RANGES OF LYAPUNOV TRANSFORMATIONS FOR OPERATOR ALGEBRAS

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1. Introduction. In this paper we shall extend results obtained in [**5**] to the W*-algebra setting.

Let $\mathcal{A}$ be a C*-algebra and let $\mathcal{A}^+$ denote the set of positive elements in $\mathcal{A}$. Given a fixed element A in $\mathcal{A}$, the Lyapunov transformation L_A corresponding to A is the mapping of $\mathcal{A}$ into itself which sends X to $AX+XA^*$. We are interested in characterizing those B in $\mathcal{A}$ for which $L_B(\mathcal{A}^+)=L_A(\mathcal{A}^+)$.

Loewy in [**6**] and [**7**] examined the case when $\mathcal{A}$ is the algebra of all $n\times n$ complex matrices, and in [**5**] the case when $\mathcal{A}=L(H)$, for any Hilbert space H, was treated. As in [**5**], [**6**] and [**7**] we shall concentrate on non-singular Lyapunov transformations, *and throughout this paper* L_A *and* L_B *will always be assumed to be invertible.* Proof of the following may be found in [**4**].

PROPOSITION 1.1. *Let A belong to the W*-algebra* $\mathcal{A}$. *Then* L_A *has a bounded inverse if and only if the spectrum of A does not intersect the imaginary axis.*

Let $\mathcal{A}=L(H)$ for some Hilbert space H. Then the main result of [**5**] is that the following are equivalent:

(i) $L_A(\mathcal{A}^+)=L_B(\mathcal{A}^+)$,

(ii) $B=(a_1+ia_2A)(a_3A+ia_4)^{-1}$, where a_i are real scalars with $a_1a_3+a_2a_4=1$.

In this paper we will show that a similar result holds when $\mathcal{A}$ is any W*-algebra, and the scalars a_i are replaced by appropriate central elements of $\mathcal{A}$. Before examining the general W*-algebra, we will show that exactly the same equivalence holds for irreducible C*-algebras.

2. The irreducible C*-case. Let $\mathcal{A}$ be any C*-algebra, and let U denote the universal representation of $\mathcal{A}$, with H denoting the Hilbert space on which U acts. It is known that the second dual $U(\mathcal{A})^{**}$ of $U(\mathcal{A})$ is a W*-algebra, and as such is isomorphic to the weak closure $\overline{U(\mathcal{A})}$ of $U(\mathcal{A})$ in $L(H)$. A proof of this may be found in [**8**]. However, for our purposes it is sufficient to notice that the map Φ which implements this isomorphism is obtained as follows.

For F in $U(\mathcal{A})^{**}$ and f in $U(\mathcal{A})^*$ we know that $f=\omega_{x,y}$ for some x, y in H. ($\omega_{x,y}(X)=\langle Xx, y\rangle$.) Hence the map $(x, y)\rightarrow F(f)=F(\omega_{x,y})$ defines a sesquilinear form on H, and so by the Riesz representation of such forms, there is a unique bounded linear operator T_F such that $F(\omega_{x,y})=\langle T_Fx, y\rangle$. That T_F lies in $\overline{U(\mathcal{A})}$ is established via the double commutant theorem.

It follows easily from this that the positive cone in $U(\mathcal{A})^{**}$ is precisely the second dual cone of $U(\mathscr{A})^+$, and that $\Phi(U\hat{}(X)) = U(X)$ for all X in $\mathscr{A}$. (Here denotes the canonical map into the second dual.)

Furthermore, routine calculations with the Arens' products show that

$$(U \circ L_A \circ U^{-1})^{**} = L_{U\hat{}(A)}$$

and so

$$\Phi(U \circ L_A \circ U^{-1})^{**}\Phi^{-1} = \bar{L}_{U(A)}.$$

(The bar indicates the natural extension of $L_{U(A)}$ to $\overline{U(\mathscr{A})}$.) Having established this notation, the next lemma follows easily.

Lemma 2.1. *Suppose A and B belong to the C^*-algebra $\mathscr{A}$ and that U is the universal representation of $\mathscr{A}$. Then if $L_A(\mathscr{A}^+) = L_B(\mathscr{A}^+)$ we also have*

$$\bar{L}_{U(A)}(\overline{U(\mathscr{A})}^+) = \bar{L}_{U(B)}(\overline{U(\mathscr{A})}^+).$$

Proof. Clearly by the preceding remarks

$$\bar{L}_{U\hat{}(A)}(\overline{U(\mathscr{A})}^+) = \bar{L}_{U(B)}(\overline{U(\mathscr{A})}^+)$$

if and only if

$$L_{U\hat{}(A)}(U(\mathscr{A})^{**+}) = L_{U\hat{}(B)}(U(\mathscr{A})^{**+}).$$

Now suppose that $F \in U(\mathscr{A})^{**+}$, $G \in U(\mathscr{A})^{**}$ and $L_{U\hat{}(A)}(G) = L_{U\hat{}(B)}(F)$. Then $L^{**}_{U(A)}(G) = L^{**}_{U(B)}(F)$ and so $G(L^*_{U(A)}(f)) = F(L^*_{U(B)}(f))$, for any f in $U(\mathscr{A})^*$. Thus if $\omega_x (= \omega_{x,x})$ is any positive functional in $U(\mathscr{A})^*$,

$$G(\omega_x) = F(L^*_{U(B)} L^{*-1}_{U(A)}(\omega_x)).$$

Finally since $L_A(\mathscr{A}^+) = L_B(\mathscr{A}^+)$, we see that $L_{U(B)} L^{-1}_{U(A)}$ maps $U(\mathscr{A})^+$ onto $U(\mathscr{A})^+$, and so $L^*_{U(B)} L^{*-1}_{U(A)}(\omega_x)$ is also a positive functional. Thus $F(L^*_{U(B)} L^{*-1}_{U(A)}(\omega_x)) \geqslant 0$ and so G lies in $U(\mathscr{A})^{**+}$, as required.

Proposition 2.2 *Let $\mathscr{A}$ be a C^*-algebra, and let π be any *-representation of $\mathscr{A}$. Suppose that $L_A(\mathscr{A}^+) = L_B(\mathscr{A}^+)$. Then*

$$L_{\pi(A)}(\overline{\pi(\mathscr{A})}^+) = L_{\pi(B)}(\overline{\pi(\mathscr{A})}^+).$$

Proof. If U is the universal representation of $\mathscr{A}$, we can apply Lemma 2.1 to conclude that $\bar{L}_{U(A)}(\overline{U(\mathscr{A})}^+) = \bar{L}_{U(B)}(\overline{U(\mathscr{A})}^+)$. Furthermore, given any *-representation π, we can find a W^*-isomorphism α of $\overline{\pi(A)}$ onto $\overline{U(A)}Q$, where Q is some projection in the centre of $\overline{U(\mathscr{A})}$, such that $\alpha(\pi(X)) = U(X)Q$ for all X in $\mathscr{A}$. Now if $L_{U(A)Q}$ denotes the "cut-down" map of $\bar{L}_{U(A)}$ to the algebra $\overline{U(\mathscr{A})}Q$ (i.e. the map which sends XQ to $(U(A)X + XU(A)^*)Q$, for all X in $\overline{U(\mathscr{A})}$), we see that $L_{U(A)Q}(\overline{U(\mathscr{A})}^+Q) = L_{U(B)Q}(\overline{U(\mathscr{A})}^+Q)$. Finally, since $\bar{L}_{\pi(A)} = \alpha^{-1}L_{U(A)Q}\alpha$ and $L_{\pi(B)} = \alpha^{-1}L_{U(B)Q}\alpha$, we conclude that $L_{\pi(A)}(\overline{\pi(\mathscr{A})}^+) = L_{\pi(B)}(\overline{\pi(\mathscr{A})}^+)$.

COROLLARY 2.3. *Let $\mathscr{A}$ be a unital C^*-algebra which has a faithful irreducible $*$-representation. Then the following are equivalent:*

(i) $L_A(\mathscr{A}^+) = L_B(\mathscr{A}^+)$,

(ii) $B = (a_1 + ia_2A)(a_3A + ia_4)^{-1}$ *for some real scalars* a_i *with* $a_1a_3 + a_2a_4 = 1$.

Proof. Let π be any faithful irreducible $*$-representation of $\mathscr{A}$ and let H_π denote the Hilbert space on which π acts. Then if $L_A(\mathscr{A}^+) = L_B(\mathscr{A}^+)$, it follows from Proposition 2.2 that $L_{\pi(A)}(\overline{\pi(\mathscr{A})}^+) = L_{\pi(B)}(\overline{\pi(\mathscr{A})}^+)$ i.e. $L_{\pi(A)}(L(H_\pi)^+) = L_{\pi(B)}(L(H_\pi)^+)$ (since π is irreducible). Thus, by Theorem 3.2 of [5], we can find real scalars a_i such that $a_1a_3 + a_2a_4 = 1$ and

$$\pi(B) = (a_1 + ia_2\pi(A))(a_3\pi(A) + ia_4)^{-1}.$$

Now (ii) follows since π is faithful.

Conversely, if B satisfies (ii), then by the same theorem quoted above

$$L_{\pi(A)}(L(H_\pi)^+) = L_{\pi(B)}(L(H_\pi)^+)$$

and, since $L^{-1}_{\pi(A)}L_{\pi(B)}$ maps $\pi(\mathscr{A})$ onto itself, (i) follows, again since π is faithful.

3. The W*-case. In this section $\mathscr{A}$ will denote a W*-algebra with centre Z. Ω will denote the maximal ideal space of Z. For any ω in Ω let $J(\omega)$ denote the smallest norm-closed two sided ideal in $\mathscr{A}$ which contains ω. $\mathscr{A}(\omega)$ will denote the quotient C*-algebra $\mathscr{A}/J(\omega)$, and $A(\omega)$ will denote the image of A in $\mathscr{A}(\omega)$.

It has been shown in [2] that

$$\|A\| = \sup\{\|A(\omega)\| : \omega \in \Omega\} \tag{1}$$

and that the mapping $\omega \to \|A(\omega)\|$ is continuous. Also

$$\mathrm{Sp}(A) = \bigcup\{\mathrm{Sp}(A(\omega)) : \omega \in \Omega\}. \tag{2}$$

(Sp($\cdot$) denotes the spectrum. A proof may be found in [4].)

Before proving the main result, we need a bound for the scalars.

LEMMA 3.1. *Let A, B in $\mathscr{A}$ implement the Lyapunov transformations L_A and L_B. Suppose that for each ω in Ω there are real scalars $a_i(\omega)$ such that $a_1(\omega)a_3(\omega) + a_2(\omega)a_4(\omega) = 1$ and*

$$B(\omega) = [a_1(\omega) + ia_2(\omega)A(\omega)][a_3(\omega)A(\omega) + ia_4(\omega)]^{-1}.$$

Then the $a_i(\omega)$ are uniformly bounded by some number K, which depends only on A and B.

Proof. First notice that since $A(\omega)$ and $B(\omega)$ commute, we can find a maximal abelian subalgebra $C(\omega)$ of $\mathscr{A}(\omega)$ in which both lie. Also Sp($A(\omega)$) and Sp($B(\omega)$) remain unaltered by passing to $C(\omega)$.

For any multiplicative linear functional ϕ on $C(\omega)$, let $a = \phi(A(\omega))$ and $b = \phi(B(\omega))$. Write $a = x_1 + ix_2$ and $b = y_1 + iy_2 (x_i, y_i \in \mathbb{R})$. Clearly

$$a_1(\omega) + ia_2(\omega)a = b(a_3(\omega)a + ia_4(\omega)) \tag{3}$$

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and so

$$[a_1(\omega)+ia_2(\omega)a][a_3(\omega)\bar{a}-ia_4(\omega)]=b\,|a_3(\omega)a+ia_4(\omega)|^2,$$

from which we see that

$$a_1(\omega)a_3(\omega)x_1+a_2(\omega)a_4(\omega)x_1=y_1\,|a_3(\omega)a+ia_4(\omega)|^2,$$

which reduces to

$$x_1=y_1\,|a_3(\omega)a+ia_4(\omega)|^2. \tag{4}$$

From here it is a routine matter to show that

$$|a_3(\omega)|\leqslant(|x_1|\,.\,|y_1|)^{-1/2}, \tag{5}$$

$$|a_4(\omega)|\leqslant(|x_1|\,.\,|y_1|^{-1})^{1/2}+(|x_1|\,.\,|y_1|)^{-1/2}\,|x_2|. \tag{6}$$

Using (3) and (4) we obtain

$$|a_1(\omega)+ia_2(\omega)a|\leqslant|b|\,(|x_1|\,.\,|y_1|^{-1})^{1/2},$$

$$|a_2(\omega)|\leqslant|b|\,(|x_1|\,.\,|y_1|)^{-1/2}, \tag{7}$$

$$|a_1(\omega)|\leqslant|b|\,\{(|x_1|\,.\,|y_1|^{-1})^{1/2}+(|x_1|\,.\,|y_1|)^{-1/2}\,|x_2|\}. \tag{8}$$

Now in the formulae (5)–(8), where $|x_i|$ appears without inversion, we may replace it with the spectral radius of $A(\omega)$ and so by $\|A(\omega)\|$, without disturbing the inequalities. Finally (1) shows that we may also substitute $\|A\|$. Similarly $|y_1|$ (but not, of course, $|y_1|^{-1}$) may be replaced by $\|B\|$, and in (7) and (8), $|b|$ may be replaced by $\|B\|$.

It remains to find upper bounds for $|x_1|^{-1}$ and $|y_1|^{-1}$. Since these represent the real parts of points in the spectra of $A(\omega)$ and $B(\omega)$, Proposition 1.1 together with (2) shows that $|x_1|$ and $|y_1|$ are bounded below by some positive number δ (which depends only on A and B). Thus we may substitute δ^{-1} for $|x_1|^{-1}$ and $|y_1|^{-1}$ in (5) ... (8), and maintain the inequalities. In this way we can find a uniform bound for the scalars $a_i(\omega)$.

We are now in a position to prove our main result.

THEOREM 3.2. *Let A and B belong to the W^*-algebra $\mathscr{A}$. Then the following are equivalent*:

(i) $L_A(\mathscr{A}^+)=L_B(\mathscr{A}^+)$,

(ii) $B=(Z_1+iAZ_2)(AZ_3+iZ_4)^{-1}$,

where Z_i are self-adjoint elements of the centre of $\mathscr{A}$ satisfying $Z_1Z_3+Z_2Z_4=I$.

Proof. (i) $\Rightarrow$ (ii). Clearly L_A and L_B map each ideal $J(\omega)$ into itself and so induce the Lyapunov transformations $L_{A(\omega)}$ and $L_{B(\omega)}$ on each $\mathscr{A}(\omega)$. Thus we see that if (i) holds, then $L_{A(\omega)}(\mathscr{A}(\omega)^+)=L_{B(\omega)}(\mathscr{A}(\omega)^+)$ for each ω in Ω. Also it follows from Proposition 1.1 and (2) that these induced Lyapunov transformations are non-singular.

Now in [3] Halpern has shown that each $\mathscr{A}(\omega)$ is a primitive C*-algebra. Thus each $\mathscr{A}(\omega)$ has a faithful irreducible *-representation. Corollary 2.3 then shows that we can find real scalars $a_i(\omega)$ with $a_1(\omega)a_3(\omega)+a_2(\omega)a_4(\omega)=1$ such that

$$B(\omega)=[a_1(\omega)+ia_2(\omega)A(\omega)][a_3(\omega)A(\omega)+ia_4(\omega)]^{-1}.$$

Lemma 3.1 shows that we can also find a constant K, independent of ω, such that $|a_i(\omega)| \leqslant K$ for $i=1,\dots,4$, and ω in Ω.

We now consider the set

$\Sigma = \{(\omega, a, b, c, d): B(\omega) = [a+ibA(\omega)][cA(\omega)+id]^{-1};\ ac+bd=1$ and a, b, c, d bounded by $K\}$.

Then Σ is a non-empty subset of $\Omega \times [-K, K]^4$ whose projection onto the first coordinate is Ω.

Also Σ is closed. For suppose $(\omega_i, a_i, b_i, c_i, d_i)$ is a net in Σ which converges to (ω, a, b, c, d) in $\Omega \times [-K, K]$. Then

$$\begin{aligned}&\|B(\omega) - [a+ibA(\omega)][cA(\omega)+id]^{-1}\| \\ &= \|[B-(a+ibA)(cA+id)^{-1}](\omega)\| \\ &= \lim_i \|[B-(a+ibA)(cA+id)^{-1}](\omega_i)\| \\ &= \lim_i \|B(\omega_i) - (a_i+ib_iA(\omega_i))(c_iA(\omega_i)+id_i)^{-1}\| = 0;\end{aligned}$$

i.e. (ω, a, b, c, d) lies in Σ. Thus Σ is a non-empty compact Hausdorff space.

Let p_i denote the projection onto the ith coordinate. Then p_i is a continuous map of Σ onto Ω and, since Ω is extremally disconnected, we may appeal to **[1]** to find a continuous selection for p_1. That is, we can find a continuous function g mapping Ω into Σ such that $p_1 \circ g(\omega) = \omega$ for all ω in Ω. (In **[1]** Gleason shows that in the category of all compact Hausdorff spaces and all continuous maps, the projective objects are precisely the extremally disconnected spaces.)

Thus each $Z_{i-1} = p_i \circ g (i=2,\dots,5)$ defines a continuous bounded real-valued function on Ω, and so defines a self-adjoint element of Z. Clearly, it follows from our choice of Σ that $B(\omega) = [Z_1 + iAZ_2][AZ_3 + iZ_4]^{-1}(\omega)$ and $(Z_1Z_3 + Z_2Z_4)(\omega) = 1$, for all ω in Ω. Thus $B = (Z_1 + iAZ_2)(AZ_3 + iZ_4)^{-1}$ with $Z_1Z_3 + Z_2Z_4 = I$ as required.

(ii) $\Rightarrow$ (i). Suppose A and B are related as in (ii). Then $A(\omega)$ and $B(\omega)$ are related as in (ii) of Corollary 2.3, and so $L_{A(\omega)}(\mathscr{A}(\omega)^+ = L_{B(\omega)}(\mathscr{A}(\omega)^+)$ for all ω in Ω. Thus if H is in $\mathscr{A}^+$ and $K = L_A^{-1}L_B(H)$ we have $K(\omega) = L_{A(\omega)}^{-1}L_{B(\omega)}(H(\omega)) \geqslant 0$ for all ω in Ω; i.e. $K \geqslant 0$. Similarly $L_B^{-1}L_A(\mathscr{A}^+) \subseteq \mathscr{A}^+$ and (i) follows.

Since a C*-algebra may have no centre at all, there can be no direct generalization of Theorem 3.2 in that direction. Nonetheless we can prove the following result.

COROLLARY 3.3. *Let A and B belong to the C*-algebra $\mathscr{A}$, and suppose that $L_A(\mathscr{A}^+) = L_B(\mathscr{A}^+)$. Then if π is any *-representation of $\mathscr{A}$ we can find self-adjoint elements Z_i in the centre of $\overline{\pi(A)}$ such that*

$$\pi(B) = (Z_1 + i\pi(A)Z_2)(\pi(A)Z_3 + iZ_4)^{-1}.$$

Proof. This follows easily from Proposition 2.2 and Theorem 3.2.

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