ON THE BERNSTEIN-SZEGÖ THEOREM FOR COMPLEX POLYNOMIALS

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Let $p(z)$ be a complex polynomial of degree less than or equal to n. Generalizing the well-known Bernstein theorem, Szegö (**3**) has shown that

$$\max_{|z|=1} |p'(z)| \leqq n \max_{|z|=1} |\operatorname{Re} p(z)|.$$

We shall give a partial generalization of this result.

THEOREM. *Let $p(z)$ be a polynomial of degree at most n. Let R be the radius of the largest disc contained in $G = \{p(z): |z| < 1\}$. Then*

$$\max_{|z|=1} |p'(z)| < eRn.$$

Since $R \leqq \max_{|z|=1} |\operatorname{Re} p(z)|$, we obtain Szegö's result, but with a worse constant. It would be interesting to see whether it is possible to replace the constant e by 1. If so, Rz^n would be an extremal for all n, and $R(z^{n/2} + \frac{1}{2}z^n)$ another extremal for even n.

The proof is based on the following result of Ahlfors (**1**) (cf., e.g., **2**, p. 321). This result corresponds to the estimate $\lambda \geqq \frac{1}{2}$ for the Landau constant.

LEMMA (Ahlfors). *If $f(z)$ is regular in $|z| < 1$, then*

$$(1 - |z|^2)|f'(z)| \leqq 2R(f) \qquad (|z| < 1),$$

where $R(f) \leqq \infty$ is the supremum of the radii of all discs contained in the plane domain $\{f(z): |z| < 1\}$.

Proof of the theorem. Let $f(z)$ map $|z| < 1$ conformally and one-to-one onto the universal covering surface of G. Then $p(z)$ is subordinate to $f(z)$; that is, $p(z) = f(\phi(z))$, where $\phi(z)$ is regular in $|z| < 1$ and satisfies $|\phi(z)| < 1$. It follows that

$$|\phi'(z)| \leqq \frac{1 - |\phi(z)|^2}{1 - |z|^2} \qquad (|z| < 1).$$

From the lemma we therefore obtain

$$|p'(z)| = |\phi'(z)f'(\phi(z))| \leqq \frac{1 - |\phi(z)|^2}{1 - |z|^2} |f'(\phi(z))| \leqq \frac{2R}{1 - |z|^2}.$$

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Since $p'(z)/z^{n-1}$ is regular at infinity, the maximum principle shows that for $0 < r < 1$

$$\max_{|z|=1} |p'(z)| \leqq \max_{|z|=r} \left|\frac{p'(z)}{z^{n-1}}\right| \leqq \frac{2R}{r^{n-1}(1-r^2)}.$$

The choice $r = [(n-1)/(n+1)]^{\frac{1}{2}}$ makes the right-hand side minimal. We obtain

$$\max_{|z|=1} |p'(z)| \leqq 2R\left(\frac{n+1}{n-1}\right)^{(n-1)/2} \frac{n+1}{2} < R\frac{n(1-1/n)}{(1-1/2n)(1-1/n)^n} < eRn.$$

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References

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