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ON THE LATTICE OF EXISTENCE VARIETIES OF LOCALLY INVERSE SEMIGROUPS

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ABSTRACT. The mapping $\mathbb{E}\colon \mathcal{V} \longmapsto \{\mathrm{E}(S) \mid S \in \mathcal{V}\}$ which assigns to each existence variety $\mathcal{V}$ of locally inverse semigroups the class of all pseudosemilattices of idempotents of members of $\mathcal{V}$ is shown to be a complete, surjective homomorphism from the lattice of existence varieties of locally inverse semigroups onto the lattice of varieties of pseudosemilattices.

1. **Introduction.** As defined by Hall [3,4,5], a class $\mathcal{V}$ of regular semigroups is an *existence variety* (or e-variety) if $\mathcal{V}$ is closed under taking direct products, regular subsemigroups and homomorphic images. The collection of all e-varieties of regular semigroups forms a complete lattice under inclusion containing the lattices of inverse semigroup and of completely regular semigroup varieties as ideal sublattices. For the subclass of all orthodox semigroups the concept of e-variety has been introduced under the term "bivariety" by Kaďourek and Szendrei [6]. For this class, the analogy to usual varieties of universal algebras is particularly strong: in each e-variety of orthodox semigroups there exists for any non-empty set X a "bifree object" on X which is an adequate analogue to the usual free object. Furthermore, e-varieties of orthodox semigroups can be characterized as classes which consist of all semigroups satisfying certain sets of "biidentities" (for definitions see [6]). The problem of whether analogues to free objects exist in other e-varieties of regular semigroups has been solved completely by Yeh [16]: using the slightly modified notion of an "e-free object", he showed that such semigroups exist in an e-variety $\mathcal{V}$ of regular semigroups if and only if $\mathcal{V}$ is contained either in the e-variety $\mathcal{LI}$ of all locally inverse semigroups or in the e-variety $\mathcal{ES}$ of all E-solid semigroups. Using the methods of [16], the author [1] introduced the concept of biidentities in a form suitable for locally inverse semigroups and established a Birkhoff-type theorem for e-varieties of locally inverse semigroups (similar to the orthodox case [6]): these can be characterized as classes consisting of all locally inverse semigroups which satisfy certain sets of biidentities. In addition, in [1] it is shown that in each e-variety $\mathcal{V}$ of locally inverse semigroups all free products exist.

For any locally inverse semigroup S, the set of all idempotents $\mathrm{E}(S)$ of S is equipped in a natural way with the binary operation of forming the sandwich element. In this way, $\mathrm{E}(S)$ is an algebraic structure of its own, called a *pseudosemilattice*. The class of

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all such algebras forms a variety. The main purpose of the present note is to obtain a natural connection between the lattice of e-varieties of locally inverse semigroups and the lattice of varieties of pseudosemilattices. It is shown that the mapping which assigns to each e-variety $\mathcal{V}$ of locally inverse semigroups the class of all pseudosemilattices of idempotents of members of $\mathcal{V}$ is a complete, surjective homomorphism. This will be established in Section 3. The tools to prove this result include two kinds of "free objects". The definitions of these, and further preliminaries are the content of the next section.

2. **Preliminaries.** For any semigroup S denote by $\mathrm{E}(S)$ the (biordered) set of idempotents of S; for $x \in S$ let $V(x) = \{y \in S \mid x = xyx, y = yxy\}$ be the set of (generalized) inverses of x in S. Given two idempotents $e,f \in \mathrm{E}(S)$ then $S(e,f) = fV(ef)e$ is the *sandwich set* of the pair (e,f). If S is regular then $S(e,f) \neq \emptyset$ for any $e,f \in \mathrm{E}(S)$ and $S(e,f) \subseteq V(ef) \cap \mathrm{E}(S)$. For the definition of (regular) biordered sets see Nambooripad [9]. A regular semigroup is *locally inverse* if the local submonoid eSe is an inverse semigroup for each $e \in \mathrm{E}(S)$. It has been shown by Nambooripad [9, Theorem 7.6] that a regular semigroup is locally inverse if and only if $|S(e,f)| = 1$ for all $e,f \in \mathrm{E}(S)$. This fact has been used by Yeh [16] to introduce another binary operation $\wedge$ on any locally inverse semigroup. Given $x, y \in S$ then $x \wedge y$ denotes the unique element of the sandwich set $S(y'y, xx')$ where $x' \in V(x)$ and $y' \in V(y)$, and $\wedge$ will be called the *sandwich operation*. The definition of $\wedge$ does not depend on the choice of the inverses x' respectively y'. The following important lemma has been proved by Yeh [16].

LEMMA 2.1. *Let S, T be locally inverse semigroups. Then each (semigroup) homomorphism $\phi: S \longrightarrow T$ respects the sandwich operation $\wedge$.*

Restricting the operation $\wedge$ on S to the set of idempotents $\mathrm{E}(S)$ one gets the *pseudosemilattice* $\mathrm{E}(S)$ *of the locally inverse semigroup S*. As a structure of its own, pseudosemilattices have been studied, for instance, by Schein [15], Nambooripad [10], Meakin [7], Meakin and Pastijn [8]. The class $\mathcal{PS}$ of all so obtained pseudosemilattices forms a variety of binary idempotent algebras [10]: the lattice of all subvarieties of $\mathcal{PS}$ will be denoted by $\mathcal{L}(\mathcal{PS})$. From Nambooripad [11], it follows that the class $\mathcal{LI}$ of all locally inverse semigroups forms an e-variety (Hall [3]): the lattice of all sub-e-varieties is denoted by $\mathcal{L}_e(\mathcal{LI})$. In order to prove the mentioned result, we need the following concepts.

DEFINITION 1. Let $\mathcal{V}$ be an e-variety of locally inverse semigroups and let X be a non-empty set. Then the *free idempotent generated semigroup in $\mathcal{V}$ on X* (or *free semiband in $\mathcal{V}$ on X*) is a semigroup $\mathrm{FI}\mathcal{V}(X) \in \mathcal{V}$, together with a mapping $\iota: X \longrightarrow \mathrm{E}\big(\mathrm{FI}\mathcal{V}(X)\big)$ satisfying the following universal property: for any $S \in \mathcal{V}$ and any mapping $\phi: X \longrightarrow \mathrm{E}(S)$ there is a unique homomorphism $\bar{\phi}: \mathrm{FI}\mathcal{V}(X) \longrightarrow S$ extending ϕ, that is, $\iota\bar{\phi} = \phi$.

The existence of $\mathrm{FI}\mathcal{V}(X)$ follows from [1, Theorem 6.7]: it is isomorphic to the $\mathcal{V}$-free product of $|X|$ copies of the trivial semigroup. It is clear that $\mathrm{FI}\mathcal{V}(X)$ is (up to isomorphic copies) uniquely determined by the cardinality $|X|$. Throughout the paper, we shall identify X and $X\iota$ and consider X as a subset of $\mathrm{FI}\mathcal{V}(X)$. Notice that $\mathrm{FI}\mathcal{V}(X)$ is the closure of X under the operations of multiplication $\cdot$ and of forming the sandwich element $\wedge$.

DEFINITION 2. Let E be a pseudosemilattice. The locally inverse semigroup $F(E)$ is the *free locally inverse semigroup with respect to the pseudosemilattice E* if it satisfies the following properties:

(1) $\mathrm{E}\big(F(E)\big) = E$
(2) $F(E)$ is generated by E (under the operation $\cdot$)
(3) for any locally inverse semigroup S and any (pseudosemilattice) homomorphism $\phi\colon E \longrightarrow \mathrm{E}(S)$ there is a homomorphism $\bar{\phi}\colon F(E) \longrightarrow S$ such that $\bar{\phi}|E = \phi$.

The existence of $F(E)$ for any pseudosemilattice E follows from a more general result of Nambooripad [9, Theorem 6.9]. Uniqueness of $F(E)$ and uniqueness of the extension $\bar{\phi}$ in Definition 2 follow immediately.

DEFINITION 3. Let E be a pseudosemilattice. Then the locally inverse semigroup $T(E)$ is characterized by the following properties:

(1) $\mathrm{E}\big(T(E)\big) = E$
(2) $T(E)$ is fundamental
(3) if S is fundamental and $\mathrm{E}(S) \cong E$ then S is isomorphic to a full regular subsemigroup of $T(E)$.

We call $T(E)$ the *Nambooripad semigroup* with respect to the pseudosemilattice E. Further, let $\overline{E}$ denote the subsemigroup of $T(E)$ generated by the idempotents E.

Existence and uniqueness of $T(E)$ and of $\overline{E}$ follow from more general results of Nambooripad [9, Section 5].

3. **The lattice of e-varieties of locally inverse semigroups.**

THEOREM 3.1. *Let X be a non-empty set and let $\hat{X}$ be the closure of X in $FI\mathcal{LI}(X)$ under the sandwich operation $\wedge$. Then $\hat{X} = \mathrm{E}\big(FI\mathcal{LI}(X)\big)$ and the algebra $(\hat{X}, \wedge)$ is the free pseudosemilattice on X.*

PROOF. $\hat{X}$ is closed under $\wedge$ and consists of idempotent elements of $\mathrm{FI}\mathcal{LI}(X)$. Hence $(\hat{X}, \wedge)$ is a pseudosemilattice. Consider the inclusion mapping $\iota\colon \hat{X} \longrightarrow \mathrm{FI}\mathcal{LI}(X)$, $e \longmapsto e$. ι is a (pseudosemilattice) homomorphism of $\hat{X}$ into $\mathrm{E}\big(\mathrm{FI}\mathcal{LI}(X)\big)$. Hence there is a (unique) homomorphism $\phi\colon F(\hat{X}) \longrightarrow \mathrm{FI}\mathcal{LI}(X)$ extending the mapping ι. ϕ respects both operations $\cdot$ and $\wedge$. On the other hand, $F(\hat{X})$ is a locally inverse semigroup. Therefore, the mapping $x \longmapsto x$ $(x \in X)$ can be (uniquely) extended to a homomorphism $\psi\colon \mathrm{FI}\mathcal{LI}(X) \longrightarrow F(\hat{X})$. Now $\psi\phi$ is a homomorphism $\mathrm{FI}\mathcal{LI}(X) \longrightarrow \mathrm{FI}\mathcal{LI}(X)$ extending the mapping $x \longmapsto x$ $(x \in X)$. Consequently, $\psi\phi$ is the identity mapping on $\mathrm{FI}\mathcal{LI}(X)$. On the other hand, the closure of X in $F(\hat{X})$ under $\cdot$ and $\wedge$ contains $\hat{X}$ and thus is a full regular subsemigroup of $F(\hat{X})$. But $F(\hat{X})$ is generated by its idempotents $\hat{X}$ under $\cdot$ so that $F(\hat{X})$ is the only full regular subsemigroup of $F(\hat{X})$. That is, $F(\hat{X})$ is generated by X under the operations $\cdot$ and $\wedge$. Hence $x\phi\psi = x$ for all $x \in X$ implies that $s\phi\psi = s$ for all $s \in F(\hat{X})$ because the homomorphism $\phi\psi$ respects both operations $\cdot$ and $\wedge$. So $\phi\psi$ is the identity mapping on $F(\hat{X})$. Hence ϕ and ψ are mutually inverse isomorphisms between $F(\hat{X})$ and $\mathrm{FI}\mathcal{LI}(X)$ so that $\hat{X} = \hat{X}\iota = \hat{X}\phi = \mathrm{E}\big(F(\hat{X})\big)\phi = \mathrm{E}\big(\mathrm{FI}\mathcal{LI}(X)\big)$. Now let G be any pseudosemilattice and $\chi: X \longrightarrow G$ be any mapping. There exists a locally inverse semigroup S such that $\mathrm{E}(S) = G$. By definition of $\mathrm{FI}\mathcal{LI}(X)$ there exists a homomorphism $\bar{\chi}: \mathrm{FI}\mathcal{LI}(X) \longrightarrow S$ such that $x\chi = x\bar{\chi}$ for each $x \in X$. Now $\bar{\chi}|\hat{X} = \bar{\chi}|\mathrm{E}\big(\mathrm{FI}\mathcal{LI}(X)\big)$ is a homomorphism $\hat{X} \longrightarrow G$ which extends χ. The extension $\bar{\chi}|\hat{X}$ is necessarily unique since $\hat{X}$ is generated by X under $\wedge$. ∎

REMARK. The free pseudosemilattice on a set has been described by Meakin [7].

Theorem 3.1 implies a similar situation for an arbitrary e-variety of locally inverse semigroups, which we now prove.

COROLLARY 3.2. *Let $\mathcal{V} \in \mathcal{L}_e(\mathcal{LI})$. Then $\mathrm{E}\big(FI\mathcal{V}(X)\big)$ is the closure of X in $FI\mathcal{V}(X)$ under $\wedge$. The pseudosemilattice $\Big(\mathrm{E}\big(FI\mathcal{V}(X)\big), \wedge\Big)$ is a relatively free pseudosemilattice on X.*

PROOF. Let $\phi: \mathrm{FI}\mathcal{LI}(X) \longrightarrow \mathrm{FI}\mathcal{V}(X)$ be the canonical homomorphism. That is, ϕ is the unique homomorphism satisfying $x\phi = x$ (the set X is interpreted as a subset of $\mathrm{FI}\mathcal{LI}(X)$ as well as of $\mathrm{FI}\mathcal{V}(X)$). Let $e \in \mathrm{E}\big(\mathrm{FI}\mathcal{V}(X)\big)$. By Lallement's Lemma, there is $f \in \mathrm{E}\big(\mathrm{FI}\mathcal{LI}(X)\big)$ such that $f\phi = e$. By Theorem 3.1, f is in the closure of X in $\mathrm{FI}\mathcal{LI}(X)$ under $\wedge$. Since ϕ respects $\wedge$, e is in the closure of $X\phi = X$ in $\mathrm{FI}\mathcal{V}(X)$ under $\wedge$.

Let $\theta: X \longrightarrow \mathrm{E}\big(\mathrm{FI}\mathcal{V}(X)\big) \subseteq \mathrm{FI}\mathcal{V}(X)$ be any mapping. There is a homomorphic extension $\bar{\theta}: \mathrm{FI}\mathcal{V}(X) \longrightarrow \mathrm{FI}\mathcal{V}(X)$ of θ. Then $\bar{\theta}|\mathrm{E}\big(\mathrm{FI}\mathcal{V}(X)\big)$ is a homomorphic extension of θ to the pseudosemilattice $\mathrm{E}\big(\mathrm{FI}\mathcal{V}(X)\big)$. The uniqueness of this extension follows from the fact that $\mathrm{E}\big(\mathrm{FI}\mathcal{V}(X)\big)$ is generated by X. In particular $\mathrm{E}\big(\mathrm{FI}\mathcal{V}(X)\big)$ is free on X in the class $\big\{\mathrm{E}\big(\mathrm{FI}\mathcal{V}(X)\big)\big\}$. By a standard universal algebra argument (see [2, §24, Theorem 5]), $\mathrm{E}\big(\mathrm{FI}\mathcal{V}(X)\big)$ is free on X in the variety generated by itself. ∎

The following is an essential definition of the paper.

DEFINITION 4. For any e-variety $\mathcal{V}$ of locally inverse semigroups let

$$\mathbb{E}\mathcal{V} = \{E \in \mathcal{PS} \mid E = \mathrm{E}(S) \text{ for some } S \in \mathcal{V}\}.$$

THEOREM 3.3. *For each e-variety $\mathcal{V}$ of locally inverse semigroups, $\mathbb{E}\mathcal{V}$ is a variety of pseudosemilattices.*

PROOF. Let $\mathcal{V}$ be an e-variety of locally inverse semigroups.

(1) Let I be an index set and $E_i \in \mathbb{E}\mathcal{V}$ for each $i \in I$. For each i there is $S_i \in \mathcal{V}$ such that $E_i = \mathrm{E}(S_i)$. Now $\prod S_i \in \mathcal{V}$ and $\mathrm{E}(\prod S_i) = \prod \mathrm{E}(S_i) = \prod E_i$ and therefore $\prod E_i \in \mathbb{E}\mathcal{V}$.

(2) Let $E \in \mathbb{E}\mathcal{V}$ and $G \subseteq E$ be a subpseudosemilattice. Let $E' = \{e' \mid e \in E\}$ be a set of cardinality $|E|$, $e \longmapsto e'$ being a bijection between E and E', and let $G' = \{g' \mid g \in G\} \subseteq E'$. Consider the free idempotent generated semigroup in $\mathcal{V}$ on E'. Now G' is a subset of E' so that $\mathrm{FI}\mathcal{V}(G')$ is a regular subsemigroup of $\mathrm{FI}\mathcal{V}(E')$. There

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is $S \in \mathcal{V}$ such that $\mathrm{E}(S) = E$. Let $\phi\colon \mathrm{FI}\mathcal{V}(E') \longrightarrow S$ denote the unique homomorphism extending the mapping $E' \longrightarrow E$, $e' \longmapsto e$. Let $T = \mathrm{FI}\mathcal{V}(G')\phi \subseteq S$. Then T is a regular subsemigroup of S. Hence $T \in \mathcal{V}$. Obviously, $G \subseteq \mathrm{E}(T)$. Let $e \in \mathrm{E}(T)$. By Lallement's Lemma there is $f \in \mathrm{E}\big(\mathrm{FI}\mathcal{V}(G')\big)$ such that $f\phi = e$. By Corollary 3.2, $f = w(e'_1, \ldots, e'_n)$ for certain $e'_i \in G'$ and w is a polynomial with respect to the binary operation $\wedge$. Now $e = f\phi = w(e'_1\phi, \ldots, e'_n\phi) = w(e_1, \ldots, e_n)$ where $e_i = e'_i\phi \in G$. Since G is closed under $\wedge$, $e \in G$. That is, $\mathrm{E}(T) = G$. In particular, $G \in \mathbb{E}\mathcal{V}$.

(3) Let $E \in \mathbb{E}\mathcal{V}$ and let $E' = E\phi$ be a homomorphic image of E. First, there is $S \in \mathcal{V}$ such that $\mathrm{E}(S) = E$. For any locally inverse semigroup T let $\mu(T)$ denote the greatest idempotent separating congruence on T. Now $S/\mu(S) \in \mathcal{V}$ and since $S/\mu(S)$ is fundamental, $S/\mu(S)$ is isomorphic to a full regular subsemigroup of the Nambooripad semigroup $T(E)$. Then $\overline{E}$ is isomorphic to a regular subsemigroup of $S/\mu(S)$ and therefore $\overline{E} \in \mathcal{V}$. Let $F(E)$ and $F(E')$ be the free locally inverse semigroups with respect to the pseudosemilattices E respectively E'. Let $\bar{\phi}\colon F(E) \longrightarrow F(E')$ be the unique homomorphic extension of $\phi\colon E \longrightarrow E'$. Let ρ be the congruence on $F(E)$ induced by $\bar{\phi}$ and let ρ^T be the greatest congruence on $F(E)$ having the same trace as ρ (see Pastijn and Petrich [14]). The greatest idempotent separating congruence on $F(E)/\rho$ is given by ρ^T/ρ (see [14]). Consequently, $F(E)/\rho^T \cong F(E')/\mu\big(F(E')\big)$. Again by [14], $\mu\big(F(E)\big) \subseteq \rho^T$ and therefore, the natural mapping $F(E) \longrightarrow F(E)/\rho^T$ can be factorized as $x \longmapsto x\mu\big(F(E)\big) \longmapsto x\rho^T$. Hence the mapping

$$F(E)/\mu\big(F(E)\big) \longrightarrow F(E)/\rho^T \longrightarrow F(E')/\mu\big(F(E')\big),$$
$$x\mu\big(F(E)\big) \longmapsto x\rho^T \longmapsto x\bar{\phi}\mu\big(F(E')\big)$$

provides a homomorphism which extends ϕ (here the pseudosemilattices $\mathrm{E}\big(F(E)\big)$ and $\mathrm{E}\Big(F(E)/\mu\big(F(E)\big)\Big)$ as well as $\mathrm{E}\big(F(E')\big)$ and $\mathrm{E}\Big(F(E')/\mu\big(F(E')\big)\Big)$ are identified). Now $F(E)/\mu\big(F(E)\big)$ is (isomorphic to) a full regular subsemigroup of $T(E)$ and $F(E')/\mu\big(F(E')\big)$ is (isomorphic to) a full regular subsemigroup of $T(E')$. In particular, there is a homomorphism ψ of a full regular subsemigroup of $T(E)$ onto a full regular subsemigroup of $T(E')$ which extends the pseudosemilattice homomorphism ϕ. Restriction of ψ to $\overline{E}$ yields $\overline{E}\psi = \overline{E'}$. Since $\overline{E} \in \mathcal{V}$ also $\overline{E'} \in \mathcal{V}$ and thus, $E' = \mathrm{E}(\overline{E'}) \in \mathbb{E}\mathcal{V}$. ∎

The kernel of the mapping $\mathbb{E}$ can be described in terms of the pseudosemilattice of idempotents of free idempotent generated semigroups.

PROPOSITION 3.4. *Let $\mathcal{V}, \mathcal{W} \in \mathcal{L}_e(\mathcal{L}I)$ and X be a countably infinite set. Then*

$$\mathbb{E}\mathcal{V} = \mathbb{E}\mathcal{W} \Leftrightarrow \mathrm{E}\big(FI\mathcal{V}(X)\big) \cong \mathrm{E}\big(FI\mathcal{W}(X)\big).$$

PROOF. Let X be a non-empty set and $\mathcal{V} \in \mathcal{L}_e(\mathcal{L}I)$. By Corollary 3.2 we know that $\mathrm{E}\big(\mathrm{FI}\mathcal{V}(X)\big)$ is a relatively free pseudosemilattice on X. Let $F(\mathbb{E}\mathcal{V})(X)$ denote the free

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object in the variety $\mathbb{E}\mathcal{V}$ on the set X. There is a locally inverse semigroup $S \in \mathcal{V}$ such that $\mathrm{E}(S) = F(\mathbb{E}\mathcal{V})(X)$. The mapping $x \longmapsto x$ can be extended to a homomorphism $\theta\colon \mathrm{FI}\mathcal{V}(X) \longrightarrow S$. Restriction of θ to $\mathrm{E}\big(\mathrm{FI}\mathcal{V}(X)\big)$ yields a surjective homomorphism $\tau\colon \mathrm{E}\big(\mathrm{FI}\mathcal{V}(X)\big) \longrightarrow \mathrm{E}(S) = F(\mathbb{E}\mathcal{V})(X)$ which extends the mapping $x \longmapsto x$ $(x \in X)$. By $\mathrm{E}\big(\mathrm{FI}\mathcal{V}(X)\big) \in \mathbb{E}\mathcal{V}$ we infer that $F(\mathbb{E}\mathcal{V})(X) \cong \mathrm{E}\big(\mathrm{FI}\mathcal{V}(X)\big)$. Since each variety is determined by its free object on a countably infinite set X we get

$$\mathbb{E}\mathcal{V} = \mathbb{E}\mathcal{W} \Leftrightarrow F(\mathbb{E}\mathcal{V})(X) \cong F(\mathbb{E}\mathcal{W})(X) \Leftrightarrow \mathrm{E}\big(\mathrm{FI}\mathcal{V}(X)\big) \cong \mathrm{E}\big(\mathrm{FI}\mathcal{W}(X)\big)$$

for the countably infinite set X. ∎

DEFINITION 5. A pseudosemilattice identity $u = v$ holds in the locally inverse semigroup S if it holds in the pseudosemilattice $\mathrm{E}(S)$ of S. It holds in the class $\mathcal{V}$ of locally inverse semigroups if it holds in each member of $\mathcal{V}$.

We now are able to prove the mentioned result.

THEOREM 3.5. *The mapping* $\mathbb{E}\colon \mathcal{L}_e(\mathcal{LI}) \longrightarrow \mathcal{L}(\mathcal{PS})$, $\mathcal{V} \longmapsto \{\mathrm{E}(S) \mid S \in \mathcal{V}\}$ *is a complete surjective homomorphism.*

PROOF. Let I be an index set and $\mathcal{V}_i \in \mathcal{L}_e(\mathcal{LI})$ for each $i \in I$. Clearly $\mathbb{E}(\bigcap \mathcal{V}_i) \subseteq \mathbb{E}\mathcal{V}_i$ for each $i \in I$ so that $\mathbb{E}(\bigcap \mathcal{V}_i) \subseteq \bigcap \mathbb{E}\mathcal{V}_i$. Conversely, let $E \in \bigcap \mathbb{E}\mathcal{V}_i$. That is, $E \in \mathbb{E}\mathcal{V}_i$ for each $i \in I$. As pointed out in the proof of Theorem 3(3), $\overline{E} \in \mathcal{V}_i$ for each i and thus $\overline{E} \in \bigcap \mathcal{V}_i$. Since $\mathrm{E}(\overline{E}) = E$ it follows that $E \in \mathbb{E}(\bigcap \mathcal{V}_i)$ and $\bigcap \mathbb{E}\mathcal{V}_i \subseteq \mathbb{E}(\bigcap \mathcal{V}_i)$.

For the join, the inclusion $\bigvee \mathbb{E}\mathcal{V}_i \subseteq \mathbb{E}(\bigvee \mathcal{V}_i)$ is trivial. For the reverse inclusion it suffices to show that each pseudosemilattice identity $w = v$ which holds in $\bigvee \mathbb{E}\mathcal{V}_i$ also holds in $\mathbb{E}(\bigvee \mathcal{V}_i)$. Suppose that $w = v$ holds in $\bigvee \mathbb{E}\mathcal{V}_i$ and thus in $\mathbb{E}\mathcal{V}_i$ for each $i \in I$. This identity therefore holds in $\mathcal{V}_i$ for each $i \in I$ (according to Definition 5) and thus also holds in $\bigcup \mathcal{V}_i$. If $S \in \bigvee \mathcal{V}_i$ then S is a homomorphic image of a regular subsemigroup of a direct product of certain members of $\bigcup \mathcal{V}_i$ (see Yeh [16, Lemma 4.8]). Therefore, if $w = v$ holds in $\bigcup \mathcal{V}_i$ then $w = v$ holds in $\bigvee \mathcal{V}_i$ and thus also holds in $\mathbb{E}(\bigvee \mathcal{V}_i)$. Consequently, $\mathbb{E}(\bigvee \mathcal{V}_i) \subseteq \bigvee \mathbb{E}\mathcal{V}_i$.

Let $\mathcal{V} \in \mathcal{L}(\mathcal{PS})$ be any variety of pseudosemilattices and let $F\mathcal{V}(X)$ be the free object in $\mathcal{V}$ on a countably infinte set X. Consider the free locally inverse semigroup $F\big(F\mathcal{V}(X)\big)$ with respect to the pseudosemilattice $F\mathcal{V}(X)$. Let $\mathcal{W}$ denote the e-variety of locally inverse semigroups generated by $F\big(F\mathcal{V}(X)\big)$. Then $\mathrm{E}\Big(F\big(F\mathcal{V}(X)\big)\Big) = F\mathcal{V}(X)$ so that $F\mathcal{V}(X) \in \mathbb{E}\mathcal{W}$. In particular, $\mathcal{V} \subseteq \mathbb{E}\mathcal{W}$. Again by [16], if $S \in \mathcal{W}$ then S is a homomorphic image of a regular subsemigroup of a direct product of a collection of semigroups all of which being isomorphic to $F\big(F\mathcal{V}(X)\big)$. If T is a homomorphic image of a locally inverse semigroup U then the pseudosemilattice $\mathrm{E}(T)$ is a homomorphic image of the pseudosemilattice $\mathrm{E}(U)$. Applied to the above situation we observe that $\mathrm{E}(S)$ divides a direct product of isomorphic copies of $F\mathcal{V}(X)$. Hence $\mathrm{E}(S) \in \mathcal{V}$. In particular, $\mathbb{E}\mathcal{W} \subseteq \mathcal{V}$. Hence $\mathbb{E}\mathcal{W} = \mathcal{V}$ and $\mathbb{E}$ is surjective. ∎

Immediately we have the following corollaries.

COROLLARY 3.6. *For each variety $\mathcal{V}$ of pseudosemilattices there are unique e-varieties of locally inverse semigroups $\mathcal{V}_E$ and $\mathcal{V}^E$ such that $\mathbb{E}^{-1}\mathcal{V}$ is the interval sublattice $[\mathcal{V}_E, \mathcal{V}^E]$ in $\mathcal{L}_e(\mathcal{LI})$.*

COROLLARY 3.7. *Let $\mathcal{V} \in \mathcal{L}(\mathcal{PS})$ and let E be any pseudosemilattice which generates $\mathcal{V}$ (for instance, $E = F\mathcal{V}(X)$ for an infinite set X). Then $\mathcal{V}_E$ is the e-variety generated by $\overline{E}$.*

PROOF. Let $\mathcal{V} \in \mathcal{L}(\mathcal{PS})$ and $E \in \mathcal{V}$ such that E generates $\mathcal{V}$. As in the proof of Theorem 3(3), $\overline{E} \in \mathcal{U}$ for each $\mathcal{U} \in \mathcal{L}_e(\mathcal{LI})$ for which $\mathbb{E}\mathcal{U} = \mathcal{V}$. Hence $\overline{E} \in \mathcal{V}_E$ and thus $\mathcal{V}_E$ contains the e-variety $\mathcal{W} = \langle \overline{E} \rangle$ which is generated by $\overline{E}$. Then also $\mathbb{E}\mathcal{W} \subseteq \mathbb{E}\mathcal{V}_E = \mathcal{V}$ since $\mathbb{E}$ is monotone. On the other hand, $E \in \mathbb{E}\mathcal{W}$ so that $\mathcal{V} \subseteq \mathbb{E}\mathcal{W}$ and thus $\mathcal{V} = \mathbb{E}\mathcal{W}$. By definition, $\mathcal{V}_E$ is the least e-variety $\mathcal{U}$ such that $\mathbb{E}\mathcal{U} = \mathcal{V}$. Therefore, we also have the reverse inclusion $\mathcal{V}_E \subseteq \mathcal{W}$. ∎

COROLLARY 3.8. *Let $\mathcal{V} \in \mathcal{L}(\mathcal{PS})$ and $B(\mathcal{V})$ be a basis of identities for $\mathcal{V}$. Denote by $\mathcal{LI}\big(B(\mathcal{V})\big)$ the class of all locally inverse semigroups which satisfy all identities from $B(\mathcal{V})$ (according to Definition 5). Then $\mathcal{V}^E = \mathcal{LI}\big(B(\mathcal{V})\big)$.*

PROOF. It is easy to see that $\mathcal{LI}\big(B(\mathcal{V})\big)$ is an e-variety. Furthermore, $\mathbb{E}\mathcal{LI}\big(B(\mathcal{V})\big) \subseteq \mathcal{V}$ since by definition all members of $\mathbb{E}\mathcal{LI}\big(B(\mathcal{V})\big)$ satisfy all identities of $B(\mathcal{V})$ and therefore all identities which hold in $\mathcal{V}$. On the other hand, $F\big(F\mathcal{V}(X)\big) \in \mathcal{LI}\big(B(\mathcal{V})\big)$ where $F\mathcal{V}(X)$ is the relatively free object on X in $\mathcal{V}$ and $F\big(F\mathcal{V}(X)\big)$ is as in Definition 2. Then $F\mathcal{V}(X) \in \mathbb{E}\mathcal{LI}\big(B(\mathcal{V})\big)$ and thus $\mathcal{V} \subseteq \mathbb{E}\mathcal{LI}\big(B(\mathcal{V})\big)$ since $\mathcal{V}$ is generated by $F\mathcal{V}(X)$ for any infinite set X. We have $\mathbb{E}\mathcal{LI}\big(B(\mathcal{V})\big) = \mathcal{V}$ and thus $\mathcal{LI}\big(B(\mathcal{V})\big) \subseteq \mathcal{V}^E$. Conversely, let $S \in \mathcal{V}^E$; then $\mathrm{E}(S) \in \mathcal{V}$ and thus S satisfies all identities of $\mathcal{V}$. In particular $S \in \mathcal{LI}\big(B(\mathcal{V})\big)$ and we have the reverse inclusion $\mathcal{V}^E \subseteq \mathcal{LI}\big(B(\mathcal{V})\big)$, too. ∎

Little is known about the lattice $\mathcal{L}(\mathcal{PS})$ of varieties of pseudosemilattices. By Schein [15], a pseudosemilattice is associative if and only if it is a normal band. Hence $\mathcal{L}(\mathcal{PS})$ contains the well known lattice of varieties of normal bands $\mathcal{L}(\mathcal{NB})$ as an ideal sublattice. $\mathcal{L}(\mathcal{NB})$ consists of the following eight varieties:

$\mathcal{T}$ trivial semigroups

$\mathcal{LZ}$ left zero semigroups

$\mathcal{RZ}$ right zero semigroups

$\mathcal{S}$ semilattices

$\mathcal{RB}$ rectangular bands

$\mathcal{LNB}$ left normal bands

$\mathcal{RNB}$ right normal bands

$\mathcal{NB}$ normal bands

For each $\mathcal{V} \in \mathcal{L}(\mathcal{NB})$, $\mathcal{V}_E = \mathcal{V}$ (the $\mathcal{V}$ on the right hand side being considered as an e-variety of locally inverse semigroups rather than a variety of bands). The corresponding

e-varieties $\mathcal{V}^E$ can be seen to be the following:

$$\mathcal{T}^E = \mathcal{G} \text{ (groups)}$$
$$\mathcal{LZ}^E = \mathcal{LG} \text{ (left groups)}$$
$$\mathcal{RZ}^E = \mathcal{RG} \text{ (right groups)}$$
$$\mathcal{S}^E = \mathcal{I} \text{ (inverse semigroups)}$$
$$\mathcal{RB}^E = \mathcal{CS} \text{ (completely simple semigroups)}$$
$$\mathcal{LNB}^E = \mathcal{LGI} \text{ (left generalized inverse semigroups, see Hall [4])}$$
$$\mathcal{RNB}^E = \mathcal{RGI} \text{ (right generalized inverse semigroups, see Hall [4])}$$
$$\mathcal{NB}^E = \mathcal{ESLI} \text{ (E-solid locally inverse semigroups)}$$

The lattice of e-varieties of E-solid locally inverse semigroups $\mathcal{L}_e(\mathcal{ESLI})$ is therefore the disjoint union of the intervals $[\mathcal{T}, \mathcal{G}]$, $[\mathcal{LZ}, \mathcal{LG}]$, $[\mathcal{RZ}, \mathcal{RG}]$, $[\mathcal{S}, \mathcal{I}]$, $[\mathcal{RB}, \mathcal{CS}]$, $[\mathcal{LNB}, \mathcal{LGI}]$, $[\mathcal{RNB}, \mathcal{RGI}]$, $[\mathcal{NB}, \mathcal{ESLI}]$.

REFERENCES

1. K. Auinger, *The bifree locally inverse semigroup on a set*, J. Algebra, to appear.

2. G. Grätzer, *Universal Algebra*, Van Nostrand, New York, 1968.

3. T. E. Hall, *Identities for existence varieties of regular semigroups*, Bull. Austral. Math. Soc. **40**(1989), 59–77.

4. ______, *Regular semigroups: amalgamation and the lattice of existence varieties*, Algebra Universalis **29**(1991), 79–108.

5. ______, *A concept of variety for regular semigroups*, Proceedings 1990 Monash Conference.

6. J. Kaďourek and M. B. Szendrei, *A new approach in the theory of orthodox semigroups*, Semigroup Forum **40**(1990), 257–296.

7. J. Meakin, *The free local semilattice on a set*, J. Pure Appl. Algebra **27**(1983), 263–273.

8. J. Meakin and F. Pastijn, *The structure of pseudosemilattices*, Algebra Universalis **13**(1981), 355–372.

9. K. S. S. Nambooripad, *Structure of regular semigroups*, Mem. Amer. Math. Soc. **224**(1979).

10. ______, *Pseudosemilattices and biordered sets I*, Simon Stevin **55**(1981), 103–110.

11. ______, *The natural partial order on a regular semigroup*, Proc. Edinburgh Math. Soc. **23**(1980), 249–260.

12. F. Pastijn, *Rectangular bands of inverse semigroups*, Simon Stevin **56**(1982), 1–97.

13. ______, *The structure of pseudo-inverse semigroups*, Trans. Amer. Math. Soc. **273**(1982), 631–655.

14. F. Pastijn and M. Petrich, *Congruences on regular semigroups*, Trans. Amer. Math. Soc. **295**(1986), 607–633.

15. B. M. Schein, *Pseudosemilattices and pseudolattices*, (Russian) Izv. Vyssh. Uchebn. Zaved. Mat. **117** (1972), 81–94; English Trans. Amer. Math. Soc. **119**(1983), 1–16.

16. Y. T. Yeh, *The existence of e-free objects in e-varieties of regular semigroups*, Internat. J. Algebra Comput., **2**(1992), 471–484.

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