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QUASI-LINEAR PARABOLIC PARTIAL DIFFERENTIAL EQUATIONS WITH DELAYS IN THE HIGHEST ORDER SPATIAL DERIVATIVES

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Abstract

A class of functional differential equations in some Hilbert space are studied. The results are applicable to many quasi-linear parabolic partial differential equations with (possibly) countably many discrete delays and finitely many distributed delays in the highest order spatial derivatives. For the linear case, an evolution operator on the *underline space* H is introduced, via which a variation of constant formula for the solution of the equation in the *underline space* H is derived. Some spectral properties of the generator of the solution semigroup defined on some appropriate space are discussed as well.

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1. Introduction

In this paper, we study a class of abstract delay equations in some Hilbert space H. One of the interesting cases which is covered by our results is the following initial-boundary value problem for a parabolic partial integro-differential equation:

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$$
(1.1)\quad
\begin{cases}
u_t(t,x) - u_{xx}(t,x) \\
\qquad = \displaystyle\int_{-r}^{0} f(t,\theta,u(t,x),u_x(t,x),u(t+\theta,x),u_x(t+\theta,x), \\
\qquad\qquad\qquad\qquad\qquad\qquad u_{xx}(t+\theta,x))\,d\theta \\
\qquad\quad + \displaystyle\sum_{i=1}^{\infty} g_i(t,u(t,x),u_x(t,x),u(t-r_i,x),u_x(t-r_i,x), \\
\qquad\qquad\qquad\qquad\qquad\qquad u_{xx}(t-r_i,x)), \\
\qquad\qquad (t,x)\in(0,\infty)\times(0,1), \\
u(t,x)=\varphi(t,x),\ (t,x)\in[-r,0]\times(0,1), \\
u(t,0)=u(t,1)=0,\ t\in(0,\infty),
\end{cases}
$$

where f and g_i are some maps and $r_i \in (0, r]$ are some positive constants. It is clear that the above equation has the following features:

There are (possibly) countably many discrete delays and finitely many (which is the same as one) distributed delays appearing in the highest order spatial derivative terms; the terms with and without delays are nonlinearly coupled.

At this point, one may realize that our investigation generalizes previous works relevant to this aspect (cf. [1–4, 11, 19]). We should note that the mentioned works only discussed the cases with at most finitely many discrete delays in the highest order derivative terms. The cases in which the delay only appears in the lower order (including zero-th order) spatial derivatives were discussed by many authors [5–7, 10, 14, 15, 18, 22]. For a brief survey of this topic, see [3]. Some other relevant results can be found in [8, 9, 12, 17].

In this paper, we will pose a set of very general conditions under which the global existence and the uniqueness of the strong solutions to a general functional differential equation is ensured. This is carried out in Section 2. Then, in Section 3, we study a delay differential equation which contains equations of form (1.1). In Section 4, a linear case is discussed. We derive a variation of constants formula which can be regarded as an extension of those given in [10]. Finally, the spectral properties of the associated solution semigroup are studied.

2. An abstract functional differential equation

In this section, we establish the existence and the uniqueness of the (mild) solution to an abstract functional differential equation. Let us start with some

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assumptions. To this end, we fix a Hilbert space H and a constant $r > 0$.

(A1) The operator $A: \mathscr{D}(A) \subset H \to H$ is linear, closed and densely defined. There exist $\delta_0 \in (0, \pi/2)$, $\omega_0 > 0$ and $M > 0$, such that

(2.1)
$$\rho(A) \supset \Sigma(\omega_0, \delta_0) \equiv \{\lambda \in \mathbb{C} | \delta_0 \le |\arg \lambda| \le \pi\} \bigcup \{\lambda \in \mathbb{C} | \operatorname{Re} \lambda \le \omega_0\},$$

$$\|(\lambda I - A)^{-1}\|_{\mathscr{L}(H)} \le \frac{M}{1 + |\lambda|}, \qquad \forall \lambda \in \Sigma(\omega_0, \delta_0). \tag{2.2}$$

By [16], we know that under (A1), the operator $-A$ generates an analytic semigroup on H and the fractional powers $A^\beta (\beta \in \mathbb{R})$ are well-defined. We let $(-\infty < a < b < +\infty)$

$$L^2(a, b; \mathscr{D}(A)) = \left\{ v(\cdot): [a, b] \to \mathscr{D}(A) \,\middle|\, \int_a^b |Av(t)|_H^2 \, dt < \infty \right\},$$
$$L^2_{\text{loc}}(-r, \infty; \mathscr{D}(A)) = \left\{ v(\cdot): [-r, \infty) \to \mathscr{D}(A) \,\middle|\, \int_{-r}^s |Av(t)|_H^2 \, dt < \infty, \right.$$
$$\left. \forall s \ge -r \right\},$$

and as in [3], we let

$$F = \left\{ x \in H \,\middle|\, \int_0^\infty |Ae^{-At} x|_H^2 \, dt < \infty \right\}.$$

For any $v(\cdot) \in L^2(a, b; \mathscr{D}(A))$ and $x \in F$, we let

$$|v(\cdot)|_{L^2(a, b; \mathscr{D}(A))} = \left[\int_a^b |Av(t)|_H^2 \, dt \right]^{1/2},$$
$$|x|_F = \left(|x|_H^2 + \int_0^\infty |Ae^{-At} x|_H^2 \, dt \right)^{1/2}.$$

From [3], we know that F is a Banach space and

$$L^2(a, b; \mathscr{D}(A)) \cap W^{1,2}(a, b; H) \hookrightarrow C([a, b]; F), \qquad \forall -\infty < a \le b < +\infty. \tag{2.3}$$

Next, for any $v(\cdot) \in L^2_{\text{loc}}(-r, \infty; \mathscr{D}(A))$ and $t \ge 0$, we let $v_t(\cdot) \in L^2(-r, 0; \mathscr{D}(A))$ be defined as $v_t(\theta) = v(t + \theta)$, $\forall \theta \in [-r, 0]$.

(A2) The mapping $G: [0, \infty) \times C([-r, 0]; \mathscr{D}(A)) \to H$ satisfies the following:

(i) For any $v(\cdot) \in L^2_{\text{loc}}(-r, \infty; \mathscr{D}(A))$ the map $t \mapsto G(t, v_t(\cdot))$ is almost everywhere defined and is in $L^2_{\text{loc}}(0, \infty; \mathscr{D}(A))$;

(ii) There exist nondecreasing functions $\omega(\cdot)$, $K(\cdot)\colon [0, r] \to [0, \infty)$, with

$$\lim_{s\downarrow 0} \omega(s) < 1, \tag{2.4}$$

such that for all $v(\cdot)$, $\hat{v}(\cdot) \in L^2_{\mathrm{loc}}(-r, \infty; \mathscr{D}(A))$, $s \in [-r, 0]$ and $t_0 \geq 0$,

$$\begin{aligned} &\left\{\int_{t_0}^{t_0+s} \left| A\int_{t_0}^{t} e^{-A(t-\tau)}[G(\tau, v_\tau(\cdot)) - G(\tau, \hat{v}_\tau(\cdot))]\,d\tau\right|_H^2 dt\right\}^{1/2} \\ &\quad\leq \omega(s)\left\{\int_{t_0}^{t_0+s} |A[v(t) - \hat{v}(t)]|_H^2\,dt\right\}^{1/2} \\ &\quad\quad + K(s)\left\{\int_{t_0-r}^{t_0} |A[v(t) - \hat{v}(t)]|_H^2\,dt\right\}^{1/2}, \end{aligned} \tag{2.5}$$

$$\left\{\int_{t_0}^{t_0+s} |G(t, v_t(\cdot))|_H^2\,dt\right\}^{1/2} \leq K(s)\left\{1 + \left[\int_{t_0-r}^{t_0+s} |Av(t)|_H^2\,dt\right]^{1/2}\right\}. \tag{2.6}$$

The above conditions on the map G are very general. In the next sections, we will see that they include many interesting concrete examples. One of the contributions of this paper is the discovery of the above conditions.

Now, let $\varphi(\cdot) \in L^2(-r, 0; \mathscr{D}(A))$ and $x \in F$. The abstract functional differential equation we are going to study is the following:

$$\begin{cases} \dot{u}(t) + Au(t) = G(t, u_t(\cdot)), & \text{a.e. } t \in (0, \infty), \\ u(0) = x, \\ u(t) = \varphi(t), & \text{a.e. } t \in [-r, 0). \end{cases} \tag{2.7}$$

Definition 2.1. A function $u(\cdot) \in L^2_{\mathrm{loc}}(-r, \infty; \mathscr{D}(A))$ is called a *mild solution* of (2.7) if it satisfies the following:

$$u(t) = \begin{cases} \varphi(t), & \text{a.e. } t \in [-r, 0); \\ e^{-At}x + \int_0^t e^{-A(t-\tau)} G(\tau, u_\tau(\cdot))\,d\tau, & t \geq 0. \end{cases} \tag{2.8}$$

We see that under (A1) and (A2), for any $u(\cdot) \in L^2_{\mathrm{loc}}(-r, \infty; \mathscr{D}(A))$, the integral term in (2.8) makes sense. The following lemma plays a very important role in the sequel.

Lemma 2.2. *Let* (A1) *hold. Then, for any* $h(\cdot) \in L^2(t_0, \infty; H)$, $\omega > -\omega_0$,

$$e^{-\omega\cdot}\int_{t_0}^{\cdot} e^{-A(\cdot-\tau)}h(\tau)\,d\tau \in L^2(t_0, \infty; \mathscr{D}(A)) \bigcap W^{1,2}(t_0, \infty; H), \tag{2.9}$$

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and

$$\int_{t_0}^{\infty}|e^{-\omega t}A\int_{t_0}^{t}e^{-A(t-\tau)}h(\tau)\,d\tau|_H^2\,dt\leq(M+1)^2\int_{t_0}^{\infty}|e^{-\omega t}h(t)|_H^2\,dt. \tag{2.10}$$

In particular, for all $s\geq 0$ *and* $h(\cdot)\in L^2(t_0, t_0+s; H)$,

$$\int_{t_0}^{t_0+s}\left|A\int_{t_0}^{t}e^{-A(t-\tau)}h(\tau)\,d\tau\right|_H^2\,dt\leq(M+1)^2\int_{t_0}^{t_0+s}|h(t)|_H^2\,dt. \tag{2.11}$$

PROOF. For any λ with $\operatorname{Re}\lambda>-\omega_0$, we know $-\lambda\in\Sigma(\omega_0, \delta_0)$. Thus, by (2.2), we have

$$\|A(\lambda I+A)^{-1}\|_{\mathscr{L}(H)}\leq 1+\frac{M|\lambda|}{1+|\lambda|}\leq M+1. \tag{2.12}$$

On the other hand, by Theorem 3.2 of [13], we know that the following problem

$$\begin{cases}\dot{u}(t)+Au(t)=h(t), & t\in[t_0, \infty),\\ u(t_0)=0,\end{cases}$$

admits a unique solution $u(\cdot)\in W^{1,2}([t_0, \infty); H)\cap L^2(t_0, \infty; \mathscr{D}(A))$. Moreover, this solution can be represented as

$$u(t)=\int_{t_0}^{t}e^{A(t-\tau)}h(\tau)\,d\tau, \qquad t\in[t_0, \infty).$$

We extend $u(\cdot)$ and $h(\cdot)$ to be zero outside of $[t_0, \infty)$ (and still denoted by themselves). Let ω be any complex number with $\operatorname{Re}\omega>-\omega_0$. By Plancherel's theorem ([21]), we get (see the proof of Theorem 3.2 in [13] and notice (2.12) above)

$$\begin{aligned}\int_{t_0}^{\infty}\left|e^{-\omega t}A\int_{t_0}^{t}e^{A(t-\tau)}h(\tau)\,d\tau\right|_H^2\,dt&=\int_{-\infty}^{\infty}|e^{-\omega t}u(t)|_{\mathscr{D}(A)}^2\,dt\\ &=\int_{-\infty}^{\infty}|A(A+i\lambda+\omega)^{-1}\mathscr{F}_t(e^{-\omega t}h(t))|_H^2\,d\lambda\\ &\leq(M+1)^2\int_{t_0}^{\infty}|e^{-\omega t}h(t)|_H^2\,dt,\end{aligned}$$

where $\mathscr{F}_t(\psi(t))$ stands for the Fourier transformation of the function $\psi(\cdot)$. Then, our conclusion follows.

Now we are ready to prove the main result of this section.

THEOREM 2.3. *Let* (A1) *and* (A2) *hold. Then, for any* $\varphi(\cdot)\in L^2(-r, 0; \mathscr{D}(A))$ *and* $x\in F$, *there exists a unique mild solution* $u(\cdot)$ *of* (2.7).

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Proof. By (2.4), we may find an $\bar{s} \in (0, r]$ such that

$$\omega(\bar{s}) < 1. \tag{2.13}$$

For any $v(\cdot) \in L^2(0, \bar{s}; \mathscr{D}(A))$, we define

$$\tilde{v}(t) = \begin{cases} \varphi(t), & t \in [-r, 0), \\ v(t), & t \in (0, \bar{s}]. \end{cases} \tag{2.14}$$

Then, we see that $\tilde{v}(\cdot) \in L^2(-r, \bar{s}; \mathscr{D}(A))$. Thus, by (A2), we can define

$$(Tv)(t) = e^{-At}x + \int_0^t e^{-A(t-\tau)} G(\tau, \tilde{v}_\tau(\cdot))\, d\tau, \quad t \in [0, \bar{s}]. \tag{2.15}$$

We observe the following (noting Lemma 2.2)

$$\begin{aligned} \left[\int_0^{\bar{s}} |A(Tv)(t)|_H^2\, dt\right]^{1/2} &\le \left[\int_0^{\bar{s}} |Ae^{-At}x|_H^2\, dt\right]^{1/2} \\ &\quad + \left[\int_0^{\bar{s}} \left|A\int_0^t e^{-A(t-\tau)} G(\tau, \tilde{v}_\tau(\cdot))\right|_H^2 dt\right]^{1/2} \\ &\le |x|_F + (M+1)\left[\int_0^{\bar{s}} |G(t, \tilde{v}_t(\cdot))|_H^2\, dt\right]^{1/2} \\ &\le |x|_F + (M+1)K(\bar{s})\left[1 + \left(\int_{-r}^0 |A\varphi(t)|_H^2\, dt\right)^{1/2}\right. \\ &\qquad \left. + \left(\int_0^{\bar{s}} |Av(t)|_H^2\, dt\right)^{1/2}\right] \\ &< \infty. \end{aligned} \tag{2.16}$$

Hence, we see that T maps $L^2(0, \bar{s}; \mathscr{D}(A))$ into itself. Next, we let $v^1(\cdot)$, $v^2(\cdot) \in L^2(0, \bar{s}; \mathscr{D}(A))$ and define $\tilde{v}^1(\cdot)$ and $\tilde{v}^2(\cdot)$ as in (2.14). Then, by (A2), we have (note $\tilde{v}^1(t) = \tilde{v}^2(t)$, a.e. $t \in [-r, 0]$)

$$\int_0^{\bar{s}} |A[(Tv^1)(t) - (Tv^2)(t)]|_H^2\, dt \le \omega(\bar{s})^2 \int_0^{\bar{s}} |A(v^1(t) - v^2(t))|_H^2\, dt,$$

that is,

$$|Tv^1(\cdot) - Tv^2(\cdot)|_{L^2(0, \bar{s}; \mathscr{D}(A))} \le \omega(\bar{s}) |v^1(\cdot) - v^2(\cdot)|_{L^2(0, \bar{s}; \mathscr{D}(A))}. \tag{2.17}$$

Hence, by (2.13), there exists a unique fixed point $u(\cdot) \in L^2(0, \bar{s}; \mathscr{D}(A))$ of the operator T. Then, we define $u(t) = \varphi(t)$ on $[-r, 0)$ and (possibly) change the values of $u(\cdot)$ on a zero-measure set such that (2.8) holds for $t \in [0, \bar{s}]$. Then we see that $u(\cdot)$ is a mild solution of (2.7) on $[-r, \bar{s}]$.

Next, we note that the mapping $t \mapsto G(t, u_t(\cdot))$ is in $L^2(0, \bar{s}; H)$. Thus, by Lemma 2.2 and (2.3), we have

$$u(\cdot) \in L^2(0, \bar{s}; \mathscr{D}(A)) \cap W^{1,2}(0, \bar{s}; H) \hookrightarrow C([0, \bar{s}]; F).$$

In particular, $u(\bar{s}) \in F$. Hence, the above procedure can be repeated. Since the step-length $\bar{s} > 0$ is independent of the initial data, the existence and the uniqueness of the global mild solution of (2.7) follow.

Next, let us collect some basic properties of the mild solution of (2.7) in the following

THEOREM 2.4. *Let* (A1) *and* (A2) *hold.*

(i) *Let* $(x, \varphi(\cdot)) \in F \times L^2(-r, 0; \mathscr{D}(A))$. *Then the mild solution* $u(\cdot)$ *of* (2.7) *satisfies the following*:

$$u(\cdot) \in L^2_{\text{loc}}(-r, \infty; \mathscr{D}(A)) \cap W^{1,2}_{\text{loc}}(0, \infty; H) \hookrightarrow C([0, \infty); F), \tag{2.18}$$

and for any $s \geq 0$, *there exists a constant* $C = C(s)$, *such that*

$$|u(\cdot)|_{L^2(0,s;\mathscr{D}(A)) \cap W^{1,2}(0,s;H)} \leq C(1 + |x|_F + |\varphi(\cdot)|_{L^2(-r,0;\mathscr{D}(A))}), \tag{2.19}$$

where

$$|u(\cdot)|^2_{L^2(0,s;\mathscr{D}(A)) \cap W^{1,2}(0,s;H)} \equiv \int_0^s |\dot{u}(t)|_H^2 \, dt + \int_0^s |Au(t)|_H^2 \, dt.$$

Also, for any $0 \leq \alpha < 1/2$ *and* $0 < \rho < 1/2 - \alpha$,

$$A^\alpha u(\cdot) \in C^\rho((0, \infty); H). \tag{2.20}$$

Moreover, if $x \in \mathscr{D}(A^{\alpha+\rho}) \cap F$, *then*

$$A^\alpha u(\cdot) \in C^\rho([0, \infty); H). \tag{2.21}$$

(ii) *In addition to* (A1) *and* (A2), *suppose there exists a function*

$$\widehat{K}(\cdot) : [0, r] \to [0, \infty),$$

such that for all $v(\cdot)$, $\hat{v}(\cdot) \in L^2_{\text{loc}}(-r, \infty; \mathscr{D}(A))$, $s \in [0, r]$ *and* $t_0 \geq 0$,

$$\int_{t_0}^{t_0+s} |G(t, v_t(\cdot)) - G(t, \hat{v}_t(\cdot))|_H^2 \, dt \leq \widehat{K}(s)^2 \int_{t_0-r}^{t_0+s} |A(v(t) - \hat{v}(t))|_H^2 \, dt. \tag{2.22}$$

Let $u(\cdot)$ *and* $\hat{u}(\cdot)$ *be the mild solutions of* (2.7) *corresponding to the initial data* $(x, \varphi(\cdot))$, $(\hat{x}, \hat{\varphi}(\cdot)) \in F \times L^2(-r, 0; \mathscr{D}(A))$, *respectively. Then, for any* $s \geq 0$, *there exists a constant* $C = C(s) > 0$, *such that*

$$\begin{aligned} &|u(\cdot) - \hat{u}(\cdot)|_{L^2(0,s;\mathscr{D}(A)) \cap W^{1,2}(0,s;H)} \\ &\qquad \leq C(|x - \hat{x}|_F + |\varphi(\cdot) - \hat{\varphi}(\cdot)|_{L^2(-r,0;\mathscr{D}(A))}). \end{aligned} \tag{2.23}$$

PROOF. (i) First of all, from (2.3) and the proof of Theorem 2.3, we see that (2.18) holds. Now, let $\bar{s} \in (0, r]$ be such that (2.13) holds. Then, let $v^0(\cdot) = 0$ and

$$v^n(\cdot) = (Tv^{n-1})(\cdot), \quad n \geq 1, \tag{2.24}$$

where T is defined as in (2.15). From (2.13) and (2.17), we see that

$$\lim_{n\to\infty} |v^n(\cdot) - u(\cdot)|_{L^2(0,\bar{s};\mathscr{D}(A))} = 0. \tag{2.25}$$

On the other hand, we have

$$\begin{aligned}
|v^n(\cdot)|_{L^2(0,\bar{s};\mathscr{D}(A))} &\leq \sum_{i=0}^{n-1} \omega(\bar{s})^i |v^1(\cdot)|_{L^2(0,\bar{s};\mathscr{D}(A))} \\
&\leq \frac{1}{1-\omega(\bar{s})}\left[|x|_F + (M+1)\left(\int_0^{\bar{s}} |G(t, \tilde{v}_t^0(\cdot))|_H^2\,dt\right)^{1/2}\right] \\
&\leq \frac{1}{1-\omega(\bar{s})}\left\{|x|_F + (M+1)K(\bar{s})\left[+\left(\int_{-r}^0 |A\varphi(t)|_H^2\,dt\right)^{1/2}\right]\right\}.
\end{aligned}$$

Thus,

$$\begin{aligned}
\left(\int_0^{\bar{s}} |Au(t)|_H^2\,dt\right)^{1/2} \leq \frac{1}{1-\omega(\bar{s})}\Bigg\{|x|_F &+ (M+1)K(\bar{s}) \\
&\times\left[1+\left(\int_{-r}^0 |A\varphi(t)|_H^2\,dt\right)^{1/2}\right]\Bigg\}.
\end{aligned} \tag{2.26}$$

Then, by noting (2.6) and the fact that $u(\cdot)$ satisfies the equation (2.7) in H, we can find a constant $a = a(\bar{s})$, such that

$$\begin{aligned}
&\left(\int_0^{\bar{s}} |\dot{u}(t)|_H^2\,dt\right)^{1/2} + \left(\int_0^{\bar{s}} |Au(t)|_H^2\,dt\right)^{1/2} \\
&\qquad \leq a\left[1 + |x|_F + \left(\int_{-r}^0 |A\varphi(t)|_H^2\,dt\right)^{1/2}\right].
\end{aligned} \tag{2.27}$$

By applying the same argument, we have (note (2.3))

$$\begin{aligned}\left(\int_{\bar s}^{2\bar s}|\dot u(t)|_H^2\,dt\right)^{1/2}&+\left(\int_{\bar s}^{2\bar s}|Au(t)|_H^2\,dt\right)^{1/2}\\
&\le a\left[1+|u(\bar s)|_F+\left(\int_{\bar s-r}^{\bar s}|Au(t)|_H^2\,dt\right)^{1/2}\right]\\
&\le a\left\{1+c\left[\left(\int_0^{\bar s}|\dot u(t)|_H^2\,dt\right)^{1/2}+\left(\int_0^{\bar s}|Au(t)|_H^2\,dt\right)^{1/2}\right]\right.\\
&\qquad\left.+\left(\int_{-r}^{0}|A\varphi(t)|_H^2\,dt\right)^{1/2}+\left(\int_0^{\bar s}|Au(t)|_H^2\,dt\right)^{1/2}\right\}.\end{aligned}$$

Thus, by induction, we obtain (2.19). Next, from the above, we see that the map

$$t\mapsto g(t)\equiv G(t,u_t(\cdot)) \tag{2.28}$$

is in $L^2_{\mathrm{loc}}(0,\infty;H)$. Thus, for any $0\le\alpha<1/2$, $0<\rho<1/2-\alpha$ and $t,h>0$, we have (note $2(\alpha+\rho)<1$)

$$\begin{aligned}|A^\alpha u(t+h)-A^\alpha u(t)|_H&\le|(e^{-Ah}-I)A^\alpha e^{-At}x|_H\\
&\quad+\left|(e^{-At}-I)\int_0^t A^\alpha e^{-A(t-r)}g(\tau)\,d\tau\right|_H+\int_t^{t+h}|A^\alpha e^{-A(t+h-r)}g(\tau)|_H\,d\tau\\
&\le C_\rho h^\rho|A^{\alpha+\rho}e^{-At}x|_H+C_\rho h^\rho\int_0^t\frac{M_{\alpha+\rho}|g(\tau)|_H}{(t-\tau)^{\alpha+\rho}}\,d\tau+\int_t^{t+h}\frac{M_\alpha|g(\tau)|_H}{(t+h-\tau)^\alpha}\,d\tau\\
&\le C_\rho h^\rho|A^{\alpha+\rho}e^{-At}x|_H+C_\rho M_{\alpha+\rho}h^\rho\left[\frac{t^{1-2(\alpha+\rho)}}{1-2(\alpha+\rho)}\right]^{1/2}\left(\int_0^t|g(\tau)|_H^2\,d\tau\right)^{1/2}\\
&\quad+\frac{M_\alpha}{(1-2\alpha)^{1/2}}h^{1-2\alpha}\left(\int_t^{t+h}|g(\tau)|_H^2\,d\tau\right)^{1/2}.\end{aligned}$$

Thus, (2.20) follows. Here, we have used the following estimates [16]:

(2.29)
$$\|A^\beta e^{-At}\|_{\mathscr{L}(H)}\le M_\beta e^{-\delta t}t^{-\beta},\qquad\forall t>0,\ \beta\ge0,\quad\text{for some }\delta>0,$$

$$|(e^{-At}-I)x|_H\le C_\beta|A^\beta x|_H,\qquad\forall x\in\mathscr{D}(A^\beta),\ \beta\ge0. \tag{2.30}$$

Now, if $x\in\mathscr{D}(A^{\alpha+\rho})\cap F$, we can take $t=0$ in the above and (2.21) follows.

(ii) Let $u(\cdot)$ and $\hat u(\cdot)$ be the mild solutions of (2.7) corresponding to initial conditions $(x,\varphi(\cdot))$, $(\hat x,\hat\varphi(\cdot))\in F\times L^2(-r,0;\mathscr{D}(A))$, respectively. Let $\bar{s} \in (0, r]$ satisfy (2.13). By (2.5), we have

$$\begin{aligned}\left\{\int_0^{\bar{s}} |A[u(\cdot) - \hat{u}(\cdot)]|_H^2 \, dt\right\}^{1/2} &\le \left[\int_0^{\bar{s}} |Ae^{-At}(x - \hat{x})|_H^2 \, dt\right]^{1/2} \\ &\quad + \left\{\int_0^{\bar{s}} \left|A \int_0^t e^{-A(t-r)}[G(\tau, u_\tau(\cdot)) - G(\tau, \hat{u}_\tau(\cdot))]\, d\tau\right|_H^2 dt\right\}^{1/2} \\ &\le |x - \hat{x}|_F + K(\bar{s}) \left[\int_{-r}^0 |A(\varphi(t) - \hat{\varphi}(t))|_H^2 \, dt\right]^{1/2} \\ &\quad + \omega(\bar{s}) \left[\int_0^{\bar{s}} |A(u(t) - \hat{u}(t))|_H^2 \, dt\right]^{1/2}.\end{aligned}$$

Thus, by (2.13), we obtain (similar to (2.26))

$$\begin{aligned}\left[\int_0^{\bar{s}} |A(u(t) - \hat{u}(t))|_H^2 \, dt\right]^{1/2} &\le \frac{1}{1 - \omega(\bar{s})}\{|x - \hat{x}|_F \\ &\quad + K(\bar{s}) \left[\int_{-r}^0 |A(\varphi(t) - \hat{\varphi}(t)|_H^2 \, dt\right]^{1/2}.\end{aligned} \tag{2.32}$$

Then, applying similar arguments used in proving (2.19), we can prove (2.23) (here, we need (2.22)).

We should note that none of conditions (2.5) or (2.22) implies the other. Next, similar to [16], let us introduce the following notion.

DEFINITION 2.5. A function $u(\cdot) \in L^2_{\mathrm{loc}}(-r, \infty; \mathscr{D}(A))$ is called a *strong solution* of (2.7), if $u(\cdot) \in W^{1,2}_{\mathrm{loc}}(0, \infty; H)$ and (2.7) holds.

PROPOSITION 2.6. *Let* (A1) *and* (A2) *hold. Then, for any* $x \in F$ *and* $\varphi(\cdot) \in L^2_{\mathrm{loc}}(-r, \infty; \mathscr{D}(A))$, *there exists a unique strong solution of* (2.7).

PROOF. From Theorem 2.4, we see that the mild solution $u(\cdot)$ of (2.7) uniquely exists and (2.18) holds. Thus by Theorem 2.9 of [16], we know that $u(\cdot)$ is a strong solution of (2.7). The uniqueness also follows easily.

Next, we study some asymptotic behavior of the mild solutions of (2.7) under some further assumptions.

THEOREM 2.7. *Let* (A1) *and* (A2) *hold. Let there exist a nondecreasing function* $K_0(\cdot): [0, \infty) \to [0, \infty)$, *such that for any* $v(\cdot) \in L^2_{\mathrm{loc}}(-r, \infty; \mathscr{D}(A))$, *there exists* $s_0 \ge 0$ *and* $K_1 \ge 0$, *whenever* $s \ge s_0$,

$$\left[\int_0^s |G(t, v_t(\cdot))|_H^2 \, dt\right]^{1/2} \le K_0(s) \left[\int_0^s |Av(t)|_H^2 \, dt\right]^{1/} + K_1, \tag{2.33}$$

and

$$K_0 \equiv \varliminf_{s\to\infty} K_0(s) < \frac{1}{M+1}, \tag{2.34}$$

where M *is determined by* (2.2). *Then, the mild solution* $u(\cdot)$ *of* (2.7) *satisfies*

$$u(\cdot) \in L^2(0,\infty;\mathscr{D}(A)) \cap W^{1,2}(0,\infty;H), \tag{2.35}$$

and for any $\alpha \in [0, 1/2)$,

$$\lim_{t\to\infty} |A^\alpha u(t)|_H = 0. \tag{2.36}$$

PROOF. From Theorem 2.3, we obtain the unique mild solution $u(\cdot)$ of (2.7) for any $x \in F$ and $\varphi(\cdot) \in L^2(-r, 0; \mathscr{D}(A))$. Then, by our assumption, there exist s_0, $K_1 \ge 0$, such that for all $s \ge s_0$, (2.33) holds for $u(\cdot)$. Thus, for $s \ge s_0$, one has (note Lemma 2.2)

(2.37)
$$\begin{aligned}\left[\int_0^s |Au(t)|_H^2\,dt\right]^{1/2} &\le |x|_F + (M+1)\left[\int_0^s |G(t, u_t(\cdot))|_H^2\,dt\right]^{1/2}\\ &\le |x|_F + (M+1)K_0(s)\left[\int_0^s |Av(t)|_H^2\,dt\right]^{1/2} + K_1.\end{aligned}$$

Hence, by (2.34), we have

$$\left[\int_0^\infty |Au(t)|_H^2\,dt\right]^{1/2} \le \frac{1}{1-K_0(M+1)}[|x|_F + K_1] < \infty. \tag{2.38}$$

Then, $u(\cdot) \in L^2(0,\infty;H)$. Thus by (2.33), we have

$$\left[\int_0^\infty |G(t, u_t(\cdot))|_H^2\,dt\right]^{1/2} \le K_0\left[\int_0^\infty |Au(t)|_H^2\,dt\right]^{1/2} + K_1 < \infty. \tag{2.39}$$

Therefore, by the equation (2.7), we see (2.35) holds. Now, we again let $g(t) = G(t, u_t(\cdot))$. Then, $g(\cdot) \in L^2(0,\infty;H)$. Thus, for any $\alpha \in [0, 1/2)$,

$$\begin{aligned}|A^\alpha u(t)|_H &\le |A^\alpha e^{-At}x|_H + \int_0^t |A^\alpha e^{-A(t-\tau)}g(\tau)|_H\,d\tau\\ &\le M_\alpha e^{-\delta t}t^{-\alpha}|x|_H + M_\alpha \int_0^t e^{-\delta(t-\tau)}(t-\tau)^{-\alpha}|g(\tau)|_H\,d\tau\\ &\le M_\alpha e^{-\delta t}t^{-\alpha}|x|_H + M_\alpha\Gamma(1-2\alpha)^{1/2}\left(\int_0^t e^{-\delta(t-\tau)}|g(\tau)|_H^2\,d\tau\right)^{1/2}\\ &\to 0 \qquad (\text{as } t\to\infty).\end{aligned}$$

Thus, (2.36) follows.

3. A class of delay differential equations

In this section, we apply the results of the previous section to a class of delay differential equations. Again, we let H be a Hilbert space, $r > 0$ be a given constant and $A: \mathscr{D}(A) \subset H \to H$ satisfy (A1) stated in Section 2. We consider the following equation:

(3.1)
$$\begin{cases} \dot{u}(t) + Au(t) = \int_{-r}^{0} g(t, \theta, u(t), u(t+\theta))\mu(d\theta), & \text{a.e. } t \in (0, \infty), \\ u(0) = x, \\ u(t) = \varphi(t), \quad \text{a.e. } t \in [-r, 0). \end{cases}$$

We let $\mathscr{L}$ and $\mathscr{B}$ be the Lebesgue σ-field and the Borel σ-field, respectively, on possibly different subintervals of $(-\infty, +\infty)$ (which can be identified from the context). Now, let us make the following further assumption:

(A3) Let $\mu(\cdot)$ be a finite regular measure defined on $([-r, 0], \mathscr{B})$, and $g: [0, \infty) \times [-r, 0] \times \mathscr{D}(A) \times \mathscr{D}(A) \to H$ be a given mapping satisfying the following:

(i) For any $x, y \in \mathscr{D}(A)$, the map $g(\cdot, \cdot, x, y)$ is strongly $\mathscr{L} \times \mathscr{B}$-measurable;

(ii) There exist nonnegative constants L_i, $i = 0, 1, \dots, 4$ with

$$\overline{\lim_{s \downarrow 0}} (M+1)\mu([-r, 0])^{1/2}[L_1 \mu([-r, 0])^{1/2} + L_2 \mu([-s, 0])^{1/2}] < 1, \tag{3.2}$$

such that for all $(t, \theta) \in [0, \infty) \times [-r, 0]$ and $x, \hat{x}, y, \hat{y} \in \mathscr{D}(A)$, we have

$$\begin{aligned} |g(t, \theta, x, y) - g(t, \theta, \hat{x}, \hat{y})|_H \le L_1|A(x - \hat{x})|_H + L_2|A(y - \hat{y})|_H \\ + L_3|x - \hat{x}|_H + L_4|y - \hat{y}|_H, \end{aligned} \tag{3.3}$$

$$|g(t, \theta, 0, 0)|_H \le L_0. \tag{3.4}$$

REMARK 3.1. If in (A3), one has $L_1 = 0$ (that is, the map $g(t, \theta, x, y) \equiv g(t, \theta, y)$) and $\mu(\cdot)$ is non-atomic at 0 (that is, $\lim_{s\downarrow 0} \mu([-s, 0]) = 0$, or equivalently, $\mu(\{0\}) = 0$), then, (3.2) is automatically true. Also, if instead of (3.3), we have

$$\begin{aligned} |g(t, \theta, x, y) - g(t, \theta, \hat{x}, \hat{y})|_H \le L_1|A^\beta(x - \hat{x})|_H + L_2|A^\beta(y - \hat{y})|_H \\ + L_3|x - \hat{x}|_H + L_4|y - \hat{y}|_H, \end{aligned} \tag{3.5}$$

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for some $\beta \in [0, 1)$, i.e., the delays do not appear in the highest order spatial derivative terms, then, by [16] (see [20] also), one has that for any $\varepsilon > 0$,

$$\begin{aligned} |A^\beta x|_H &\le C|x|_H^{1-\beta}|Ax|_H^\beta \\ &\le \varepsilon|Ax|_H + C_\varepsilon|x|_H, \quad \forall x \in \mathscr{D}(A). \end{aligned} \tag{3.6}$$

Thus, we also see that for suitably chosen constants L_i's, one has (3.2).

It is not hard to see that condition (3.2) essentially means that the parabolicity of the equation is not ruined by the (delay) perturbation terms. Since μ can be a very general regular measure, (3.1) includes equations with countably many discrete delays and finitely many distributed delays in the "highest order spatial derivative" and "lower order spatial derivative" terms. At this stage, we note that equation (3.1) contains a very wide class of delay equations. We will make some comments on the constant L_i a little later.

In the following, we will assume $L_3 = L_4 = 0$ for notational simplicity. It is not hard to see that these terms cause no difficulty in the proofs. Also, the statement of the results is not changed after setting $L_3 = L_4 = 0$. The following lemma is basic for the integrals in (3.1) to have meaning.

LEMMA 3.2. *Let* (A3) *hold. Let* $v(\cdot) \in L^2_{\mathrm{loc}}(-r, \infty; \mathscr{D}(A))$ *with* $Av(\cdot)$ *Borel measurable. Then, the integral*

$$\int_{-r}^0 g(\cdot, \theta, v(\cdot), v(\cdot + \theta))\mu(d\theta) \in L^2_{\mathrm{loc}}(0, \infty; H), \tag{3.7}$$

and for any $0 \le t_0 \le t_1 < \infty$,

$$\begin{aligned} &\left[\int_{t_0}^{t_1} \left|\int_{-r}^0 g(t, \theta, v(t), v(t+\theta))\mu(d\theta)\right|_H^2 dt\right]^{1/2} \\ &\le \mu([-r, 0])\left\{L_0(t_1 - t_0)^{1/2} + L_1\left(\int_{t_0}^{t_1} |Av(t)|_H^2\, dt\right)^{1/2}\right. \\ &\qquad \left. + L_2\left(\int_{t_0-r}^{t_1} |Av(t)|_H^2\, dt\right)^{1/2}\right\}. \end{aligned} \tag{3.8}$$

PROOF. Since $Av(\cdot)$ is Borel measurable, we can find a sequence of Borel measurable simple functions $w_k(\cdot) = \sum_i w_k^i \chi_{E_k^i}(\cdot)$ with $w_k^i \in H$ and $E_k^i \in \mathscr{B}$ (the Borel σ-field on $[-r, \infty)$), such that

$$\lim_{k\to\infty} |w_k(t) - Av(t)|_H = 0, \qquad \forall t \in [-r, \infty). \tag{3.9}$$

We define $v_k(\cdot) = A^{-1}w_k(\cdot)$. Then, we see that the map $(t, \theta) \mapsto g(t, \theta, v_k(t), v_k(t+\theta))$ is $\mathscr{L} \times \mathscr{B}$-measurable. Thus, from (3.3), we know that $(t, \theta) \mapsto$

$g(t, \theta, v(t), v(t+\theta))$ is $\mathscr{L} \times \mathscr{B}$-measurable. To complete the proof, it suffices to prove (3.8). By (3.3) and (3.4), one can obtain

$$\begin{aligned}
&\left[\int_{t_0}^{t_1}\left|\int_{-r}^{0} g(t, \theta, v(t), v(t+\theta))\mu(d\theta)\right|_H^2 dt\right]^{1/2} \\
&\quad \leq \left\{\int_{t_0}^{t_1}\left[\int_{-r}^{0}(L_0 + L_1|Av(t)|_H + L_2|Av(t+\theta)|_H)\mu(d\theta)\right]_H^2 dt\right\}^{1/2} \\
&\quad \leq L_0\mu([-r, 0])(t_1 - t_0)^{1/2} + L_1\mu([-r, 0])\left(\int_{t_0}^{t_1}|Av(t)|_H^2\, dt\right)^{1/2} \\
&\qquad + L_2\mu([-r, 0])^{1/2}\left(\int_{t_0}^{t_1}\int_{-r}^{0}|Av(t+\theta)|_H^2\mu(d\theta)\, dt\right)^{1/2} \\
&\quad \leq \mu([-r, 0])\left\{L_0(t_1 - t_0)^{1/2} + L_1\left(\int_{t_0}^{t_1}|Av(t)|_H^2\, dt\right)^{1/2}\right. \\
&\qquad\qquad \left. + L_2\left(\int_{t_0 - r}^{t_1}|Av(t)|_H^2\, dt\right)^{1/2}\right\}.
\end{aligned}$$

Hence, (3.8) follows.

Next, let us make some observations. We note that for any $v(\cdot) \in L^2_{\text{loc}}(-r, \infty; \mathscr{D}(A))$, $Au(\cdot)$ is Lebesgue measurable, in general. However, we can find a Borel measurable function $\bar{w}(\cdot) \in L^2_{\text{loc}}(-r, \infty; H)$, such that

$$Av(t) = \bar{w}(t), \quad \text{a.e. } t \in [-r, \infty).$$

(Here, a.e. is with respect to the Lebesgue measure.) Let

$$\bar{v}(t) = A^{-1}\bar{w}(t), \quad t \in [-r, \infty).$$

Then, $A\bar{v}(\cdot)$ is Borel measurable and

$$v(t) = \bar{v}(t), \quad \text{a.e. } t \in [-r, \infty).$$

We call $\bar{v}(\cdot)$ a Borel correction of $v(\cdot)$. Now, if $\hat{v}(\cdot)$ is another Borel measurable correction of $v(\cdot)$, then

$$\hat{v}(t) = \bar{v}(t), \quad \text{a.e. } t \in [-r, \infty).$$

Hence, for all given $\theta \in [-r, 0]$,

$$g(t, \theta, \bar{v}(t), \bar{v}(t+\theta)) = g(t, \theta, \hat{v}(t), \hat{v}(t+\theta)), \quad \text{a.e. } t \in [-r, \infty).$$

Thus, we have

$$
\begin{aligned}
(3.10)\qquad &\int_0^t e^{-A(t-\tau)} \int_{-r}^0 g(\tau, \theta, \hat{v}(\tau), \hat{v}(\tau+\theta))\mu(d\theta)\, d\tau \\
&= \int_{-r}^0 \int_0^t e^{-A(t-\tau)} g(\tau, \theta, \hat{v}(\tau), \hat{v}(\tau+\theta))\, d\tau \mu(d\theta) \\
&= \int_{-r}^0 \int_0^t e^{-A(t-\tau)} g(\tau, \theta, \bar{v}(\tau), \bar{v}(\tau+\theta))\, d\tau \mu(d\theta) \\
&= \int_0^t e^{-A(t-\tau)} \int_{-r}^0 g(\tau, \theta, \bar{v}(\tau), \bar{v}(\tau+\theta))\mu(d\theta)\, d\tau.
\end{aligned}
$$

In the sequel, only the integrals of the above type will be involved. Hence, hereafter, without loss of generality, we may assume that for any $v(\cdot) \in L^2_{\mathrm{loc}}(-r, \infty; \mathscr{D}(A))$, $Av(\cdot)$ is Borel measurable. Then, the following definition makes sense.

Definition 3.3. Let $x \in H$ and $\varphi(\cdot) \in L^2(-r, 0; \mathscr{D}(A))$. A function $u(\cdot) \in L^2_{\mathrm{loc}}(-r, \infty; \mathscr{D}(A))$ is called a mild solution of (3.1), if it satisfies

$$
(3.11)\qquad u(t) = \begin{cases} \varphi(t), & \text{a.e. } t \in [-r, 0); \\ e^{-At}x + \displaystyle\int_0^t e^{-A(t-\tau)} \left[\int_{-r}^0 g(\tau, \theta, u(\tau+\theta))\mu(d\theta)\right] d\tau, & t \geq 0. \end{cases}
$$

A function $u(\cdot) \in L^2_{\mathrm{loc}}(-r, \infty; \mathscr{D}(A))$ is called a *strong solution* of (3.1) if $u(\cdot) \in W^{1,2}_{\mathrm{loc}}(0, \infty; H)$ and (3.1) holds.

Now, let us prove the following

Theorem 3.4. *Let* (A1) *and* (A3) *hold. Then, for any $x \in F$ and $\varphi(\cdot) \in L^2(-r, 0; \mathscr{D}(A))$, there exists a unique mild solution $u(\cdot)$ of* (3.1). *Moreover, this solution is also a strong solution.*

Proof. For any $\varphi(\cdot) \in C([-r, 0]; \mathscr{D}(A))$, we define

$$
(3.12)\qquad G(t, \varphi(\cdot)) = \int_{-r}^0 g(t, \theta, \varphi(0), \varphi(\theta))\mu(d\theta).
$$

We will show that G satisfies (A2). By Lemma 3.2 and our convention, we see that (i) of (A2) hold. Now, let $v(\cdot)$, $\hat{v}(\cdot) \in L^2_{\mathrm{loc}}(-r, \infty; \mathscr{D}(A))$,

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$s \in [0, r]$, $t_0 \geq 0$. Noting Lemma 2.2 (with $\omega = 0$), we have

$$
\begin{aligned}
I &\equiv \left\{ \int_{t_0}^{t_0+s} \left| A \int_{t_0}^{t} e^{-A(t-\tau)} [G(\tau, v_\tau(\cdot)) - G(\tau, \hat{v}_\tau(\cdot))] \, d\tau \right|_H^2 dt \right\}^{1/2} \\
&\leq (M+1) \left\{ \int_{t_0}^{t_0+s} \left[\int_{-r}^{0} (L_1 |A(v(t) - \hat{v}(t))|_H \right. \right. \\
&\qquad \left. \left. + L_2 |A(v(t+\theta) - \hat{v}(t+\theta))|_H) \mu(d\theta) \right]^2 dt \right\}^{1/2} \\
&\leq (M+1) \left\{ L_1 \mu([-r, 0]) \left[\int_{t_0}^{t_0+s} |A(v(t) - \hat{v}(t))|_H^2 \, dt \right]^{1/2} \right. \\
&\qquad + L_2 \mu([-r, 0])^{1/2} \left[\int_{t_0}^{t_0+s} \int_{-r}^{0} |A(v(t+\theta) \right. \\
&\qquad \left. \left. - \hat{v}(t+\theta))|_H^2 \mu(d\theta) \, dt \right]^{1/2} \right\}.
\end{aligned} \tag{3.13}
$$

Now, let us set

$$w(t) = |A(v(t) - \hat{v}(t))|_H^2,$$

and estimate the following (noting $s \in [0, r]$):

$$
\begin{aligned}
&\int_{t_0}^{t_0+s} \int_{-r}^{0} w(t+\theta) \mu(d\theta) \, dt \\
&\quad \leq \int_{t_0}^{t_0+r} \int_{-r}^{t_0-t} w(t+\theta) \mu(d\theta) \, dt + \int_{t_0}^{t_0+s} \int_{t_0-t}^{0} w(t+\theta) \mu(d\theta) \, dt \\
&\quad = \int_{-r}^{0} \int_{t_0}^{t_0-\theta} w(t+\theta) \, dt \mu(d\theta) + \int_{-s}^{0} \int_{t_0-\theta}^{t_0+s} w(t+\theta) \, dt \mu(d\theta) \\
&\quad \leq \mu([-r, 0]) \int_{t_0-r}^{t_0} w(t) \, dt + \mu([-s, 0]) \int_{t_0}^{t_0+s} w(t) \, dt.
\end{aligned} \tag{3.14}
$$

Hence, by (3.13), we have

$$
\begin{aligned}
I &\leq \omega(s) \left[\int_{t_0}^{t_0+s} |A(v(t) - \hat{v}(t))|_H^2 \, dt \right]^{1/2} \\
&\quad + (M+1) L_2 \mu([-r, 0]) \left[\int_{t_0-r}^{t_0} |A(v(t) - \hat{v}(t))|_H^2 \, dt \right]^{1/2},
\end{aligned} \tag{3.15}
$$

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with

$$\omega(s)=(M+1)\mu([-r,0])^{1/2}[L_1\mu([-r,0])^{1/2}+L_2\mu([-s,0])^{1/2}]. \tag{3.16}$$

Then, by (3.2), we see that (2.4) holds. From (3.8) we have

$$\begin{aligned}&\left[\int_{t_0}^{t_0+s}|G(t,v_t(\cdot))|_H^2\,dt\right]^{1/2}\\ &\qquad\le\mu([-r,0])(L_0\sqrt{s}+L_1+L_2)\left[1+\left(\int_{t_0-r}^{t_0+s}|Av(t)|_H^2\,dt\right)^{1/2}\right].\end{aligned} \tag{3.17}$$

Hence, we may choose $K(\cdot)$ suitably, such that (2.5) and (2.6) hold. Then, Theorem 2.3 and Proposition 2.6 apply.

If L_3 and L_4 are not zero, then, instead of $\omega(s)$ in (3.15), we have $\omega(s)+\hat{\omega}(s)$, with the function $\hat{\omega}(\cdot)\colon[-r,0]\to[0,\infty)$ having the property that

$$\lim_{s\to 0}\hat{\omega}(s)=0. \tag{3.18}$$

Thus, the appearance of L_3 and L_4 does not affect the result of Theorem 3.4. It is not hard for us to realize that the constants L_i actually can be replaced by some functions of t and θ. More precisely, (ii) of (A3) can be replaced by the following:

(ii$'$) There exist $\mathscr{L}\times\mathscr{B}$-measurable functions $a_i\colon[0,\infty)\to[0,\infty)$ and constants $\bar{a}_i\ge 0$ $(i=0,1,2,3,4)$, with the following properties:

$$\left[\int_{t_0}^{t_0+r}\left(\int_{-r}^{0}a_0(t,\theta)\mu(d\theta)\right)^2dt\right]^{1/2}\le\bar{a}_0,\qquad\forall t_0\ge 0, \tag{3.19}$$

$$\int_{-r}^{0}a_1(t,\theta)\mu(d\theta)\le\bar{a}_1,\qquad\int_{-r}^{0}a_3(t,\theta)\mu(d\theta)\le\bar{a}_3,\quad\text{a.e. } t\in[0,\infty), \tag{3.20}$$

$$\left(\int_{-r}^{0}a_2(t,\theta)^2\mu(d\theta)\right)^{1/2}\le\bar{a}_2,\qquad\left(\int_{-r}^{0}a_4(t,\theta)^2\mu(d\theta)\right)^{1/2}\le\bar{a}_4,\quad\text{a.e. } t\in[0,\infty), \tag{3.21}$$

$$\varlimsup_{s\downarrow 0}(M+1)[\bar{a}_1+\bar{a}_2\mu([-s,0])^{1/2}]<1, \tag{3.22}$$

such that for all $(t,\theta)\in[0,\infty)\times[-r,0]$ and $x,\hat{x},y,\hat{y}\in\mathscr{D}(A)$, we have

$$\begin{aligned}|g(t,\theta,x,y)-g(t,\theta,\hat{x},\hat{y})|_H&\le a_1(t,\theta)|A(x-\hat{x})|_H\\ &\quad+a_2(t,\theta)|A(y-\hat{y})|_H\\ &\quad+a_3(t,\theta)|x-\hat{x}|_H+a_4(t,\theta)|y-\hat{y}|_H,\end{aligned} \tag{3.23}$$

$$|g(t,\theta,0,0)|_H \le a_0(t,\theta). \tag{3.24}$$

Due to the above theorem, hereafter we will not distinguish the mild and the strong solutions and simply call them solutions.

As in the previous section, we can also obtain a result about the asymptotic behavior of the solution $u(\cdot)$ of (3.1). We omit the details here.

4. Linear Case

In this section, we consider linear delay equations with delays appear in the "highest order spatial derivative" terms, namely, we are interested in studying the following type of equation:

$$\begin{cases} \dot{u}(t) + Au(t) = L(u_t(\cdot)) + f(t), & \text{a.e. } t > 0, \\ u(0) = x, \\ u(t) = \varphi(t), \quad \text{a.e. } t \in [-r, 0], \end{cases} \tag{4.1}$$

where we assume that A satisfies (A1) and $L\colon C([-r,0];\mathscr{D}(A)) \to H$ is a linear operator satisfying the following:

(L1) There exist a nondecreasing function $\omega(\cdot)\colon [0,r] \to [0,\infty)$, with

$$\lim_{s\downarrow 0} \omega(s) < 1, \tag{4.2}$$

and a constant K, such that for any $v(\cdot) \in L^2_{\mathrm{loc}}(-r,\infty;\mathscr{D}(A))$, the function $t \mapsto L(v_t(\cdot))$ is in $L^2_{\mathrm{loc}}(0,\infty;H)$ and

$$\begin{aligned} &\int_0^s \left| A\int_0^t e^{-A(t-\tau)} L(v_\tau(\cdot))\,d\tau \right|_H^2 dt \\ &\quad \le \omega(s)^2 \int_0^s |Av(t)|_H^2\,dt + K^2 \int_{-r}^0 |Av(t)|_H^2\,dt, \qquad \forall s \in [0,r]. \end{aligned} \tag{4.3}$$

Let us look at an important case in which (L1) holds. Suppose $\{A(\theta) : \theta \in [-r,0]\}$ is a family of closed linear operators satisfying

$$\mathscr{D}(A(\theta)) \supset \mathscr{D}(A), \qquad \forall \theta \in [-r,0], \tag{4.4}$$

and there exist a finite regular measure μ defined on the Borel σ-field $\mathscr{B}[r,0]$ and a function

$$\bar{a}(\cdot) \in L^2_\mu([-r,0]) = \left\{ a(\cdot)\colon [-r,0] \to \mathbb{R} \,\middle|\, \int_{-r}^0 a(\theta)^2\,d\mu(d\theta) < \infty \right\},$$

with the property that

$$\lim_{s\downarrow 0} (M+1) \left(\int_{-r}^0 \bar{a}(\theta)^2 \mu(d\theta) \right)^{1/2} \mu([-s,0])^{1/2} < 1, \tag{4.5}$$

(the constant M is determined by (A1)) such that

$$(4.6)\qquad |A(\theta)x|_H \le \bar{a}(\theta)|Ax|_H, \quad \forall x \in \mathscr{D}(A).$$

Then, for any $\varphi(\cdot) \in C([-r, 0]; \mathscr{D}(A))$, we set

$$(4.7)\qquad L(\varphi(\cdot)) = \int_{-r}^{0} A(\theta)\varphi(\theta)\mu(d\theta).$$

We can check that such an operator L satisfies (L1) for

$$\omega(s) = (M+1)\left(\int_{-r}^{0} \bar{a}(\theta)^2\mu(d\theta)\right)^{1/2} \mu([-s, 0])^{1/2},$$

and

$$K = (M+1)\left(\int_{-r}^{0} \bar{a}(\theta)^2\mu(d\theta)\right)^{1/2} \mu([-r, 0])^{1/2} \ (= \omega(r)).$$

A similar observation to that made in Section 3 shows that (4.6) can be replaced by

$$(4.6')\qquad |A(\theta)x|_H \le \bar{a}(\theta)|Ax|_H + \hat{a}(\theta)|x|_H, \qquad \forall x \in \mathscr{D}(A),$$

with the function $\bar{a}(\cdot)$ being the same as that in (4.6), and some function $\hat{a}(\cdot) \in L^2_\mu([-r, 0])$ (without condition (4.5)).

It is not hard to see that the above case covers many interesting cases such as finitely many or countably many discrete delays and/or finitely many distributed delays appearing at all possible orders (no more than the order of the main operator A) of spatial derivative terms. Thus, this is a very wide class of linear delay differential equations. In particular, (4.1) contains the cases discussed in [3, 10, 22].

By the result of Section 2, we have the following

PROPOSITION 4.1. *Let* (A1) *and* (L1) *hold. Let* $f(\cdot) \in L^2_{\text{loc}}(0, \infty; H)$. *Then, for any* $(x, \varphi(\cdot)) \in F \times L^2(-r, 0; \mathscr{D}(A))$, *there exists a unique solution*

$$u(\cdot) \in L^2_{\text{loc}}(-r, \infty; \mathscr{D}(A)) \cap W^{1,2}_{\text{loc}}(0, \infty; H) \hookrightarrow C([0, \infty); F)$$

of (4.1) *satisfying the following estimate:*

$$(4.8)\qquad \begin{aligned} &\int_0^s |\dot{u}(t)|_H^2\, dt + \int_0^s |Au(t)|_H^2\, dt \\ &\quad \le C(s)\left\{|x|_F^2 + \int_{-r}^0 |A\varphi(t)|_H^2\, dt + \int_0^s |f(t)|_H^2\, dt\right\}, \qquad \forall s \ge 0, \end{aligned}$$

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with some nondecreasing function $C(s)$ *independent of the data* $(x, \varphi(\cdot), f(\cdot))$.

We note that since the problem is linear, an assumption similar to (2.22) does not have to be made, and the condition (4.3) is enough to obtain the estimate (4.8). The first purpose of this section is to derive the variation of constant formula for system (4.1). To this end, let us make the following assumption

(L2) The operator L satisfies

$$L: C([-r, 0]; \mathscr{D}(A^2)) \to \mathscr{D}(A), \tag{4.9}$$

and the operator

$$\widetilde{L} \equiv ALA^{-1}: C([-r, 0]; \mathscr{D}(A)) \to H$$

satisfies (L1) with some $\tilde{\omega}(\cdot)$ and $\widetilde{K}$. Moreover, for all $v(\cdot) \in L^2_{\mathrm{loc}}(-r, \infty; \mathscr{D}(A))$,

$$\int_0^s |L(v_t(\cdot))|_H^2 \, dt \le K^2 \int_{-r}^s |Av(t)|_H^2 \, dt, \qquad \forall s \in [0, r], \tag{4.10}$$

$$\int_0^s L(v_\tau(\cdot)) \, d\tau = L\left(\int_0^s v_\tau(\cdot) \, d\tau\right), \qquad \forall s \in [0, \infty). \tag{4.11}$$

Again, for the operator L defined by (4.7), we see that (4.10) and (4.11) are automatically true and in order the other conditions of (L2) hold, it suffices to assume that

$$|AA(\theta)A^{-1}x|_H \le \tilde{a}(\theta)|Ax|_H, \qquad \forall x \in \mathscr{D}(A), \ \theta \in [-r, 0], \tag{4.12}$$

that is,

$$\|AA(\theta)A^{-2}\|_{\mathscr{L}(H)} \le \tilde{a}(\theta), \qquad \forall \theta \in [-r, 0], \tag{4.13}$$

for some $\tilde{a}(\cdot)$ satisfying (4.5).

We notice that in (L1), (4.6) means

$$\|A(\theta)A^{-1}\|_{\mathscr{L}(H)} \le \bar{a}(\theta), \qquad \forall \theta \in [-r, 0]. \tag{4.14}$$

It is not hard to understand that (4.13) implies (in the case of delay partial differential equations) that the coefficients of the spatial differential operator $A(\theta)$ have certain regularity in the spatial variables.

With the aid of (L2), we have a strengthened version of Proposition 4.1.

PROPOSITION 4.2. *Let* (A1) *and* (L2) *hold. Let* $f(\cdot) \in L^2_{\mathrm{loc}}(0, \infty; \mathscr{D}(A))$. *Then, for any* $(x, \varphi(\cdot)) \in A^{-1}F \times L^2(-r, 0; \mathscr{D}(A^2))$, *there exists a unique solution* $u(\cdot)$ *of* (4.1) *satisfying*

$$u(\cdot) \in L^2_{\mathrm{loc}}(-r, \infty; \mathscr{D}(A^2)) \cap W^{1,2}_{\mathrm{loc}}(0, \infty; \mathscr{D}(A)) \hookrightarrow C([0, \infty); A^{-1}F),$$

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and the estimate (4.8) *also holds* (*with a possibly different function* $C(\cdot)$).

Proof. Consider the following problem

$$\text{(4.15)} \qquad \begin{cases} \dot{v}(t) + Av(t) = \widetilde{L}(v(t(\cdot)) + Af(t), & \text{a.e. } t > 0, \\ v(0) = Ax, \\ v(t) = A\varphi(t), & \text{a.e. } t \in [-r, 0]. \end{cases}$$

By Proposition 4.1, there exists a unique solution $v(\cdot)$ of (4.15) with

$$v(\cdot) \in L^2_{\text{loc}}(-r, \infty; \mathscr{D}(A)) \cap W^{1,2}_{\text{loc}}(0, \infty; H) \hookrightarrow C([0, \infty); F).$$

Thus, by setting

$$u(t) = A^{-1}v(t), \qquad \forall t \in [-r, \infty),$$

and the uniqueness of the solution of (4.1), which is ensured by (4.10), we obtain our results. $\square$

At this moment, let us make the following simple observation: Condition (4.11) will play an important role in deriving the variation of constant formula and it is not needed in Proposition 4.2.

Now, let us consider the following homogeneous problem

$$\text{(4.16)} \qquad \begin{cases} \dot{w}(t) + Aw(t) = L(w_t(\cdot)), & \text{a.e. } t > 0, \\ w(0) = x, \\ w(t) = 0, & \text{a.e. } t \in [-r, 0]. \end{cases}$$

Let us assume (A1) and (L2). Then, from the proof of Proposition 4.2, we know that (4.16) has a unique solution $w(\cdot)$, provided $x \in F$. Thus, we can define a family of linear operators as follows: For any $x \in F$,

$$\text{(4.17)} \qquad \Phi(t)x = w(t), \qquad \forall t \in [-r, \infty).$$

Then, it is clear that

$$\text{(4.18)} \quad \Phi(\cdot): F \to L^2_{\text{loc}}(-r, \infty; \mathscr{D}(A)) \cap W^{1,2}_{\text{loc}}(0, \infty; H) \hookrightarrow C([0, \infty); F),$$

and

(4.19)

$$\Phi(\cdot): A^{-1}F \to L^2_{\text{loc}}(-r, \infty; \mathscr{D}(A^2)) \cap W^{1,2}_{\text{loc}}(0, \infty; \mathscr{D}(A)) \\ \hookrightarrow C([0, \infty); A^{-1}F).$$

From the definition, we also see that $\Phi(\cdot)$ satisfies the following

$$\text{(4.20)} \qquad \Phi(t) = \begin{cases} 0, & t < 0, \\ e^{-At}I_F + \int_0^t e^{-A(t-\tau)}L(\Phi_\tau(\cdot))\,d\tau, & t \geq 0, \end{cases}$$

where

$$L(\Phi_\tau(\cdot))x \equiv L(\Phi_\tau(\cdot)x).$$

Equivalently, we have

$$\begin{cases} \dot{\Phi}(t) + A\Phi(t) = L(\Phi_t(\cdot)), & \text{a.e. } t > 0, \\ \Phi(0) = I_F, \\ \Phi(t) = 0, \quad t \in [-r, 0), \end{cases} \tag{4.21}$$

The main result of this section is the following

Theorem 4.3. *Let* (A1) *and* (L2) *hold. Then,*

(i) *For any* $h(\cdot) \in L^2_{\text{loc}}(0, \infty; H)$, *the map*

$$t \mapsto \int_0^t \Phi(t-\tau)h(\tau)\,d\tau$$

is in $L^2_{\text{loc}}(0, \infty; \mathscr{D}(A)) \cap W^{1,2}_{\text{loc}}(0, \infty; H) \hookrightarrow C([0, \infty); F)$, *and there exists a nondecreasing function* $C(\cdot)\colon \mathbb{R}^+ \to \mathbb{R}^+$, *such that* $\forall h(\cdot) \in L^2_{\text{loc}}(0, \infty; H)$, $T \geq 0$,

$$\int_0^T \left| A \int_0^t \Phi(t-\tau)\,d\tau \right|_H^2 dt \leq C(T) \int_0^T |h(t)|_H^2\,dt. \tag{4.22}$$

(ii) (*The Variation of Constants Formula*) *For any* $x \in F$, $\dot{\varphi}(\cdot) \in L^2(-r, 0; \mathscr{D}(A))$ *and* $f(\cdot) \in L^2_{\text{loc}}(0, \infty; H)$, *the corresponding solution of* (4.1) *is given by*

(4.23)

$$u(t) = \begin{cases} \varphi(t), & t < 0, \\ \Phi(t)x + \displaystyle\int_0^t \Phi(t-\tau)L(\varphi_\tau(\cdot))\,d\tau + \int_0^t \Phi(t-\tau)f(\tau)\,d\tau, & t \geq 0, \end{cases}$$

where $\varphi(\cdot)$ *is extended to be zero in* $(0, \infty)$.

Proof. Let us first prove (ii) for the following case:

(4.24)

$$x \in A^{-1}F, \qquad \varphi(\cdot) \in C([-r, 0]; \mathscr{D}(A^2)), \quad f(\cdot) \in C([0, \infty); \mathscr{D}(A^2)).$$

By (L2) and Proposition 4.2, we know that the maps

$$t \mapsto \Phi(t)x,$$

$$t \mapsto \int_0^t \Phi(t-\tau)L(\varphi_\tau(\cdot))\,d\tau,$$

$$t \mapsto \int_0^t \Phi(t-\tau)f(\tau)\,d\tau$$

are all in $L^2_{\mathrm{loc}}(0, \infty; \mathscr{D}(A))$. Thus, the operator L is applicable to them. Next, by (4.23) and (4.20), for $t \geq 0$, we have

$$
\begin{aligned}
u(t) &\equiv \Phi(t)x + \int_0^t \Phi(t-\tau)L(\varphi_\tau(\cdot))\,d\tau + \int_0^t \Phi(t-\tau)f(\tau)\,d\tau \\
&= e^{-At}x + \int_0^t e^{-A(t-\tau)}[L(\Phi_\tau(\cdot)x) + L(\varphi_\tau(\cdot)) + f(\tau)]\,d\tau \\
&\quad + \int_0^t \int_0^{t-\tau} e^{-A(t-\tau-s)}[L(\Phi_s(\cdot)))L(\varphi_\tau(\cdot)) + L(\Phi_s(\cdot)f(\tau))]\,ds\,d\tau.
\end{aligned} \tag{4.25}
$$

Observe that

$$
\begin{aligned}
&\int_0^t e^{-A(t-\tau)}L(u_\tau(\cdot))\,d\tau \\
&\qquad = \int_0^t e^{-A(t-\tau)}L(u(\tau+\cdot)\chi_{[-\tau,\infty)})\,d\tau + \int_0^t e^{-A(t-\tau)}L(\varphi_\tau(\cdot))\,d\tau,
\end{aligned} \tag{4.26}
$$

while, by noting that $\Phi(t) = 0$, for $t < 0$, we have

(4.27)

$$
\begin{aligned}
&\int_0^t e^{-A(t-\tau)}L(u(\tau+\cdot)\chi_{[-\tau,\infty)})\,d\tau \\
&\quad = \int_0^t e^{-A(t-\tau)}L\left(\Phi(\tau+\cdot)x + \int_0^{\tau+\cdot}\Phi(\tau+\cdot-s)[L(\varphi_s(\cdot)) + f(s)]\,ds\right)d\tau \\
&\quad = \int_0^t e^{-A(t-\tau)}L(\Phi(\tau+\cdot)x\,d\tau \\
&\qquad + \int_0^t e^{-A(t-\tau)}L\left(\int_0^{\tau+\cdot}\Phi(\tau+\cdot-s)[L(\varphi_s(\cdot)) + f(s)]\,ds\right)d\tau.
\end{aligned}
$$

On the other hand, for $g(s) \equiv L(\varphi_s(\cdot)) + f(s)$, which is in $L^2_{\mathrm{loc}}(0, \infty; \mathscr{D}(A))$, by some direct computation, in which the condition (4.11) has to be used, we have

$$
\begin{aligned}
&\int_0^t e^{-A(t-\tau)}L\left(\int_0^{\tau+\cdot}\Phi(\tau+\cdot-s)g(s)\,ds\right)d\tau \\
&\qquad = \int_0^t \int_0^{t-\tau} e^{-A(t-\tau-s)}L(\Phi_s(\cdot)g(\tau))\,ds\,d\tau.
\end{aligned} \tag{4.28}
$$

Thus, combining (4.25–4.28), we obtain

$$
\begin{aligned}
&\int_0^t e^{-A(t-\tau)}[L(\Phi_\tau(\cdot)x) + L(\varphi_\tau(\cdot))]\,d\tau \\
&\quad + \int_0^t \int_0^{t-\tau} e^{-A(t-\tau-s)}[L(\Phi_s(\cdot))L(\varphi_\tau(\cdot)) + L(\Phi_s(\cdot)f(\tau))]\,ds\,d\tau \\
&\quad = u(t) - e^{-At}x - \int_0^t e^{-A(t-\tau)}f(\tau)\,d\tau, \qquad t \geq 0.
\end{aligned}
$$

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Thus, by the uniqueness, we see that (ii) is proved for the case (4.24).

Now, for any $h(\cdot) \in L^2_{\mathrm{loc}}(0, \infty; H)$, we can find a sequence $h_n(\cdot) \in C([0, \infty); \mathscr{D}(A^2))$, such that

$$\lim_{n\to\infty} \int_0^T |h_n(t) - h(t)|_H^2 \, dt = 0, \qquad \forall T \geq 0. \tag{4.29}$$

Then, using the above, we see that

$$u_n(t) \equiv \begin{cases} 0, & t \in [-r, 0); \\ \int_0^t \Phi(t-\tau) h_n(\tau)\, d\tau, & t \in [0, \infty) \end{cases}$$

is the unique solution of (4.1) corresponding to the data

$$x = 0, \qquad \varphi(\cdot) = 0, \qquad f(\cdot) = h_n(\cdot).$$

By Proposition 4.2, we see that (since our problem is linear)

$$\begin{aligned} &\int_0^T |\dot{u}_n(t) - \dot{u}_m(t)|_H^2 \, dt + \int_0^T |A(u_n(t) - u_m(t))|_H^2 \, dt \\ &\qquad \leq C(T) \int_0^T |h_n(t) - h_m(t)|_H^2 \, dt, \quad \forall T \geq 0. \end{aligned} \tag{4.30}$$

Thus, we see that (i) follows. Finally, for any

$$x \in F, \qquad \varphi(\cdot) \in L^2(-r, 0; \mathscr{D}(A)), \qquad f(\cdot) \in L^2_{\mathrm{loc}}(0, \infty; H),$$

we can find a sequence $(x_n, \varphi_n(\cdot), f_n(\cdot))$ satisfying (4.24), with the property

$$\lim_{n\to\infty} |x_n - x|_F = 0, \qquad \lim_{n\to\infty} \int_{-r}^0 |A(\varphi_n(t) - \varphi(t))|_H^2 \, dt = 0,$$

$$\lim_{n\to\infty} \int_0^T |f_n(t) - f(t)|_H^2 \, dt = 0, \qquad \forall T \geq 0.$$

Then we let $u_n(\cdot)$ and $u(\cdot)$ be the solution of (4.1) corresponding to $(x_n, \varphi_n(\cdot), f_n(\cdot))$ and $(x, \varphi(\cdot), f(\cdot))$, respectively. By Proposition 4.1, we see

$$\begin{aligned} &\int_0^T |\dot{u}_n(t) - \dot{u}(t)|_H^2 \, dt + \int_0^s |A(u_n(t) - u(t))|_H^2 \, dt \\ &\qquad \leq C(T) \left\{ |x_n - x|_F^2 + \int_{-r}^0 |A(\varphi_n(t) - \varphi(t))|_H^2 \, dt \right. \\ &\qquad\qquad \left. + \int_0^T |f_n(t) - f(t)|_H^2 \, dt \right\}, \qquad \forall T \geq 0. \end{aligned} \tag{4.31}$$

Hence, our conclusion follows from the proved (i) and the uniqueness of the solutions.

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Next, we would like to study some properties of the operator $\Phi(\cdot)$. To this end, we let S be defined as follows: For any $(x, \varphi(\cdot)) \in F \times L^2_{\text{loc}}(-r, 0; \mathscr{D}(A))$, let the corresponding solution of (4.1) be $u(\cdot)$. Then,

$$S(t)\begin{pmatrix} x \\ \varphi(\cdot) \end{pmatrix} \equiv \begin{pmatrix} u(t) \\ u_t(\cdot) \end{pmatrix}, \qquad \forall t \geq 0. \tag{4.32}$$

We have the following proposition, the proof of which is obvious.

Proposition 4.4. *The operator $S(t)$ is a C_0-semigroup on $Z \equiv F \times L^2(-r, 0; \mathscr{D}(A))$.*

From the above proposition, we have

Proposition 4.5. *The operator $\Phi(\cdot)$ satisfies*

$$\Phi(t+s) = \Phi(t)\Phi(s) + \int_0^t \Phi(t-\tau)L(G_{s+\tau}(\cdot)\chi_{[-r,\tau)}(\cdot))\,d\tau, \tag{4.33}$$

where

$$L(\Phi_{s+\tau}(\cdot)\chi_{[r,-\tau)}(\cdot))x \equiv L(\Phi(s+\tau+\cdot)x\chi_{[-r,-\tau)}(\cdot)), \qquad \forall x \in F. \tag{4.34}$$

Proof. Since $S(t)$ is a C_0-semigroup on Z, we see that for any $x \in F$ and $t, s \geq 0$, by (4.23),

$$\begin{aligned} \begin{pmatrix} \Phi(t+s)x \\ \Phi(t+s+\cdot)x \end{pmatrix} &= S(t+s)\begin{pmatrix} x \\ 0 \end{pmatrix} \\ &= S(t)S(s)\begin{pmatrix} x \\ 0 \end{pmatrix} = S(t)\begin{pmatrix} \Phi(s)x \\ \Phi(s+\cdot)x \end{pmatrix} \\ &= \begin{pmatrix} \Phi(t)\Phi(s)x + \int_0^t \Phi(t-\tau)L(\Phi(s+\tau+\cdot)x\chi_{[-r,-\tau)}(\cdot))\,d\tau \\ * \quad * \quad * \end{pmatrix}. \end{aligned} \tag{4.35}$$

Thus, (4.33) follows.

From the above proposition, we see that in general the evolution operator $\Phi(\cdot)$ is not a semigroup in the underline space H.

The rest of this section is devoted to a study of the generator of the semigroup $S(t)$.

Theorem 4.6. *Let* (L1) *and* (L2) *hold. Let the operator L be continuous from $W^{1,2}([-r, 0]; \mathscr{D}(A))$ to H. Then, the generator Λ of the semigroup $S(t)$ is characterized by*

$$\mathscr{D}(\Lambda) = \{(x, \varphi(\cdot)) \in \mathscr{D}(A) \times W^{1,2}([-r, 0]; \mathscr{D}(A)) \mid \varphi(0) = x,\ -Ax + L(\varphi(\cdot)) \in F\}, \tag{4.36}$$

and

$$\Lambda\begin{pmatrix} x \\ \varphi(\cdot)\end{pmatrix} = \begin{pmatrix} -Ax + L(\varphi(\cdot)) \\ \dot\varphi(\cdot)\end{pmatrix}, \qquad \forall (x, \varphi(\cdot)) \in \mathscr{D}(\Lambda). \tag{4.37}$$

The proof is almost the same as that given in [3].

Next, we define an operator

$$\Delta_\lambda x = \lambda x + Ax - L(e^{\lambda\cdot} x), \qquad \forall x \in \mathscr{D}(A), \lambda \in \mathbb{C}. \tag{4.38}$$

Then, we see that $\Delta_\lambda : \mathscr{D}(A) \to H$. In the finite-dimensional case, the equation

$$\det \Delta_\lambda = 0$$

serves as the characteristic equation ([8]). Thus, we can use the operator Δ_λ to characterize the spectrum of the operator Λ. For the infinite dimensional case with

$$L(\varphi(\cdot)) = \sum_{i=0}^{k} A_i \varphi(0 - r_i) + \int_{-r}^{0} \widehat{A}(\theta)\varphi(\theta)\,d\theta,$$

where $A_i \in \mathscr{L}(H)$, $0 \le i \le k$ and $0 = r_0 < \cdots < r_k = r < \infty$, the operator Δ_λ was also used to characterize the spectrum of Λ (see [22]). In [4], the case with

$$L(\varphi(\cdot)) = A_1\varphi(0 - r) + \int_{-r}^{0} a(s)A_2(s)\varphi(s)\,ds,$$

where $A_i \in \mathscr{L}(\mathscr{D}(A), H)$, $i = 1, 2$; $a(\cdot) \in L^2(-r, 0)$ was studied. A set of technical conditions were assumed to obtain the characterization of the spectrum of the operator Λ via Δ_λ. In our case, we take a direct approach to the problem, which is a little different from [4]. Before stating the result, let us introduce the following space:

$$W_0^{1,2}([-r, 0]; \mathscr{D}(A)) = \{v \in W^{1,2}([-r, 0]; \mathscr{D}(A)) | v(0) = 0\}.$$

We assume the following further hypothesis:

$$L(W_0^{1,2}([-r, 0]; \mathscr{D}(A))) + F = H. \tag{4.39}$$

We see, in particular, that if L maps $W_0^{1,2}([-r, 0]; \mathscr{D}(A))$ onto H, then (4.39) holds. More concretely, let us observe the following case:

$$L(\varphi(\cdot)) = \int_{-r}^{0} a(\theta)A\varphi(\cdot)\mu(d\theta), \qquad \forall \varphi(\cdot) \in L^2(-r, 0; \mathscr{D}(A)), \tag{4.40}$$

with $a(\cdot)$ being Borel measurable and bounded, satisfying the property that

$$c_0 \equiv \int_{-r}^{0} a(\theta)b(\theta)\mu(d\theta) \neq 0, \tag{4.41}$$

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for some scalar function $b(\cdot) \in W^{1,2}([-r, 0])$ satisfying $b(0) = 0$. Then, we claim that condition (4.39) holds. In fact, for any $h \in H$, we let

$$\varphi(\cdot) = \frac{b(\cdot)}{c_0} A^{-1} h.$$

Hence we see $\varphi(\cdot) \in W_0^{1,2}([-r, 0]; \mathscr{D}(A))$ and

$$L(\varphi(\cdot)) = h.$$

We see that (4.41) is very general.

Next, we let $\rho(\Lambda)$, $\sigma_p(\Lambda)$, $\sigma_c(\Lambda)$ and $\sigma_r(\Lambda)$ be the resolvent, the point, the continuous and the residual spectrum of the operator Λ, respectively ([21]). We denote the null space and the range of the operator Δ_λ by $\mathscr{N}(\Delta_\lambda)$ and $\mathscr{R}(\Delta_\lambda)$, respectively. Then, we have the following result.

LEMMA 4.7. *Let* (4.39) *hold. Then, for any* $\lambda \in \mathbb{C}$,

$$\mathscr{N}(\lambda - \lambda) \neq 0 \Leftrightarrow \mathscr{N}(\Delta_\lambda) \neq 0; \tag{4.42}$$

$$\mathscr{R}(\lambda - \Lambda) = Z \Leftrightarrow \mathscr{R}(\Delta_\lambda) = H; \tag{4.43}$$

$$\overline{\mathscr{R}(\lambda - \Lambda)}^{H \times L^2(-r, 0; \mathscr{D}(A))} \supseteq Z \Leftrightarrow \overline{\mathscr{R}(\Delta_\lambda)}^{H} = H. \tag{4.44}$$

PROOF. Let

$$\begin{pmatrix} x \\ \varphi(\cdot) \end{pmatrix} \in \mathscr{D}(\Lambda) \backslash \{0\},$$

such that

$$(\lambda - \Lambda) \begin{pmatrix} x \\ \varphi(\cdot) \end{pmatrix} = 0.$$

This implies that

$$\varphi(\cdot) = e^{\lambda \cdot} x.$$

Thus $x \neq 0$ and

$$\Delta_\lambda x = 0.$$

Conversely, let $x \in \mathscr{D}(A)$, $x \neq 0$, such that

$$\Delta_\lambda x \equiv \lambda x + Ax - L(e^{\lambda \cdot} x) = 0.$$

Then, it is easy to see that

$$\begin{pmatrix} x \\ e^{\lambda \cdot} x \end{pmatrix} \in \mathscr{D}(\Lambda) \backslash \{0\},$$

and

$$(\lambda - \Lambda) \begin{pmatrix} x \\ e^{\lambda \cdot} x \end{pmatrix} = 0.$$

This proves (4.42). Now, we let

$$\mathscr{R}(\lambda - \Lambda) = Z.$$

Then, for any $h \in H$, by condition (4.39), we can find $y \in F$ and $w(\cdot) \in W_0^{1,2}([-r, 0]; \mathscr{D}(A))$, such that

$$h = y - L(w(\cdot)).$$

Let

$$\psi(\cdot) = \dot{w}(\cdot) + \lambda w(\cdot) \in L^2(-r, 0; \mathscr{D}(A)).$$

Then, we let $(x, \varphi(\cdot)) \in \mathscr{D}(\Lambda)$, such that

$$(\lambda - \Lambda)\begin{pmatrix} x \\ \varphi(\cdot) \end{pmatrix} = \begin{pmatrix} y \\ \psi(\cdot) \end{pmatrix}.$$

This gives

$$\Delta_\lambda x = h.$$

Conversely, let $\mathscr{R}(\Delta_\lambda) = H$. Then, for any $(y, \psi(\cdot)) \in Z$, there exists an $x \in \mathscr{D}(A)$, such that

$$\Delta_\lambda x \equiv \lambda x + Ax - L\left(\int_0^{\cdot} e^{\lambda \cdot} x\right) = y - L\left(\int_0^{\cdot} e^{\lambda(\cdot - r)} \psi(\tau)\, d\tau\right).$$

Then we let

$$\varphi(\cdot) = e^{\lambda \cdot} x - \int_0^{\cdot} e^{\lambda(\cdot - \tau)} \psi(\tau)\, d\tau.$$

It is easy to show that

$$\begin{pmatrix} x \\ \varphi(\cdot) \end{pmatrix} \in \mathscr{D}(\Lambda),$$

and

$$(\lambda - \Lambda)\begin{pmatrix} x \\ \varphi(\cdot) \end{pmatrix} = \begin{pmatrix} y \\ \psi(\cdot) \end{pmatrix}.$$

Thus, (4.43) follows. Finally, let

$$\overline{\mathscr{R}(\lambda - \Lambda)}^{H \times L^2(-r, 0; \mathscr{D}(A))} \supseteq Z.$$

Then, for any $h \in H$, there exist a $y \in F$ and $w(\cdot) \in W_0^{1,2}([-r, 0]; \mathscr{D}(A))$, such that

$$h = y - L(w(\cdot)).$$

Let

$$\psi(\cdot) = \dot{w}(\cdot) + \lambda w(\cdot) \in L^2(-r, 0; \mathscr{D}(A)).$$

Then, one can find a sequence $(x_n, \varphi_n(\cdot)) \in \mathscr{D}(\Lambda)$, such that

$$(\lambda - \Lambda)\begin{pmatrix} x_n \\ \varphi_n(\cdot) \end{pmatrix} \to \begin{pmatrix} y \\ \psi(\cdot) \end{pmatrix}, \quad \text{in } H \times L^2(-r, 0; \mathscr{D}(A)).$$

This implies

$$\Delta_\lambda x_n \to y - L(w(\cdot)) = h, \quad \text{in } H.$$

Conversely, let

$$\overline{\mathscr{R}(\Delta_\lambda)}^H = H.$$

Then, for any $(y, \psi(\cdot)) \in Z$, there exists a sequence $x_n \in \mathscr{D}(A)$, such that

$$\Delta_\lambda x_n \to y - L\left(\int_0^{\cdot} e^{\lambda(\cdot-\tau)}\psi(\tau)\,d\tau\right), \quad \text{in } H.$$

Let

$$\varphi_n(\cdot) = e^{\lambda\cdot}x_n - \int_0^{\cdot} e^{\lambda(\cdot-\tau)}\psi(\tau)\,d\tau.$$

Then, we see that $(x_n, \varphi_n(\cdot)) \in \mathscr{D}(\Lambda)$ and

$$(\lambda - \Lambda)\begin{pmatrix} x_n \\ \varphi_n(\cdot)\end{pmatrix} \to \begin{pmatrix} y \\ \psi(\cdot)\end{pmatrix}, \quad \text{in } H \times L^2(-r, 0; \mathscr{D}(A)).$$

This completes the proof.

From the above lemma, we can easily obtain the following

THEOREM 4.8. *Let* (4.39) *holds. Then,*

(4.45) $$\lambda \in \rho(\Lambda) \Leftrightarrow \Delta_\lambda : \mathscr{D}(A) \to H \text{ is bijective};$$

(4.46) $$\lambda \in \sigma_p(\Lambda) \Leftrightarrow \mathscr{N}(\Delta_\lambda) \neq 0;$$

(4.47) $$\lambda \in \sigma_c(\Lambda) \Rightarrow \mathscr{N}(\Delta_\lambda) = \{0\}, \quad \mathscr{R}(\Delta_\lambda) \neq H, \quad \overline{\mathscr{R}(\Delta_\lambda)}^H = H;$$

(4.48) $$\lambda \in \sigma_r(\Lambda) \Leftarrow \mathscr{N}(\Delta_\lambda) = \{0\}, \quad \overline{\mathscr{R}(\Delta_\lambda)}^H \neq H.$$

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