AXES OF POINTS OF CERTAIN LINEAR SYSTEMS OF POLARITIES

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To H. S. M. Coxeter on his sixtieth birthday

1. Introduction. In this issue honouring Professor Coxeter, I am pleased to present some results of an investigation that was prompted by questions which he himself raised over a decade ago.

With respect to a linear system of polarities in complex projective three-space, the polars of a fixed point Q form an axial pencil of planes. The axis of the pencil is called *the axis of point Q with respect to the linear system of polarities*. Since there are ∞^3 axes and ∞^4 lines in the space, not every line is an axis. The following discussion answers the questions of how many and which lines are axes with respect to the linear systems of polarities that have a fixed self-polar tetrahedron. The axes of points of fixed planes are also discussed. In general, the axes are determined as belonging to pencils, bundles, complexes, congruences, and quadric cones, and these are located relative to the invariant self-polar tetrahedron.

2. Preliminaries. A convenient way to specify a polarity in complex projective three-space (S_3) is by means of a self-polar tetrahedron and another pair of corresponding elements. If polarity Γ has a self-polar tetrahedron $ABCD$ and a pair of corresponding elements, point P and plane π, then Γ is completely determined by this information. We write

$$\Gamma = (ABCD)(P\pi).$$

In referring to the planes of the self-polar tetrahedron $ABCD$, we adopt the following notation:

$$\alpha = BCD, \qquad \beta = ACD, \qquad \gamma = ABD, \qquad \delta = ABC.$$

Thus, Γ establishes the correspondence

$$A \leftrightarrow \alpha, \qquad B \leftrightarrow \beta, \qquad C \leftrightarrow \gamma, \qquad D \leftrightarrow \delta.$$

We shall also require the following linear construction (**2**; or **3**, p. 124) for the polar plane χ of an arbitrary point X with respect to polarity $(ABCD)$ $(P\pi)$ (see Fig. 1). Writing

$$DP \cdot \delta = P_\delta, \qquad DX \cdot \delta = X_\delta, \qquad \text{and } \pi \cdot \delta = p_\delta,$$

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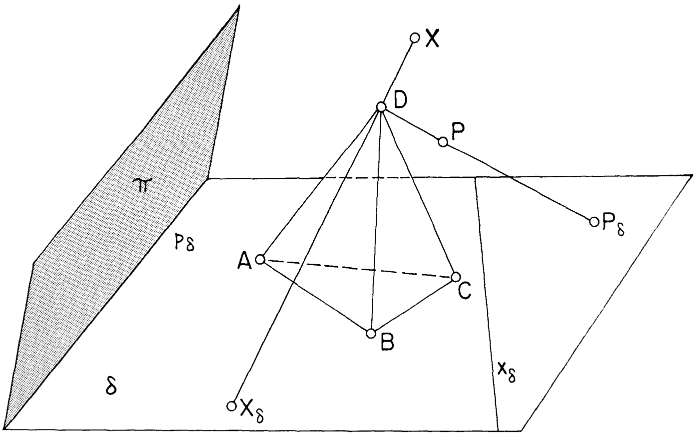

FIGURE 1

we note that DP is the polar line of p_δ. Hence, the polarity induced in plane δ is $(ABC)(P_\delta\, p_\delta)$. Using Coxeter's construction (**1**, 5.64), the polar of X_δ in plane δ is line

$$x_\delta = [AP_\delta \cdot (a \cdot P_\delta X_\delta)(p_\delta \cdot AX_\delta)][BP_\delta \cdot (b \cdot P_\delta X_\delta)(p_\delta \cdot BX_\delta)].$$

Performing a similar construction in plane α yields the polar x_α of the point X_α. The plane determined by $x_\alpha\, x_\delta$ is χ, the polar of X. In order to prove this, it is sufficient to remark that X is conjugate to every point on x_α and to every point on x_δ.

We shall be studying linear systems of polarities in which points A, B, C, D, and P are fixed, while plane π varies in a pencil through line l_P. There will be four cases, according as l_P meets zero, one, two, or three edges of $ABCD$. These will be denoted $\mathscr{P}_0, \mathscr{P}_1, \mathscr{P}_2, \mathscr{P}_3$, respectively.

The final convention we establish is the following: l_X denotes the axis of point X. This is consistent with calling l_P the axis of the pencil of planes π.

3. The $\mathscr{P}_0$ system. Let θ be any plane through AD other than β or γ. As X varies on θ, $X_\alpha = AX \cdot \alpha$ varies on a line through D. Focusing on the polarities induced in plane α, we use Coxeter's result (**2**, p. 96) that $X'_\alpha = l_X \cdot \alpha$ also varies on a line through D. Similarly, X'_δ varies on a line through A.

If X is in general position, l_X does not meet any edge of $ABCD$ in a $\mathscr{P}_0$ system of polarities (**3**, p. 124). Thus $X'_\alpha \neq X'_\delta$ and $l_X = X'_\alpha X'_\delta$, and we have located all axes of points in general position.

If X is on a face, say α, of $ABCD$, then it is clear that l_X consists of the bundle of lines through A.

This provides our first characterization of the set of axes.

THEOREM 1. *The axes l_X of points X, in general position, with respect to a $\mathscr{P}_0$ system may be regarded as the lines of plane pencils, each centre of which is a point* *on a face, say α, of the fixed self-polar tetrahedron $ABCD$, and each plane of which meets the opposite vertex, in this case A. The axes of points in a face constitute a bundle of lines through the opposite vertex.*

Since the set of axes of points in general position have been characterized as sets of plane pencils, it is reasonable to conjecture that they constitute a linear complex "modulo" some aberration resulting from points on the self-polar tetrahedron. However, this conjecture is proved false by the following theorem, because the lines of a complex through every point of space form a plane pencil.

THEOREM 2. *The axes through any point in general position constitute a cone of lines through the vertices of the fixed self-polar tetrahedron.*

Proof. Let X be a point in general position and l_X its axis. Consider two points, Q and R on l_X. Their polar planes, with respect to a polarity of the system, intersect in the polar line of l_X with respect to this polarity. As the polarity varies in the linear system the polar planes of Q and R describe projective pencils about l_Q and l_R, respectively. Thus, the lines of intersection of corresponding planes—that is, the polar lines of l_X—describe a cone of lines whose vertex is X. But l_Q and l_R are also lines of the cone, as are all the axes of points on l_X.

To show that the cone passes through the vertices of the self-polar tetrahedron, we consider the cone as generated by the polar lines of l_X. Once again, we write $X'_\alpha = l_X \cdot \alpha$ and $X'_\delta = l_X \cdot \delta$. As the polarity varies in the system, the polar of X'_δ in plane δ varies in a pencil about a point X_δ. And, in every polarity, the polar planes of X'_α pass through A. Therefore, the cone of lines meets plane δ in a conic which is generated as the intersection of corresponding lines of projective pencils through X_δ and A. Hence, A lies on the cone; similarly, B, C, and D are also on the cone.

4. The $\mathscr{P}_1$ system. Let us suppose that l_P is on AB. Then self-dual systems of polarities are induced on planes ABC and ABD (**1**, p. 76), which implies that every axis meets AB.

We proceed toward a more complete description of the ∞^3 axes.

Every polarity of the system induces the same involution of points on AB. Hence a point R on AB possesses the same polar plane ρ for every polarity of the system; in fact, $\rho = AB \cdot R'$, where R' is defined by $H(AB, RR')$. This means that the points R of AB do not have axes associated with them. The points on the remaining five edges of $ABCD$ have their respective opposite edges as axes. This is consistent with the fact that all axes meet AB, for the five edges other than CD (which is opposite AB) all meet AB. In particular, it is important to note that AB itself is an axis.

Referring to Figure 2, let X be a point in general position on a plane θ through CD. All X on θ give rise to axes that pass through the point X'_δ which is defined by

$$\theta \cdot AB = Y \quad \text{and} \quad H(AB, YX'_\delta).$$

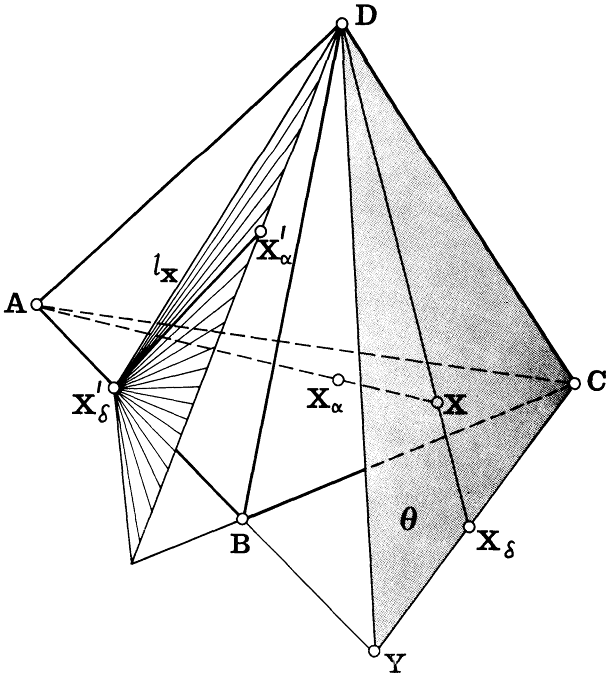

FIGURE 2

As X varies on a line (in θ) through D, $X_\alpha = AX \cdot \alpha$ also varies on a line through D; this again establishes X'_α on a line through D. Thus the axes $l_X = X'_\alpha X'_\delta$ constitute a pencil of lines with centre X'_δ, with the plane of the pencil passing through D. By considering all points of θ to be on lines through D, we see that the axes of the points on θ form a bundle of lines with centre X'_δ. But X'_δ is the harmonic conjugate of $\theta \cdot AB$ with respect to A and B; so each point of AB can be obtained as the centre of a bundle of axes by selecting different positions for θ. Hence,

THEOREM 3. *The set of axes of a $\mathscr{P}_1$ system of polarities is a special linear complex of lines that all pass through an edge of the fixed self-polar tetrahedron.*

The axes of points of a fixed plane in general position was characterized in (**3**, p. 128) for a $\mathscr{P}_0$ system: they are the ∞^2 chords of the twisted cubic of poles of the fixed plane with respect to the $\mathscr{P}_0$ system. We now present a similar characterization for the $\mathscr{P}_1$ system.

THEOREM 4. *Relative to a $\mathscr{P}_1$ system, the axes of points of a fixed plane in general position can be characterized as the ∞^2 lines which are generators of cones whose vertices are on one edge of the self-polar tetrahedron and which meet the conic that is the locus of poles of the fixed plane. Only one of the cones degenerates to a flat pencil.*

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Proof. Let the $\mathscr{P}_0$ system be defined by l_P meeting AB. Call τ the fixed plane and T a general point on it. We shall obtain our characterization of axes by considering the points of τ as ranges $t = CDT \cdot \tau$. As usual, we write $DT \cdot \delta = T_\delta$. The axis l_T of T meets AB in T'_δ. Thus, every point of t has its axis meeting T'_δ. As T varies on t, $T_\alpha = AT \cdot \alpha$ varies on a line t_α. This implies that $T'_\alpha = l_T \cdot \alpha$ varies in a conic through B, C, and D, as was shown by Coxeter **(1**, 6.81**)**. Therefore, the axes of the points of t form a quadric cone with vertex T'_δ. The cone meets C and D, and also A and B since line AB is a generator.

Let R and S be two points of τ, such that l_R and l_S are skew. Clearly, there exist such pairs of points. As the polarity varies in the $\mathscr{P}_1$ system, the polar planes of R and S vary in projective pencils. Hence, their lines of intersection form a regulus, with l_R and l_S members of the conjugate regulus. Lines l_R and l_S are therefore embedded in the quadric surface generated. The locus of poles of τ is a conic **(3**, p. 126**)** which is a section of the regulus. Hence, we conclude that the locus of poles of τ is a section of the quadric cone of axes of points on τ.

We have argued with T, and consequently t, in general position on plane τ. Special cases can arise only if t_α passes through one of the vertices B, C, or D. The last two of these possibilities cannot occur as long as τ is in general position; however, it is possible for t_α to be incident with B. This occurs when t meets AB, so that T'_δ is the companion of $\tau \cdot AB$ in the involution on AB. T'_α also varies on a line t'_α through B, so the axes of points on t are the lines of a pencil with vertex T'_δ in plane $T'_\delta\, t_\alpha$. It is not surprising that this special case occurs precisely when T'_δ is in the plane of the conic formed by the poles of τ. In fact, T'_δ is a point of the conic **(3**, pp. 126–127**)**.

In the event that τ is in special position, say incident with point C, then reasoning similar to the foregoing shows that the axes of points on τ are the lines of pencils whose vertices (T'_δ) are on AB and whose planes meet C. The same characterization of axes also holds if τ passes through the line CD.

5. The $\mathscr{P}_2$ system. Let the linear system of polarities be defined by having l_P meet AB and CD. Then all the axes must meet both AB and CD. We shall now show that every line meeting AB and CD is an axis.

Any point not on AB or CD can be located on a line l that meets AB and CD. Let $l \cdot AB = E$ and $l \cdot BC = F$. Then the axis of every point on l is line $E'F'$ where $H(AB, EE')$ and $H(CD, FF')$. Clearly, each point on AB other than A and B can be obtained as an E'; similarly for the points of CD. But A, B, C, and D can also be obtained as points of axes, as the following argument shows. There are ∞^2 choices for self-polar tetrahedra that can be used to define the system **(3**, p. 125**)**. Any tetrahedron $\bar{A}\bar{B}\bar{C}\bar{D}$ is self-polar for all polarities of the system if $\bar{A}\bar{B}$ is a pair of the unique involution induced on AB, and $\bar{C}\bar{D}$ is a pair of the unique involution induced on CD. Thus, if some tetrahedron with vertices entirely different from $ABCD$ is used to define the system, then the foregoing proves that A, B, C, and D lie on axes. In addition, the lines AB and CD are also axes of points (on the opposite edges). This gives

THEOREM 5. *The $\mathscr{P}_2$ system admits only ∞^2 lines as axes of points. These form a general linear congruence.*

COROLLARY. *If the $\mathscr{P}_2$ system is defined by $(ABCD)$ $(P\pi)$, with π varying about l_P such that l_P meets AB and CD, then AB and CD are the directrices of the linear congruence of axes.*

If τ is a fixed plane and the $\mathscr{P}_2$ system is defined as in the previous proof, then the locus of poles of τ is a line t which meets both AB and CD. Line t must be incident with each axis l_T of point T on τ. Hence, the axes of all points of τ are the lines which meet AB, CD, and t simultaneously. Hence, we have

THEOREM 6. *Relative to a $\mathscr{P}_2$ system, the axes of points of a fixed plane τ form a degenerate regulus consisting of two flat pencils in different planes that intersect in the locus of poles of τ.*

(In the $\mathscr{P}_2$ system we have described, the pencil in ABt has vertex $AB \cdot t$ and the pencil in CDt has vertex $CD \cdot t$.)

6. The $\mathscr{P}_3$ system. If the system is defined by having l_P in plane δ, then the polarity induced in δ is the same for every polarity in the system. The points of δ do not have axes, but every other point of S_3 has an axis in δ. The fact that every line y in δ is an axis is easily shown. Call Y_δ the pole of y in the polarity induced in δ. Then every point of DY_δ (other than Y_δ) has y as its axis. Hence,

THEOREM 7. *A $\mathscr{P}_3$ system of polarities admits only ∞^2 lines as axes, namely the entire set of lines in the face of the self-polar tetrahedron in which the polarity is the same for each member of the system.*

All possible positions for axes, relative to a $\mathscr{P}_3$ system, are exhausted by the axes of points of a single plane. This is established in the proof of the following theorem.

THEOREM 8. *Relative to a $\mathscr{P}_3$ system $(ABCD)$ $(P\pi)$ in which l_P is in plane δ, the axes of points of a fixed plane in general position are the ∞^2 lines of δ.*

Proof. Let y be an arbitrary line in δ, and Y_δ the pole of y with respect to the polarity induced in δ. If the fixed plane τ is in general position, then Y_δ can always be obtained as the projection of a point of τ from centre D. This point has y as its axis.

The proof of Theorem 8 is not valid if τ meets D, in which case the axes form a pencil of lines in δ. The vertex of the pencil is the pole of $\tau \cdot \delta$ with respect to the polarity in δ.

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