

CAYLEY GRAPHS AND GRAPHICAL REGULAR REPRESENTATIONS

SHASHA ZHENG 

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Studying highly symmetric graphs, especially vertex-transitive graphs, is an interesting and challenging topic in algebraic graph theory. In this thesis, we focus our attention on Cayley graphs, a family of vertex-transitive graphs. A graphical regular representation (GRR for short) of a group G is a Cayley graph of G whose full automorphism group is equal to the right regular permutation representation of G .

In 1982, Babai, Godsil, Imrich and Lovász conjectured that except for Cayley graphs of two infinite families of groups—abelian groups of exponent greater than 2 and generalised dicyclic groups—almost all finite Cayley graphs are GRRs (see [1, Conjecture 2.1]). In the study of GRRs of valency three, Xia and Fang [4] conjectured that with finitely many exceptions, every finite nonabelian simple group has a cubic GRR; and Spiga [3] conjectured that except for two-dimensional projective special linear groups and a finite number of other cases, every finite nonabelian simple group admits a cubic GRR with connection set containing only one involution. In 1998, Xu [7] introduced the concept of normal Cayley graphs: a Cayley graph of a group is called normal if the right regular permutation representation of the group is normal in the full automorphism group of the graph. In [7], Xu conjectured that except for Cayley graphs of Hamiltonian 2-groups, almost all finite Cayley graphs are normal. The purpose of this thesis is to study these problems.

The main work of this thesis consists of two parts, which are presented in Chapter 3 and Chapter 4, respectively. In the first part, we study cubic GRRs of some families of classical groups and, based on some previously known results, confirm the conjecture of Fang and Xia, and a modified version of Spiga’s conjecture. In the second part, we

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estimate the number of GRRs of a given group with large enough order and confirm the conjecture of Babai, Godsil, Imrich and Lovász as well as the conjecture of Xu.

Some of the research has been published in [2, 6] or can be found in [5].

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SHASHA ZHENG, School of Mathematics and Statistics,
The University of Melbourne, Parkville 3010, Victoria, Australia
e-mail: zhesz@student.unimelb.edu.au