ON ORDERED GEOMETRIES

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In Theorem 2.20 of his Geometric Algebra, Artin shows that any ordering of a plane geometry is equivalent to a weak ordering of its skew field. Referring to his Theorem 1.16 that every weakly ordered field with more than two elements is ordered, he deduces his Theorem 2.21 that any ordering of a Desarguian plane with more than four points is (canonically) equivalent to an ordering of its field. We should like to present another proof of this theorem stimulated by Lipman's paper [this Bulletin, vol. 4, 3, pp. 265-278]. Our proof seems to bypass Artin's Theorem 1.16; cf. the postscript.

1. Our starting point is an affine geometry, i.e. a set $\underline{G}$ of "points" $A, B, \ldots, O, X, \ldots$ coordinatized by a left linear vector space of more than one dimension over a skew field $K = \{\alpha, \beta, \ldots, \lambda, \ldots\}$. Thus any ordered pair of points A, B determines a vector $B-A$ and we can add this vector to the point C obtaining the point

$$D = C + (B-A) = B + (C-A) .$$

D is the one and only solution of

$$D - C = B - A .$$

If an arbitrary point O is designated as the origin, we obtain a one-one correspondence between the points $X \in \underline{G}$ and their radius vectors $X - O$. If $X - O = \lambda(A-O)$ or $Y - O = (A-O) + (B-O)$ [i.e. $Y = A + (B-O)$], write $X = \lambda A$ and $Y = A + B$ respectively. Thus the points λA and $A+B$ are defined only if the origin O has been agreed upon.

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If $A \neq B$, denote the straight line through A and B by $A \times B$.

2. We assume that our geometry is ordered. Thus a relation $\langle ABC \rangle$, "B lies between A and C", is defined for collinear mutually distinct points A, B, C. This relation is supposed to satisfy the following conditions:

(i) $\langle ABC \rangle$ implies $\langle CBA \rangle$.

(ii) If A, B, C are collinear and mutually distinct, then one and only one of the three relations holds

$$\langle ABC \rangle, \quad \langle BCA \rangle, \quad \langle CAB \rangle.$$

(iii) If $\langle ABC \rangle$ and $\langle BCD \rangle$, then $\langle ABD \rangle$ (and therefore also $\langle ACD \rangle$).

(iv) If $\langle ABC \rangle$ and $\langle ACD \rangle$, then $\langle BCD \rangle$ (and hence also $\langle ABD \rangle$).

(v) Two of the relations

$$\langle BAC \rangle, \quad \langle CAD \rangle, \quad \langle DAB \rangle$$

exclude the third.

(vi) Betweenness is invariant with respect to parallel projection.

These conditions are not independent. The reader may verify that (i), (ii), (iii), and (v) imply (iv).

If $\langle ABC \rangle$ or $\langle ACB \rangle$ or $B = C \neq A$, B and C are said to lie on the same side of A. Thus if A, B, C are collinear, B and C do not lie on the same side of A if and only if $A = B$ or $A = C$ or $\langle CAB \rangle$. Conditions (i)-(v) imply that the property of lying on the same side of A is an equivalence relation. We also note: If $\langle BAC \rangle$ and $\langle BAD \rangle$, C and D must lie on the same side of A.

3. We now choose two arbitrary points O and A in $\underline{G}$;

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$O \neq A$. Thus the straight line $\ell = O \times A$ has the parametric representation

$$\ell : \quad X = \lambda A , \quad \lambda \in \underline{K}$$

We define

(1) $\lambda > 0 \leftrightarrow A$ and λA lie on the same side of O .

Since A and $1 . A = A$ lie on the same side of O , (1) implies

(2) $$1 > 0 .$$

Our definition seems to depend on the choice of O and A . We first verify that it actually is independent of A . Let

$$B \notin \ell ; \quad \ell' = O \times B .$$

The dilatation $X \to \lambda X$ maps $A \times B$ onto $\lambda A \times \lambda B$. Thus

$$\lambda A \times \lambda B \parallel A \times B \qquad \text{for all } \lambda$$

and (vi) implies

(3) $$\left\{ \begin{array}{l} B \text{ and } \lambda B \text{ lie on the same side of } O \\ \qquad\qquad \updownarrow \\ A \text{ and } \lambda A \text{ lie on the same side of } O . \end{array} \right.$$

If $C \in \ell$, $C \neq O$, then $\lambda C \times \lambda B \parallel C \times B$. Applying (3) twice, we obtain

(4) $$\left\{ \begin{array}{l} C \text{ and } \lambda C \text{ lie on the same side of } O \\ \qquad\qquad \updownarrow \\ B \text{ and } \lambda B \text{ lie on the same side of } O \\ \qquad\qquad \updownarrow \\ A \text{ and } \lambda A \text{ lie on the same side of } O \end{array} \right.$$

(4) shows the independence of definition (1) of the choice of A . We now verify its independence of O .

Let $O' \neq O$. On account of the preceding result we may choose $A \notin O \times O'$. Construct the straight line ℓ' through

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O' parallel to $\mathcal{l} = O \times A$ and project $\mathcal{l}$ onto $\mathcal{l}'$ parallel to $O \times O'$. The image of the point $X \in \mathcal{l}$ is the point $X' \in \mathcal{l}'$ satisfying $X' - O' = X - O$. If A' is the projection of A, we therefore have

$$(5) \quad X-O=\lambda(A-O) \leftrightarrow X'-O'=X-O=\lambda(A-O)=\lambda(A'-O') \leftrightarrow X'-O'=\lambda(A'-O').$$

Hence by (vi)

$$\begin{gathered} A' \text{ and } O' + \lambda(A'-O') \text{ lie on the same side of } O' \\ \updownarrow \\ A \text{ and } O + \lambda(A-O) \text{ lie on the same side of } O. \end{gathered}$$

This completes our proof.

4. Let O, A, X, O'' be four points on the line $\mathcal{l}$: $O \neq A$. Choose O' outside $\mathcal{l}$. Let $\mathcal{l}'$ be the straight line through O' parallel to $\mathcal{l}$. Project first $\mathcal{l}$ parallel to $O \times O'$ onto $\mathcal{l}'$ then $\mathcal{l}'$ parallel to $O' \times O''$ back onto $\mathcal{l}$. Suppose the points A and X are mapped first onto A' and X' and these points are then projected onto A'' and X'' respectively. Then

$$A - O = A' - O' = A'' - O'' \text{ and } X - O = X' - O' = X'' - O''.$$

Hence

$$X = O + \lambda(A-O) \quad \leftrightarrow \quad X'' = O'' + \lambda(A''-O'')$$

and the mapping

$$X \to X'' = X + (O''-O)$$

of $\mathcal{l}$ onto itself is a translation. Being the product of two parallel projections it preserves betweenness.

5. If one straight line contains exactly two points, then every line will. Conditions (i)-(vi) are trivially satisfied. The skew-field $\underline{K}$ becomes the prime field with two elements. Definition (1) becomes trivial.

From now on assume that every straight line contains three or more points. If $\langle OAB \rangle$, translation yields by 4.

$$\langle A, 2A, A+B \rangle \quad \text{and} \quad \langle B, A+B, 2B \rangle .$$

If $\underline{K}$ had the characteristic two, this would imply

$$\langle A, O, A+B \rangle \quad \text{and} \quad \langle B, A+B, O \rangle ,$$

hence $\langle O, A+B, B \rangle$ and $\langle AOB \rangle$, contradiction. Thus the characteristic of $\underline{K}$ is different from two.

6. Suppose the points P, B, C are collinear and mutually distinct. By 5., the midpoint O of the segment BC exists. Taking O as the origin, we have $C = -B$. Let $Q = -P$.

Let $B' \notin \mathcal{l} = B \times C$. Project $\mathcal{l}$ parallel to $B \times B'$ onto $\mathcal{l}' = B' \times O$. Denote the projection of C (of P, of Q) by C' (P', Q'). By 3.

$$C' = -B' , \quad Q' = -P' .$$

Suppose $P = \lambda B$. Then $Q = -\lambda B$. Hence

$$P' = \lambda B' , \quad Q' = -\lambda B'$$

and

$$B - C' = B' - C = B + B' , \quad P - Q' = P' - Q = \lambda(B+B') .$$

Therefore

$$B \times C' \parallel B' \times C \parallel P \times Q' \parallel P' \times Q .$$

Thus projection of $\mathcal{l}'$ onto $\mathcal{l}$ parallel to $B \times C'$ maps B' onto C, C' onto B, P' onto Q, Q' onto P. In other words, we have been able to interchange B and C, P and Q by means of two consecutive parallel projections. Hence Postulate (vi) yields

(6) $\langle PBC \rangle$ implies $\langle QCB \rangle$, hence $\langle BCQ \rangle$ and $\langle PBQ \rangle$.

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7. Write $\lambda < 0$ if $\lambda \geq 0$ is false. Thus

$$\lambda < 0 \leftrightarrow <\lambda A , O , A> .$$

By 3., this definition is independent of the choice of O and A; $O \neq A$.

Let

$$\lambda > 0 , \mu > 0 \; [\lambda < 0 , \mu < 0] ; \quad \lambda \neq \mu .$$

Thus λA, μA and A lie on the same side of O [O separates A from λA and μA]. Hence λA and μA lie on the same side of O. On account of $\lambda A \neq \mu A$, we may assume e.g. $<O , \lambda A , \mu A>$.

Applying (6) with $P = O$, $B = \lambda A$, $C = \mu A$, we obtain that $\lambda A + \mu A = (\lambda+\mu)A$ and λA lie on the same side of O. Hence $(\lambda+\mu)A$ and A lie on the same side [on opposite sides] of O. We thus have

$$(7) \quad \begin{cases} \lambda > 0 , \mu > 0 , \quad \lambda \neq \mu \text{ implies } \lambda + \mu > 0 \\ \lambda < 0 , \mu < 0 , \quad \lambda \neq \mu \text{ implies } \lambda + \mu < 0 \end{cases}$$

Suppose $\lambda > 0$, $-\lambda > 0$ [$\lambda < 0$, $-\lambda < 0$]. Then (7) yields $0 > 0 [0 < 0]$; contradiction. Hence $\lambda > 0$ implies $-\lambda < 0$ and $\lambda < 0$ implies $-\lambda > 0$, or

$$(8) \qquad \lambda < 0 \leftrightarrow -\lambda > 0 .$$

This formula contains the trichotomy law: Every $\lambda \in \underline{K}$ satisfies one and only one of the relations

$$\lambda > 0 , \lambda = 0 , -\lambda > 0 .$$

8. By (2) and (8), $-1 < 0$. Hence

$$<-B , O , B> \qquad \text{for every } B \neq O .$$

Betweenness being invariant under translations, this leads to

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(9) $\qquad <O\ ,\ B\ ,\ 2B> \qquad$ for every $B \neq O$.

Given $A \neq O$ and $\lambda > 0$. Choose $B = \lambda A$; thus $2B = 2\lambda A$. Since A and B as well as B and $2B$ lie on the same side of O , A and $2B$ lie on the same side of O , i.e. $2\lambda > 0$. The first line of (7) can therefore be improved to

(10) $\qquad \lambda > 0\ ,\ \mu > 0$ implies $\lambda + \mu > 0$.

9. Let $\lambda > 0$, $\mu > 0$. By (4), μA and $\lambda(\mu A)$ lie on the same side of O . Furthermore A and μA lie on the same side of O . Hence A and $\lambda\mu A$ lie on the same side of O . This proves

(11) $\qquad \lambda > 0\ ,\ \mu > 0$ implies $\lambda\mu > 0$.

By (10) , (11) , and the trichotomy law, $\underline{K}$ is ordered.

10. We note in passing that our betweenness relation is preserved by dilations.

Let a dilation be given by

$$X' = \alpha X + O' .$$

Thus $\alpha \neq 0$, O is mapped onto O' , and $X' - O' = \alpha(X - O)$.

Suppose now that $B = \lambda A$, Then

$$B' - O' = \alpha(B-O) = \alpha\lambda(A-O) = \alpha\lambda\alpha^{-1} \cdot \alpha(A-O) = \alpha\lambda\alpha^{-1} \cdot (A' - O') .$$

If $<BOA>$, 7. implies $\lambda < 0$. Therefore $\alpha\lambda\alpha^{-1} < 0$ and hence $<B'O'A'>$.

11. We now start conversely from an ordered skew-field $\underline{K}$ and a geometry $\underline{G}$ over $\underline{K}$. Let O , A , B denote any three mutually distinct collinear points. Thus $B = \lambda A$ for some λ . Define

$$<BOA> \leftrightarrow \lambda < 0 .$$

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Let the points O , A , B be mapped by parallel projection onto O' , A' , B' respectively. Let ℓ'' be the straight line through O' parallel to $\ell = O \times A$. Our parallel projection can be obtained by first projecting ℓ onto ℓ'' mapping O , A , B onto O' , A'' , B'' respectively and then ℓ'' onto $\ell' = O' \times A'$. Applying first the argument leading to (5) and then <u>3.</u> , we obtain

$$B-O = \lambda(A-O) \leftrightarrow B''-O' = \lambda(A''-O') \leftrightarrow B'-O' = \lambda(A'-O') .$$

This yields condition (vi) :

$$\langle BOA \rangle \leftrightarrow \langle B'O'A' \rangle .$$

Let A , B , C be collinear and mutually distinct ; $O \in A \times B$, $E \in A \times B$, $O \neq E$. Suppose

$$(12) \qquad A = \alpha E , \quad B = \beta E , \quad C = \gamma E .$$

Thus α , β , γ are mutually distinct and

$$A - B = (\alpha-\beta)E , \quad C - B = (\gamma-\beta)E$$

or

$$A - B = (\alpha-\beta)(\gamma-\beta)^{-1}(C-B) .$$

This yields

$$\langle ABC \rangle \leftrightarrow (\alpha-\beta)(\gamma-\beta)^{-1} < 0 .$$

This condition is satisfied if and only if

either $\alpha-\beta>0$, $\beta-\gamma>0$ or $\alpha-\beta<0$, $\beta-\gamma<0$.

We thus have

$$(13) \qquad \langle ABC \rangle \leftrightarrow \text{either } \gamma<\beta<\alpha \text{ or } \alpha<\beta<\gamma .$$

<u>12.</u> Obviously, (13) implies Condition (i).

Since three mutually distinct elements α , β , γ of <u>K</u>

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satisfy exactly one of the six inequalities

$$\alpha < \beta < \gamma\,, \quad \beta < \gamma < \alpha\,, \quad \gamma < \alpha < \beta$$

$$\gamma < \beta < \alpha\,, \quad \alpha < \gamma < \beta\,, \quad \beta < \alpha < \gamma\,,$$

Condition (ii) is satisfied.

Condition (iii): Let <ABC> , <BCD> . We use the notations (12). Let $D = \sigma E$. Then

either $\quad \alpha < \beta < \gamma \quad$ or $\quad \gamma < \beta < \alpha$

and either $\quad \beta < \gamma < \sigma \quad$ or $\quad \sigma < \gamma < \beta$.

Hence

either $\quad \alpha < \beta < \gamma < \sigma \quad$ or $\quad \sigma < \gamma < \beta < \alpha$.

In particular

either $\quad \alpha < \beta < \sigma \quad$ or $\quad \sigma < \beta < \alpha$.

This yields <ABD> .

The proof of (iv) is similar.

Finally for Condition (v): <BAC> and <CAD> imply

either $\quad \beta < \alpha < \gamma \quad$ or $\quad \gamma < \alpha < \beta$

and either $\quad \gamma < \alpha < \sigma \quad$ or $\quad \sigma < \alpha < \gamma$.

Therefore

either $\quad \beta < \alpha$ and $\sigma < \alpha$ or $\alpha < \beta$ and $\alpha < \sigma$.

This excludes <DAB> .

13. Postscript. The referee has made the following comment: "The proof [of Artin's Theorem 2.21] not only bypasses Artin's Theorem 1.16 but actually contains it. It is only necessary to

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observe that the one property of ordered fields used in <u>11</u> and <u>12</u> is that

$$(\alpha-\beta)(\gamma-\beta)^{-1} < 0 \quad \rightarrow \text{ either } \alpha < \beta < \gamma \text{ or } \gamma < \beta < \alpha .$$

This is quickly checked for weakly ordered fields where the notation is so chosen that $0 < 1$. For then $\gamma - \beta > 0$ or $\gamma - \beta < 0$ according as the transformation $\sigma \rightarrow \sigma(\gamma-\beta)$ preserves or reverses order. Hence from $(\alpha-\beta)(\gamma-\beta)^{-1} < 0$ we get

$$\text{either} \quad \gamma - \beta > 0 > \alpha - \beta \quad \text{or} \quad \gamma - \beta < 0 < \alpha - \beta$$

whence

$$\text{either} \quad \alpha < \beta < \gamma \qquad \text{or} \quad \gamma < \beta < \alpha$$

and conversely."

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