It suffices to show that for a given k the set $\cup\{\mathcal{X}^{[i]};\, 1 \leqslant i \leqslant k\}$ is h-independent. Let $\widetilde{\mathcal{X}} = \{x_{i_1}, \dots, x_{i_t}\}$ be such that $x_{i_\tau} \in \langle \mathcal{X}^{[i_\tau]} \rangle$, $x_{i_\tau} \neq 0$, and $1 \leqslant i_1 < \dots < i_t \leqslant m$. We know from Lemma 2.2 that x_{i_τ} is h-regular and $\mathrm{h}(x_{i_\tau}) + \mathrm{e}(x_{i_\tau}) = i_\tau$. Hence Lemma 2.3 implies that $\widetilde{\mathcal{X}}$ is h-independent and Lemma 2.1 extends h-independence from $\widetilde{\mathcal{X}}$ to $\mathcal{X}$.

For (ii) and (iii) we note that $\mathcal{U}^{[k]} = \mathcal{U}_k$ and $\mathcal{Z}^{[s]} = \mathcal{Z}^{s-1}$. ∎

Using the preceding lemma we can relate a set $\mathcal{X}$ and its h-independence to a corresponding set $\mathcal{U}$ of height zero elements and to a subset $\mathcal{Z}$ of the socle of M.

Proposition 3.2. *Let $\mathcal{X} = \{x_\lambda \mid \lambda \in \Lambda\}$ be a an independent subset of M such that $\mathrm{h}(x_\lambda) = s_\lambda$, $\mathrm{e}(x_\lambda) = k_\lambda$, $\lambda \in \Lambda$. Let $\mathcal{U} = \{u_\lambda \mid \lambda \in \Lambda\}$ be a corresponding set of height zero elements of M such that $x_\lambda = p^{s_\lambda} u_\lambda$, $\lambda \in \Lambda$. Then the following statements are equivalent.*

(i) *$\mathcal{X}$ is h-independent.*

(ii) *$\mathcal{U}$ is p-independent.*

(iii) *The set $\mathcal{Z} = \{z_\lambda = p^{k_\lambda - 1} x_\lambda \mid \lambda \in \Lambda\}$ is h-independent.*

Proof: Since $\mathcal{X}$ is independent we have $x_\lambda \neq x_\mu$ and $u_\lambda \neq u_\mu$, if $\lambda \neq \mu$. For $\lambda \in \Lambda$ define $\pi x_\lambda = u_\lambda$. Then $\pi : \mathcal{X} \to \mathcal{U}$ is a bijection. Note that x_λ is h-regular if and only if $u_\lambda = \pi x_\lambda$ is h-regular.

(i) $\Rightarrow$ (ii) Because of Lemma 3.1 it suffices to prove that the sets $\mathcal{U}_k$ in (3.14) are p-independent. Consider an element

$$v = \alpha_1 u_1 + \cdots \alpha_m u_m \in \langle \mathcal{U}_k \rangle$$

with

$$r = \mathrm{h}(\alpha_1) = \cdots = \mathrm{h}(\alpha_t) < \mathrm{h}(\alpha_{t+1}) \leqslant \cdots \leqslant \mathrm{h}(\alpha_m) < k. \tag{3.15}$$

Then

$$\alpha_j = p^r \gamma_j, \ \gamma_j \sim 1, \quad \text{for} \quad j = 1, \ldots, t. \tag{3.16}$$

Let $x_j \in \mathcal{X}$ be such that $\pi x_j = u_j$ and $x_j = p^{\mu_j} u_j$, $j = 1, \ldots, t$. Then $\mu_j < k = \mathrm{e}(u_j)$. Hence $k - \mu_j - 1 \geqslant 0$ and

$$p^{k-1}\gamma_j u_j = p^{k-1-\mu_j}\gamma_j p^{\mu_j} u_j = p^{k-1-\mu_j}\gamma_j x_j \neq 0, \ j = 1, \ldots, t. \tag{3.17}$$

Because of (3.15) and (3.16) we have

$$\begin{aligned} p^{k-r-1}v = p^{k-r-1}(\alpha_1 u_1 + \cdots + \alpha_t u_t) &= p^{k-1}(\gamma_1 u_1 + \cdots + \gamma_t u_t) \\ &= p^{k-1-\mu_1}\gamma_1 x_1 + \cdots + p^{k-1-\mu_t}\gamma_t x_t. \end{aligned}$$

Recall that $\widetilde{\mathcal{X}} = \{x_1, \ldots, x_t\} \subseteq \mathcal{X}$ is h-independent. Hence it follows from (3.17) that $p^{k-1-r}v \neq 0$. In particular we have $v \neq 0$. Thus $\mathcal{U}_k$ is independent. We also obtain

$$\begin{aligned} \mathrm{h}(p^{k-1-r}v) &= \min\{\mathrm{h}(p^{k-1-\mu_j}\gamma_j x_j) \mid 1 \leqslant j \leqslant t\} \\ &= \min\{\mathrm{h}(p^{k-1}\gamma_j u_j) \mid 1 \leqslant j \leqslant t\} = k - 1. \end{aligned}$$

Hence $\mathrm{h}(v) \leqslant r$, which implies

$$\mathrm{h}(v) = r = \min\{\mathrm{h}(\alpha_i) \mid 1 \leqslant i \leqslant m\}.$$

Thus $\mathcal{U}_k$ is h-independent.

(ii) $\Rightarrow$ (i) Assume that $\mathcal{U}$ is p-independent. Let us focus on an element $x = \alpha_1 x_1 + \cdots + \alpha_m x_m \in \langle \mathcal{X} \rangle$ with $\mathrm{h}(x_i) = s_i$ and $u_i = \pi x_i$, $1 \leqslant i \leqslant m$. From $x_i = p^{s_i} u_i$ and $\mathrm{h}(\alpha_i x_i) = \mathrm{h}(\alpha_i p^{s_i})$ we obtain

$$\mathrm{h}(x) = \mathrm{h}\left(\sum \alpha_i p^{s_i} u_i\right) = \min\{\mathrm{h}(\alpha_i p^{s_i})\} = \min\{\mathrm{h}(\alpha_i) + \mathrm{h}(x_i)\},$$

which shows that $\mathcal{X}$ is h-independent.

(ii) $\Leftrightarrow$ (iii) For $z_\lambda = p^{k_\lambda - 1} x_\lambda$ set $\widetilde{\pi} z_\lambda = u_\lambda$. Then $\widetilde{\pi} : \mathcal{Z} \to \mathcal{U}$ is a bijection and we can apply the first part of the proposition to the case where $\mathcal{X} = \mathcal{Z}$. ∎

We are now ready to derive our main result as an immediate consequence of Proposition 3.2.

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Proof of Theorem 1.2:

(i) $\Rightarrow$ (ii) Let S be a pure submodule of M with a basis $\mathcal{U} = \{u_\lambda \mid \lambda \in \Lambda\}$ such that W has a basis $\mathcal{X} = \{p^{s_\lambda} u_\lambda \mid \lambda \in \Lambda\}$. We know from Lemma 1.1 that the set $\mathcal{U}$ is p-independent. Hence it follows from Proposition 3.2 that $\mathcal{X}$ is an h-independent basis of W.

(ii) $\Rightarrow$ (i) Let $\mathcal{X} = \{x_\lambda \mid \lambda \in \Lambda\}$ be an h-independent basis of W and let $\mathcal{U} = \{u_\lambda \mid \lambda \in \Lambda\}$ be a set of h-regular elements of height zero such that $x_\lambda = p^{s_\lambda} u_\lambda$. Then it follows from Proposition 3.2 that $\mathcal{U}$ is p-independent. Hence, by Lemma 1.1 the submodule $S = \langle \mathcal{U} \rangle$ is pure and W is stacked in S. $\square$

Before turning to the proof of Corollary 1.3 we want to show that Kaplanski's condition (1.1) is necessary for the existence of an h-independent basis of M.

Proposition 3.3. *If W has an h-independent basis then W satisfies* (1.1).

Proof: It is obvious that (1.1) is equivalent to

$$p^n W \cap p^{n+r} M \subseteq p^n (W \cap p^r M), \; n \geqslant 0, \; r \geqslant 0. \tag{3.18}$$

Take an element $x \in p^n W \cap p^{n+r} M$. We can assume that r is maximal. Then $x = p^n w$ for some $w \in W$, and $\mathrm{h}(x) = n + r$. Now let $\mathcal{X}$ be an h-independent basis of W. Then $w = \alpha_1 x_1 + \cdots + \alpha_m x_m \in \langle \mathcal{X} \rangle$. Assume $\mathrm{e}(\alpha_i x_i) > n$, $i = 1, \ldots, t$, and $\mathrm{e}(\alpha_i x_i) \leqslant n$, $i > t$. Set $\widetilde{w} = \alpha_1 x_1 + \cdots + \alpha_t x_t$. Then $\widetilde{w} \in W$ and $x = p^n \widetilde{w}$, and we obtain

$$\begin{aligned} n + r = \mathrm{h}(p^n w) &= \min\{\mathrm{h}(p^n \alpha_i x_i);\, i = 1, \ldots, t\} \\ &= n + \min\{\mathrm{h}(\alpha_i x_i);\, i = 1, \ldots, t\} = n + \mathrm{h}(\widetilde{w}). \end{aligned}$$

Hence $\mathrm{h}(\widetilde{w}) = r$. We have $\widetilde{w} \in p^r M$, and we conclude that $x \in p^n (W \cap p^r M)$. $\square$

Proof of Corollary 1.3: If M is of bounded order then M is a direct sum of cyclic submodules (see for example, [**7**, p. 88]) and each pure submodule is a direct summand of M. Hence the equivalence of (i) and (ii) follows immediately from Theorem 1.2. We refer to [**7**, p. 65]) for the fact that (i) and (iii) are equivalent provided that M is of bounded order. $\square$

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