SPACE FORMS OF GRASSMANN MANIFOLDS

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1. Introduction. We shall consider the classification problem for space forms of (Riemannian manifolds which are covered by) real, complex, and quaternionic Grassmann manifolds. In the particular case of the real Grassmann manifold of oriented 1-dimensional subspaces of a real Euclidean space, this is the classical "spherical space form problem" of Clifford and Klein. We shall not consider space forms of the Cayley projective plane because it is easy to see that there are no non-trivial ones.

The space forms are completely determined for the quaternionic (Theorem 1), complex (Theorem 2), and even-dimensional real (Theorem 3) Grassmann manifolds; the problem of determining the space forms of the odd-dimensional real Grassmann manifolds is reduced (Theorems 4 and 5) to the spherical space form problem. While our methods are quite elementary, they rest on some elaborate results from the theory of finite groups **(2, 5** and **6)** and the theory of Lie groups **(1** and **4)**. A few of our shorter calculations parallel calculations in **(4)**, but are given here in more suitable form for the convenience of the reader.

2. Definitions and conventions. In order to establish definitions and notation, we recall some well-known facts concerning Grassmann manifolds.

2.1. Let F be one of the fields $\mathbf{R}$ (real), $\mathbf{C}$ (complex), or $\mathbf{H}$ (quaternion). Given integers $0 < q < n$, $\mathbf{G}_{q,n}(F)$ denotes the *Grassmann manifold* over F consisting of all q-dimensional subspaces of the left vector space F^n of dimension n over F, both the subspaces and F^n being oriented if $F = \mathbf{R}$. We endow F^n with positive definite Hermitian (relative to the conjugation of F over $\mathbf{R}$) form, and let $\mathbf{U}(n, F)$ denote the group of all linear transformations of F^n which preserve this form. $\mathbf{U}(n, F)$ is the orthogonal group $\mathbf{O}(n)$ if $F = \mathbf{R}$, the unitary group $\mathbf{U}(n)$ if $F = \mathbf{C}$, or the symplectic group $\mathbf{Sp}(n)$ if $F = \mathbf{H}$. Acting linearly on F^n, $\mathbf{U}(n, F)$ acts on $\mathbf{G}_{q,n}(F)$ in a natural fashion; it is well known that $\mathbf{U}(n, F)$ acts transitively on the elements of $\mathbf{G}_{q,n}(F)$. This gives $\mathbf{G}_{q,n}(F)$ a topology and the structure of a differentiable manifold, for **(7,** § 4.11**)** $\mathbf{U}(n, F)$ is a Lie group and we can identify $\mathbf{G}_{q,n}(F)$ with a coset space of $\mathbf{U}(n, F)$ by choosing $P \in \mathbf{G}_{q,n}(F)$ and sending $g \in \mathbf{U}(n, F)$ to $g(P)$; excluding $\mathbf{G}_{2,4}(\mathbf{R})$, the isotropy subgroup $\{g \in \mathbf{U}(n, F) : g(P) = P\}$ acts irreducibly on the tangent space to $\mathbf{G}_{q,n}(F)$ at P, so any two $\mathbf{U}(n, F)$-invariant Riemannian metrics (differentiable fields of positive definite inner products on the tangent

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spaces) on $\mathbf{G}_{q,n}(F)$ are proportional (**1**). Thus we may regard $\mathbf{G}_{q,n}(F)$ as a Riemannian manifold (differentiable manifold together with a Riemannian metric) in an essentially canonical fashion.

We remark that $\mathbf{G}_{q,n}(\mathbf{R})$ is a double covering of the manifold of non-oriented q-dimensional subspaces of $\mathbf{R}^n$, and is often called the *oriented* real Grassmann manifold. $\mathbf{G}_{1,n}(\mathbf{R})$ is the sphere $\mathbf{S}^{n-1}$, and the associated non-oriented Grassmann manifold is the real projective space $\mathbf{P}^{n-1}(\mathbf{R})$. $\mathbf{G}_{1,n}(\mathbf{C})$ is the complex projective space $\mathbf{P}^{n-1}(\mathbf{C})$, and $\mathbf{G}_{1,n}(\mathbf{H})$ is the quaternionic projective space $\mathbf{P}^{n-1}(\mathbf{H})$.

Representing the manifolds $\mathbf{G}_{q,n}(F)$ as coset spaces, one easily checks that they are simply connected, except for $\mathbf{G}_{1,2}(\mathbf{R})$. If r is the dimension of F over $\mathbf{R}$, then $\mathbf{G}_{q,n}(F)$ has dimension $rq(n-q)$.

2.2. If M is a Riemannian manifold, then an *isometry* of M is a differentiable homeomorphism $f : M \to M$ whose *tangent maps* $f_* : M_x \to M_{f(x)}$ are linear isometries, where M_x denotes the tangent space to M at $x \in M$. The set of all isometries of M forms a Lie group, denoted $\mathbf{I}(M)$ and called the *full group of isometries* of M; the identity component, denoted $\mathbf{I}_0(M)$, is called the *connected group of isometries* of M.

Our choice of Riemannian metric for $\mathbf{G}_{q,n}(F)$ shows that the transformation by any element of $\mathbf{U}(n, F)$ is an isometry. Replacing $\mathbf{U}(n, F)$ with its quotient by the normal subgroup consisting of all elements that induce the identity transformation, we obtain a subgroup of $\mathbf{I}(\mathbf{G}_{q,n}(F))$ which contains $\mathbf{I}_0(\mathbf{G}_{q,n}(F))$. Calculation of $\mathbf{I}(\mathbf{G}_{q,n}(F))$ is not immediate, but É. Cartan has given a very clever method for making this calculation (**1**; see **4**, § 2.4 for an exposition of this method, **4**, §§ 5.5.5, 5.5.7–8, 5.5.11 for an exposition of the calculations of the explicit forms of the $\mathbf{I}(\mathbf{G}_{q,n}(F))$). We shall draw on these results as we need them.

It will sometimes be convenient to replace $\mathbf{G}_{q,n}(F)$ by $\mathbf{G}_{n-q,n}(F)$. This can be done without difficulty because we have, for appropriate normalizations of the metrics, an isometry $\beta : \mathbf{G}_{q,n}(F) \to \mathbf{G}_{n-q,n}(F)$ given by $\beta(P) = P^\perp$, where $P^\perp$ denotes the orthogonal complement of P in F^n; in case $F = \mathbf{R}$, $P^\perp$ is oriented such that the Grassmann product $P \wedge P^\perp$ gives the original orientation of $\mathbf{R}^n$. In particular, $\beta \in \mathbf{I}(\mathbf{G}_{q,2q}(F))$.

2.3. A *Riemannian covering* is a covering $\pi : M \to N$ where M and N are connected Riemannian manifolds and π is a local isometry; then every deck transformation of the covering (homeomorphism $\gamma : M \to M$ with $\pi = \pi \cdot \gamma$) is an isometry of M, the group Γ of all deck transformations is a discrete subgroup of $\mathbf{I}(M)$ which acts freely (the identity element $1 \in \Gamma$ is the only element of Γ which has a fixed point on M), and, if M is simply connected, then N can be identified with the quotient M/Γ. Conversely, if Δ is a discrete subgroup of $\mathbf{I}(M)$ which acts freely on M, then M/Δ has a unique Riemannian structure such that the projection $M \to M/\Delta$ is a Riemannian covering.

We shall study the manifolds that admit a Riemannian covering by a

Grassmann manifold; thus the classification problem for such manifolds is reduced to a problem on discrete subgroups of $\mathbf{I}(\mathbf{G}_{q,n}(F))$. For Grassmann manifolds $\mathbf{G}_{1,n}(\mathbf{R})$ this is, of course, the spherical space form problem of Clifford and Klein.

3. Quaternionic Grassmann manifolds. Let M be the quaternionic Grassmann manifold $\mathbf{G}_{q,n}(\mathbf{H})$ consisting of all q-dimensional subspaces of $\mathbf{H}^n$. Then $\mathbf{U}(n, \mathbf{H})/\{\pm I\} = \mathbf{Sp}(n)/\{\pm I\}$ (I = identity matrix) is the connected group of isometries $\mathbf{I}_0(M)$; $\mathbf{I}(M) = \mathbf{I}_0(M)$ if $q \neq n - q$, and

$$\mathbf{I}(M) = \mathbf{I}_0(M) \cup \beta \cdot \mathbf{I}_0(M)$$

if $q = n - q$ (recall $\beta : P \to P^\perp$). $\mathbf{I}_0(M)$ has isotropy subgroup

$$\{\mathbf{Sp}(q) \times \mathbf{Sp}(n-q)\}/\{\pm I\}$$

which contains a maximal torus, whence **(3**, Th. **4)** every element of $\mathbf{I}_0(M)$ has a fixed point on M.

Let Γ be a subgroup of $\mathbf{I}(M)$ which acts freely on M. Γ has at most one element in each component of $\mathbf{I}(M)$ because $\Gamma \cap \mathbf{I}_0(M) = \{1\}$. Thus $\Gamma = \{1\}$ if $q \neq n - q$, and Γ is either $\{1\}$ or of the form $\{1, \beta g\}$ if $q = n - q$. In the latter case, $(\beta g)^2 \in \Gamma \cap \mathbf{I}_0(M)$, giving $1 = \beta g \beta g = \beta^2 g^2 = g^2$; in terms of linear transformations, $g^2 = \pm I$. $g^2 = -I$ is impossible, for then g would have matrix diag. $\{\mathbf{i}, \mathbf{i}, \ldots, \mathbf{i}\}$ in some orthonormal basis $\{e_j\}$ of $\mathbf{H}^n$, where $\{1, \mathbf{i}, \mathbf{j}, \mathbf{k}\}$ is the usual basis of $\mathbf{H}$ over $\mathbf{R}$; then $v_j = e_j + \mathbf{j} \cdot e_{q+j}$ and $v_j g$ have inner product $\langle v_j, v_j \cdot g\rangle = \langle e_j + \mathbf{j} \cdot e_{q+j}, \mathbf{i} \cdot e_j + \mathbf{ji} \cdot e_{q+j}\rangle = 0$, so the subspace P of $\mathbf{H}^n$ with basis $\{v_1, \ldots, v_q\}$ satisfies $g(P) = P^\perp$, and is thus a fixed point for βg. Thus $g^2 = I$, and consequently g has matrix

$$\begin{pmatrix} I_u & \\ & -I_v \end{pmatrix}$$

in some orthonormal basis $\{f_j\}$ of $\mathbf{H}^n$, where $u + v = n$ and I_w denotes the $w \times w$ identity matrix. As $n = 2q$, βg has a fixed point on M if and only if $u = v$. Thus we may assume $v < u$.

We have now proved:

THEOREM 1. *Let N be a Riemannian manifold that admits a Riemannian covering by the quaternionic Grassmann manifold $\mathbf{G}_{q,n}(\mathbf{H})$. If $n \neq 2q$, then N is isometric to $\mathbf{G}_{q,n}(\mathbf{H})$. If $n = 2q$, then N is isometric to $\mathbf{G}_{q,n}(\mathbf{H})$ or to one of the q manifolds $\mathbf{G}_{q,n}(\mathbf{H})/\Delta_v$, where Δ_v is the subgroup $\{1, \beta g_v\}$ of $\mathbf{I}(\mathbf{G}_{q,n}(\mathbf{H}))$ and g_v has matrix*

$$\begin{pmatrix} I_{n-v} & \\ & -I_v \end{pmatrix}$$

with $0 \leqslant v < q$.

We remark that Δ_u and Δ_v are conjugate in $\mathbf{I}(\mathbf{G}_{q,2q}(\mathbf{H}))$ if and only if $u = v$, so the corresponding quotient manifolds of $\mathbf{G}_{q,2q}(\mathbf{H})$ are isometric if and only if $u = v$. Note also that Δ_u is cyclic of order 2.

4. Complex Grassmann manifolds. Let M be the complex Grassmann manifold $\mathbf{G}_{q,n}(\mathbf{C})$, consisting of all q-dimensional subspaces of $\mathbf{C}^n$. Then $\mathbf{I}_0(M)$ is the group $\mathbf{U}(n, \mathbf{C})/\{\text{scalars}\} = \mathbf{U}(n)/\{\text{scalars}\} = \mathbf{SU}(n)/\{\exp(2\pi\iota k/n)I\}$, where $\iota^2 = -1$, and where $\mathbf{SU}(n)$ is the special (determinant 1) unitary group. Choosing an orthonormal basis $\{e_j\}$ of $\mathbf{C}^n$, the conjugation of $\mathbf{C}$ over $\mathbf{R}$ induces a conjugation of $\mathbf{C}^n$ which sends q-dimensional subspaces into q-dimensional subspaces, and thus induces a transformation $\alpha : M \to M$.

$$\mathbf{I}(M) = \mathbf{I}_0(M) \cup \alpha \cdot \mathbf{I}_0(M)$$

if $q \neq n - q$, and $\mathbf{I}(M) = \mathbf{I}_0(M) \cup \alpha\cdot\mathbf{I}_0(M) \cup \beta\cdot\mathbf{I}_0(M) \cup \beta\alpha\cdot\mathbf{I}_0(M)$ if $q = n - q$. $\mathbf{I}_0(M)$ has isotropy subgroup

$$\{\mathbf{SU}(n) \cap [\mathbf{U}(q) \times \mathbf{U}(n-q)]\}/\{\exp(2\pi\iota k/n)I\}$$

which contains a maximal torus, whence (**3**, Th. 4) every element of $\mathbf{I}_0(M)$ has a fixed point on M.

Let Γ be a subgroup of $\mathbf{I}(M)$ which acts freely on M. $\Gamma \cap \mathbf{I}_0(M) = \{1\}$, so Γ has at most one element in each component of $\mathbf{I}(M)$.

Let $\gamma \in \Gamma \cap \alpha\cdot\mathbf{I}_0(M)$; $\gamma = \alpha g$ with $g \in \mathbf{I}_0(M)$. As in (**4**, § 5.5.5), $\gamma^2 \in \Gamma \cap \mathbf{I}_0(M)$, whence, in terms of linear transformations of $\mathbf{C}^n$, $cI = \alpha g \alpha g = \alpha^2 \cdot {}^t g^{-1} \cdot g = {}^t g^{-1}\cdot g$ for some $c \in \mathbf{C}$ with $c^n = 1$; $g = \exp(X)$ and $cI = \exp(Y)$ where X is skew-Hermitian and Y is pure imaginary and scalar, and we have

$$\exp({}^tX + Y) = c\,{}^tg = g = \exp(X).$$

Thus $\exp(X - {}^tX - Y) = I = {}^tI = \exp({}^tX - X - Y)$, and it follows that $c^2 I = \exp(2\,{}^tX - 2X) = I$, so ${}^tg = \pm g$. Now $g \neq {}^tg$, for we could then choose $h \in \mathbf{SU}(n)$ with $hg\,{}^th = I$; then

$${}^th^{-1}\cdot\gamma\cdot{}^th = {}^th^{-1}\alpha h^{-1} hg\,{}^th = {}^th^{-1}\alpha h^{-1} = {}^th^{-1}\cdot{}^th\alpha = \alpha$$

leaves fixed the q-dimensional subspaces of $\mathbf{C}^n$ with basis $\{e_1, \ldots, e_q\}$, and it follows that γ has a fixed point on M. Thus ${}^tg = -g$, so $n = 2m$ for some integer m and we can choose $h \in \mathbf{SU}(n)$ with

$$hg\,{}^th = J = \begin{pmatrix} 0 & I_m \\ -I_m & 0 \end{pmatrix}.$$

Replacing Γ by its conjugate ${}^th^{-1}\cdot\Gamma\cdot{}^th$, we replace γ by

$${}^th^{-1}\cdot\alpha h^{-1}hg\,{}^th = {}^th^{-1}\cdot{}^th\alpha J = \alpha J;$$

thus we may assume $g = J$. q is odd, for if q were even (say $q = 2t$) then the subspace spanned by $\{e_1, e_{m+1}, \ldots, e_t, e_{m+t}\}$ would be a fixed point for γ on M; this implies that $n - q$ is odd, for n is even. On the other hand, if q and $n - q$ are odd, it is clear that αJ has no fixed point on M.

Now suppose $q = n - q$ and let $\gamma \in \Gamma \cap \beta\alpha\cdot\mathbf{I}_0(M)$. $\gamma = \beta\alpha g$ with $g \in \mathbf{I}_0(M)$ and $\gamma^2 = \beta\alpha g\beta\alpha g = (\alpha g)^2 \in \Gamma \cap \mathbf{I}_0(M)$. Using the fact that β is central in

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$\mathbf{I}(M)$ we see that, as before, in terms of linear transformations of $\mathbf{C}^n$, we may replace Γ by a conjugate and assume either that $g = I$ or that $g = J$. The latter is impossible, for (note that $q = m$) the subspace of $\mathbf{C}^n$ with basis $\{e_1, \ldots, e_m\}$ would be a fixed point for $\gamma = \beta\alpha J$ on M, and the former is impossible because the subspace of $\mathbf{C}^n$ with basis $\{e_1 + \iota e_{m+1}, \ldots, e_m + \iota e_{2m}\}$ would be a fixed point for $\gamma = \beta\alpha$ on M. Thus Γ has no element in $\beta\alpha \cdot \mathbf{I}_0(M)$.

Let $q = n - q$ and $\gamma \in \Gamma \cap \beta \cdot \mathbf{I}_0(M)$; $\gamma = \beta g$ with $g \in \mathbf{I}_0(M)$ and $\gamma^2 = g^2 \in \Gamma \cap \mathbf{I}_0(M)$. In terms of linear transformations, $g \in \mathbf{SU}(n)$ with $g^2 = \exp(2\pi\iota k/n)I$. We replace g by $\exp(2\pi\iota l/n)g$, thus replacing g^2 by $\exp(2\pi\iota(k + 2l)/n)I$, and assume that $k = 0$ or 1. As β is central in $\mathbf{I}(M)$, we replace g by $-g$ if necessary, conjugate γ by an element of $\mathbf{I}_0(M)$, and assume that

$$g = \begin{pmatrix} aI_u & \\ & -aI_v \end{pmatrix}$$

with $u + v = n$, $v \leqslant u$ and $a = \exp(\pi\iota k/n)$. Now n is even and $\det g = 1$, so u, v, and k have the same parity. It is clear that βg has a fixed point on M if and only if $u = v$.

Finally, observe that Γ lies in either $\mathbf{I}_0(M) \cup \alpha \cdot \mathbf{I}_0(M)$ or $\mathbf{I}_0(M) \cup \beta \cdot \mathbf{I}_0(M)$ because $\Gamma \cap \beta\alpha \cdot \mathbf{I}_0(M)$ is empty.

We have now proved:

THEOREM 2. *Let N be a Riemannian manifold that admits a Riemannian covering by the complex Grassmann manifold $\mathbf{G}_{q,n}(\mathbf{C})$. Then:*

(1) *N is isometric to $\mathbf{G}_{q,n}(\mathbf{C})$; or*

(2) *both q and $n - q$ are odd, and N is isometric to $\mathbf{G}_{q,n}(\mathbf{C})/\Lambda$ where Λ is the subgroup $\{1, \alpha J\}$ of $\mathbf{I}(\mathbf{G}_{q,n}(\mathbf{C}))$ and J has matrix*

$$\begin{pmatrix} 0 & I_m \\ -I_m & 0 \end{pmatrix}$$

with $n = 2m$; or

(3) *$2q = n$ and N is isometric to one of the $[\frac{1}{2}(q + 1)]$ manifolds $\mathbf{G}_{q,n}(\mathbf{C})/\Delta_{2v}$ where Δ_{2v} is the subgroup $\{1, \beta g_{2v}\}$ of $\mathbf{I}(\mathbf{G}_{q,n}(\mathbf{C}))$, and g_{2v} has matrix*

$$\begin{pmatrix} I_{n-2v} & \\ & -I_{2v} \end{pmatrix}$$

with $0 \leqslant 2v < q$; or

(4) *$2q = n$ and N is isometric to one of the $[\frac{1}{2}(q + 1)]$ manifolds $\mathbf{G}_{q,n}(\mathbf{C})/\Psi_v$, where Ψ_v is the subgroup $\{1, \beta h_{2v-1}\}$ of $\mathbf{I}(\mathbf{G}_{q,n}(\mathbf{C}))$, and h_{2v-1} has matrix*

$$\begin{pmatrix} aI_{n-2v+1} & \\ & -aI_{2v-1} \end{pmatrix}$$

with $a = \exp(\pi\iota/n)$ and $1 \leqslant 2v - 1 < q$.

We remark that each of the groups Λ, Δ_{2v}, and Ψ_u is cyclic of order 2, but that any two distinct ones lying in $\mathbf{I}(\mathbf{G}_{q,n}(\mathbf{C}))$ are non-conjugate, so the corresponding quotients of $\mathbf{G}_{q,n}(\mathbf{C})$ are not isometric.

5. Even-dimensional real Grassmann manifolds. Let M be a real Grassmann manifold $\mathbf{G}_{q,n}(\mathbf{R})$, consisting of all oriented q-dimensional subspaces of an oriented $\mathbf{R}^n$. M has dimension $q(n-q)$; we shall now assume M to be even-dimensional, deferring the odd-dimensional case to § 6. As mentioned at the end of § 2.2, we can replace $\mathbf{G}_{q,n}(\mathbf{R})$ by $\mathbf{G}_{n-q,n}(\mathbf{R})$; $q(n-q)$ being even, this allows us to assume that q is even. Having made this assumption, it is clear that $\{\pm I\}$ is the kernel of the action of $\mathbf{U}(n, \mathbf{R}) = \mathbf{O}(n)$ on M. Let $\mathbf{I}'(M)$ denote the subgroup $\mathbf{O}(n)/\{\pm I\}$ of $\mathbf{I}(M)$ induced by $\mathbf{O}(n)$, and let $\mathbf{SO}(n)$ denote the special (determinant 1) orthogonal group. If $n-q$ is odd, then $\mathbf{I}_0(M) = \mathbf{SO}(n) = \mathbf{I}'(M)$; if $n-q$ is even, then

$$\mathbf{I}_0(M) = \mathbf{SO}(n)/\{\pm I\}$$

has index 2 in $\mathbf{I}'(M)$.

We have a transformation $\omega : M \to M$ defined by $P \to -P$ (opposite orientation); ω is an isometry of order 2, and the quotient $M/\{1, \omega\}$ is the non-oriented Grassmann manifold. If $q = n - q$, we have the isometry β of M given by $P \to P^\perp$, where $P^\perp$ is oriented such that $P \wedge P^\perp$ gives the original orientation of $\mathbf{R}^n$; as q is even, we have $P \wedge P^\perp = P^\perp \wedge P$, so $\beta^2 = 1$. Now let $\mathbf{I}''(M) = \mathbf{I}'(M) \cup \omega\cdot\mathbf{I}'(M)$ if $q \neq n-q$, and let

$$\mathbf{I}''(M) = \mathbf{I}'(M) \cup \omega\cdot\mathbf{I}'(M) \cup \beta\cdot\mathbf{I}'(M) \cup \beta\omega\cdot\mathbf{I}'(M)$$

if $q = n - q$. $\mathbf{I}(M) = \mathbf{I}''(M)$ if $M \neq \mathbf{G}_{4,8}(\mathbf{R})$. If $M = \mathbf{G}_{4,8}(\mathbf{R})$, then M has an isometry δ of order 3 such that conjugation of $\mathbf{I}_0(M)$ by δ is the triality automorphism, and $\mathbf{I}(M) = \mathbf{I}''(M) \cup \delta\cdot\mathbf{I}''(M) \cup \delta^2\cdot\mathbf{I}''(M)$.

Let Γ be a subgroup of $\mathbf{I}(M)$ which acts freely on M. $\mathbf{SO}(n)$ has isotropy subgroup $\mathbf{SO}(q) \times \mathbf{SO}(n-q)$, and $\mathbf{SO}(q) \times \mathbf{SO}(n-q)$ contains a maximal torus of $\mathbf{SO}(n)$ because q is even; it follows that an isotropy subgroup of $\mathbf{I}_0(M)$ contains a maximal torus, so every element of $\mathbf{I}_0(M)$ has a fixed point on M. Thus $\Gamma \cap \mathbf{I}_0(M) = \{1\}$, Γ has at most one element in any component of $\mathbf{I}(M)$, and $\gamma \to \gamma\cdot\mathbf{I}_0(M)$ gives an isomorphism of Γ onto a subgroup of $\mathbf{I}(M)/\mathbf{I}_0(M)$. C. T. C. Wall and I have proved (**4**, §§ 5.5.9–10) that a self-homeomorphism of $\mathbf{G}_{4,8}(\mathbf{R})$ of order 3 has a fixed point; thus Γ is conjugate to a subgroup of $\mathbf{I}''(M)$ in case $M = \mathbf{G}_{4,8}(\mathbf{R})$. Thus we may assume $\Gamma \subset \mathbf{I}''(M)$ generally.

Let $g \in \mathbf{O}(n)/\{\pm I\} = \mathbf{I}'(M)$. If $g \in \mathbf{I}_0(M)$, then it has a fixed point. If $g \notin \mathbf{I}_0(M)$, then $n - q$ is even and g has matrix $\pm\mathrm{diag}\{R(a_1), \ldots, R(a_u), 1, -1\}$ in some orthonormal basis $\{f_j\}$ of $\mathbf{R}^n$, where $2u \geqslant q$ and

$$R(a) = \begin{pmatrix} \cos a & \sin a \\ -\sin a & \cos a \end{pmatrix}.$$

Then the oriented subspace of $\mathbf{R}^n$ with ordered basis $\{f_1, \ldots, f_q\}$ is a fixed point for g. This proves $\Gamma \cap \mathbf{I}'(M) = \{1\}$.

Following ordered bases, one checks that ω commutes with every element of $\mathbf{I}'(M)$. Let $\gamma = \omega g \in \Gamma \cap \omega\cdot\mathbf{I}'(M)$; then $\gamma^2 = \omega g \omega g = g^2 \in \Gamma \cap \mathbf{I}_0(M)$;

in terms of linear transformations, $g^2 = \pm I$. If $g^2 = I$, we choose an orthonormal basis of $\mathbf{R}^n$ in which g has matrix

$$\begin{pmatrix} I_{n-v} & \\ & -I_v \end{pmatrix},$$

observe that ωg has a fixed point if $g \neq \pm I$, and conclude that $\gamma = \omega$. If $g^2 = -I$, then $n = 2m$ and g has matrix

$$\begin{pmatrix} 0 & I_m \\ -I_m & 0 \end{pmatrix}$$

in some orthonormal basis; it is then easily seen that $\gamma = \omega g$ has no fixed point.

We now need:

LEMMA. *Let $q = n - q$. Then β commutes with ω and every element of $\mathbf{I}_0(M)$, $g\beta g^{-1} = \beta\omega$ if $g \in \mathbf{I}'(M)$ is represented by a transformation of determinant -1, and then $g\beta \to b$, $\omega \to b^2$ and $g \to a$ gives an isomorphism of $\mathbf{I}''(M)/\mathbf{I}_0(M)$ onto the group $\{a, b : a^2 = b^4 = 1,\ aba = b^{-1}\}$.*

To prove the first statement, one checks that β centralizes $\mathbf{I}_0(M)$ and then observes that the commutator $[\omega, \beta]$ is a central element of the centreless group $\mathbf{I}_0(M)$. Cartan's construction of $\mathbf{I}(M)$ **(1)** shows that $\mathbf{I}''(M)/\mathbf{I}_0(M)$ is a non-abelian group of order 8 with several elements of order 2; it is thus abstractly isomorphic to the group $\{a, b\}$. As g induces an outer automorphism of $\mathbf{I}_0(M)$, it must normalize the group $\{\omega, \beta\}$; as $g\omega = \omega g$, the second statement follows, and the third statement becomes clear. This completes the proof of the lemma.

Let $\gamma = \beta g \in \Gamma \cap \beta\cdot\mathbf{I}_0(M)$. Then $\gamma^2 = \beta g\beta g = \beta^2 g^2 = g^2 \in \Gamma \cap \mathbf{I}_0(M)$, so $g^2 = \pm I$. If $g^2 = -I$ we can produce a fixed point; thus we may assume that

$$g = \begin{pmatrix} I_{n-v} & \\ & -I_v \end{pmatrix}, \qquad v < q, v \text{ even},$$

in some orthonormal basis of $\mathbf{R}^n$. Similarly, we have the same conclusion for g if $\omega\beta g \in \Gamma \cap \omega\beta\cdot\mathbf{I}_0(M)$.

Let $\gamma = \beta g \in \Gamma \cap \beta\cdot\mathbf{I}'(M)$, $g \notin \mathbf{I}_0(M)$. Then

$$\gamma^2 = \beta g\beta g = \omega g^2 \in \Gamma \cap \omega\cdot\mathbf{I}_0(M),$$

and an argument above shows that $g^2 = \pm I$ or $g^4 = -I$. The latter would imply $g \in \mathbf{I}_0(M)$; thus

$$g = \begin{pmatrix} I_{n-v} & \\ & -I_v \end{pmatrix}, \qquad v < q, \qquad v \text{ odd},$$

in some orthonormal basis of $\mathbf{R}^n$, and $\gamma^2 = \omega$. Similarly, if

$$\gamma = \omega\beta g \in \Gamma \cap \omega\beta\cdot\mathbf{I}'(M)$$

with $g \notin \mathbf{I}_0(M)$, we conjugate by g and conclude that

$$g = \begin{pmatrix} I_{n-v} & \\ & -I_v \end{pmatrix}, \qquad v < q, \qquad v \text{ odd},$$

in some orthonormal basis, and $\gamma^2 = \omega$.

If $q = n - q$, then Γ has 1, 2, or 4 elements because $\Gamma \cap \mathbf{I}'(M) = \{1\}$. If Γ is cyclic of order 4, then $\Gamma = \{1, \beta g_v, \omega, \omega\beta g_v\}$ where g_v is conjugate by an element of $\mathbf{I}_0(M)$ to

$$\begin{pmatrix} I_{n-v} & \\ & -I_v \end{pmatrix}, \qquad v \text{ odd}, \qquad v < q.$$

If Γ is cyclic of order 2, then the possibilities are $\{1, \omega\}$, $\{1, \omega J\}$, $\{1, \beta g_v\}$, and $\{1, \omega\beta g_v\}$ with v even, $v < q$, $J^2 = -I$. But the latter two are conjugate by g_1.

The only other non-trivial possibility is that Γ is the product of two cyclic groups of order 2. Then $\Gamma = \{1, \omega a_1, \beta a_2, \omega\beta a_3\}$ with $a_i \in \mathbf{I}'(M)$, $a_2^2 = a_3^2 = I$, and either $a_1 = I$ or $a_1^2 = -I$. As Γ has no element of order 4, each $a_i \in \mathbf{I}_0(M)$. We may assume $a_3 = a_1 a_2$, and we observe that $a_i a_j = \pm a_j a_i$ because Γ is abelian and because ω and β commute with each other and with each a_i. $a_2 a_3 = -a_3 a_2$ would imply that a_3 exchanges the eigenspaces of $+1$ and -1 of a_2, whence these eigenspaces would have the same dimension and βa_2 would have a fixed point. Thus $a_2 a_3 = a_3 a_2$. Similarly $a_1 a_2 = a_2 a_1$ and $a_1 a_3 = a_3 a_1$. Now $a_1 = a_1 a_2 a_2 = a_3 a_2$ implies $a_1^2 = a_3^2 a_2^2 = I$, so $a_1 = I$ and $a_2 = a_3$. Thus Γ is $\mathbf{I}_0(M)$-conjugate to $\{1, \omega, \beta g_v, \omega\beta g_v\}$ where g_v has matrix

$$\begin{pmatrix} I_{n-v} & \\ & -I_v \end{pmatrix}, \qquad v \text{ even}, \qquad v < q.$$

We have now proved:

THEOREM 3. *Let N be an even-dimensional Riemannian manifold that admits a Riemannian covering by the (oriented) real Grassmann manifold $\mathbf{G}_{q,n}(\mathbf{R})$. Then:*

(1) *N is isometric to $\mathbf{G}_{q,n}(\mathbf{R})$; or*

(2) *N is isometric to the non-oriented Grassmann manifold $\mathbf{G}_{q,n}(\mathbf{R})/\Omega$ where $\Omega = \{1, \omega\} \subset \mathbf{I}(\mathbf{G}_{q,n}(\mathbf{R}))$; or*

(3) *$n = 2m$ and N is isometric to $\mathbf{G}_{q,n}(\mathbf{R})/\Xi$ where $\Xi = \{1, \omega J\} \subset \mathbf{I}(\mathbf{G}_{q,n}(\mathbf{R}))$ and J has matrix*

$$\begin{pmatrix} 0 & I_m \\ -I_m & 0 \end{pmatrix};$$

or

(4) *$2q = n$ and N is isometric to one of the $\frac{1}{2}q$ manifolds $\mathbf{G}_{q,n}(\mathbf{R})/\Delta_{2u}$, $0 \leqslant u < \frac{1}{2}q$, where $\Delta_{2u} = \{1, \beta g_{2u}\} \subset \mathbf{I}(\mathbf{G}_{q,n}(\mathbf{R}))$ and g_v has matrix*

$$\begin{pmatrix} I_{n-v} & \\ & -I_v \end{pmatrix};$$

or

(5) $2q = n$ *and* N *is isometric to one of the* q *manifolds* $\mathbf{G}_{q,n}(\mathbf{R})/\Sigma_v$, $0 \leqslant v < q$, where $\Sigma_v = \{1, \omega, \beta g_v, \omega\beta g_v\} \subset \mathbf{I}(\mathbf{G}_{q,n}(\mathbf{R}))$. Σ_v *is cyclic of order* 4 *if* v *is odd, and* $\Sigma_{2u} = \Omega \times \Delta_{2u}$ *as a direct product of subgroups.*

6. Odd-dimensional real Grassmann manifolds. 6.1. Let M be an odd-dimensional real Grassmann manifold $\mathbf{G}_{q,n}(\mathbf{R})$, consisting of all oriented q-dimensional subspaces of an oriented $\mathbf{R}^n$; the dimension of M being $q(n-q)$, both q and $n-q$ are odd, and n is even. Now $-I \in \mathbf{SO}(n)$ changes the orientation of every q-dimensional subspace of $\mathbf{R}^n$, so $\mathbf{O}(n)$ acts effectively (the kernel of the action is trivial) on M. If $q \neq n-q$, then $\mathbf{O}(n) = \mathbf{I}(M)$. If $q = n-q$, then we have the isometry $\beta : P \to P^{\perp}$ of M; q being odd, $P \wedge P^{\perp} = -P^{\perp} \wedge P$, so $\beta^2 = -I \in \mathbf{SO}(n)$; then $\mathbf{I}(M) = \mathbf{O}(n) \cup \beta\cdot\mathbf{O}(n)$ except in the trivial case $M = \mathbf{G}_{1,2}(\mathbf{R}) = \mathbf{S}^1$.

6.2. Lemma 1. *Let* $g \in \mathbf{O}(n)$ *with* $\det g = -1$. *Then* g *has a fixed point on* M.

Proof. As n is even and q is odd, $\mathbf{R}^n$ has an orthonormal basis $\{e_j\}$ in which g has matrix $\operatorname{diag}\{R(a_1), \ldots, R(a_u), 1, -1, R(b_1), \ldots, R(b_v)\}$ where $q = 2u+1$, $n-q = 2v+1$, and $R(c)$ is defined as in § 5; the oriented subspace of $\mathbf{R}^n$ with ordered basis $\{e_1, \ldots, e_q\}$ is a fixed point for g on M. This completes the proof of Lemma 1.

Lemma 2. *Let* $g \in \mathbf{SO}(n)$. *Then* g *has a fixed point on* M *if and only if* g *has an eigenvalue* $+1$.

Proof. Suppose that g has a fixed point $P \in M$; then P is a g-invariant subspace of $\mathbf{R}^n$. As $q = \dim P$ is odd and g preserves the orientation of P, we can find $v \in P$ with $g(v) = v \neq 0$.

Now suppose that g has an eigenvalue $+1$. We choose, as for Lemma 1, an orthonormal basis $\{f_j\}$ of $\mathbf{R}^n$ in which g has matrix

$$\operatorname{diag}\{R(a_1), \ldots, R(a_u), 1, 1, R(b_1), \ldots, R(b_v)\}$$

and observe that the oriented subspace of $\mathbf{R}^n$ with ordered basis $\{f_1, \ldots, f_q\}$ is a fixed point for g on M. This completes the proof of Lemma 2.

Theorem 4. *Let* $\mathfrak{n}$ *be the class of all odd-dimensional Riemannian manifolds which admit a Riemannian covering by the (oriented) real Grassmann manifold* $\mathbf{G}_{q,n}(\mathbf{R})$ *with* $n \neq 2q$, *and let* $\mathfrak{G}$ *be the class of all finite subgroups of* $\mathbf{O}(n)$ *which acts freely on the unit sphere* $\mathbf{S}^{n-1} \subset \mathbf{R}^n$. *Then* $\mathfrak{n}$ *is the class of all Riemannian manifolds isometric to a quotient* $\mathbf{G}_{n,q}(\mathbf{R})/\Gamma$ *with* $\Gamma \in \mathfrak{G}$.

For $\mathbf{I}(\mathbf{G}_{q,n}(\mathbf{R})) = \mathbf{O}(n)$ because of our hypothesis that $n \neq 2q$; thus the Theorem is an immediate consequence of Lemmas 1 and 2.

6.3. Let Γ be a finite subgroup of $\mathbf{I}(M)$ which acts freely on M. If $\Gamma \subset \mathbf{O}(n)$, then Γ lies in the class $\mathfrak{G}$ of Theorem 4, which is fairly well known. Now assume that $\Gamma \not\subset \mathbf{O}(n)$; $q = n - q$ and Γ meets $\beta\cdot\mathbf{O}(n)$. One of our main tools for treating Γ is:

LEMMA 3. *Every $\gamma \in \Gamma \cap \beta\cdot\mathbf{O}(n)$ has order divisible by* 4.

Proof. Let $\gamma = \beta g$. Then $\gamma^2 = \beta g \beta g = \beta^2 g^2 = -g^2$ has order t, whence γ has order $2t$. If t is odd, then we have $\beta h = (\beta g)^t \in \Gamma$ of order 2, whence $h^2 = -I$; this is impossible, for we could then choose a well-oriented orthonormal basis $\{v_j\}$ of $\mathbf{R}^n$ in which h has matrix

$$\begin{pmatrix} 0 & -I_q \\ I_q & 0 \end{pmatrix}.$$

and the oriented subspace of $\mathbf{R}^n$ with ordered basis $\{v_{q+1}, \ldots, v_n\}$ would be a fixed point for βh on M. Thus t is even, and the proof of Lemma 3 is completed.

Lemma 3 will be used with:

LEMMA 4. *Let $\beta g \in \Gamma \cap \beta\cdot\mathbf{O}(n)$ and $h \in \mathbf{O}(n)$. Then $hgh^{-1} \neq -g$ and $hgh^{-1} \neq -g^{-1}$.*

Proof. Suppose that $hgh^{-1} = -g$ or that $hgh^{-1} = -g^{-1}$; we shall obtain a contradiction by constructing a fixed point for βg on M. Playing with eigenvectors, it is easy to see that h maps the eigenspace of $+1$ for g into the eigenspace of -1, and maps the eigenspace of -1 for g into the eigenspace of $+1$; thus those eigenspaces have the same dimension and h preserves the orthogonal complement V of their sum. V is a g-invariant orthogonal direct sum $U \oplus W$ where g has square $-I$ on U and has no eigenvalue $\pm\iota$ on W. It is not difficult to see that U and W are h-invariant. We can now write W as a g-invariant orthogonal direct sum of subspaces W_i and, for each W_i, choose an orthonormal basis in which g has matrix

$$\operatorname{diag}\{R(a_i), \ldots, R(a_i)\},$$

such that $-\pi/2 < a_0 < a_1 < \ldots < a_m < \pi/2$ and $a_i \neq 0$. Passing to the complexification of W and looking at eigenvectors, we see that

$$h(W_i) \subset W_{m-i} \qquad \text{and} \qquad R(a_i) = -R(\pm a_{m-i}).$$

This proves the existence of an orthonormal basis of $\mathbf{R}^n$ in which the matrix of g is of the form $\operatorname{diag}\{J_1, \ldots, J_k\}$ where each block J_u is of one of the forms

$$R(\pi/2), \qquad \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad \text{or} \qquad \begin{pmatrix} R(a) & 0 \\ 0 & -R(a) \end{pmatrix}.$$

Construction of the desired fixed points for βg is now clear. This completes the proof of Lemma 4.

LEMMA 5. *If u and v are prime integers, then every subgroup of Γ of order uv is cyclic.*

Proof. Let Δ be such a subgroup. If $\Delta \subset \mathbf{O}(n)$, then, using Lemmas 1 and 2, the result is due to H. Zassenhaus (**5**, Satz 3). If Δ meets $\beta \cdot \mathbf{O}(n)$, then Lemma 3 shows that Δ has an element of order 4, so $u = v = 2$ and Δ is cyclic. This completes the proof of Lemma 5.

LEMMA 6. *Every Sylow subgroup of* Γ *is cyclic, and every* 2-*Sylow subgroup of* Γ *is generated by an element of* $\beta \cdot \mathbf{O}(n)$.

Proof. From Lemma 5, we see (**2**, p. 120 or **6**, p. 149) that the odd Sylow subgroups of Γ are cyclic and that either the 2-Sylow subgroups are cyclic or the 2-Sylow subgroups are generalized quaternionic. Now, using Lemma 3, let us choose a 2-Sylow subgroup Δ of Γ which has an element $\beta g \in \beta \cdot \mathbf{O}(n)$. As Δ meets $\beta \cdot \mathbf{O}(n)$, it is clear that $\beta \cdot \mathbf{O}(n)$ contains a generator of Δ if Δ is cyclic.

Suppose that Δ is a generalized quaternionic group. Then Δ has generators A and B with defining relations

$$A^{2^{\alpha-1}} = I, \qquad A^{2^{\alpha-2}} = B^2, \qquad BAB^{-1} = A^{-1},$$

where $\alpha > 2$ is an integer. Every element of Γ of order 2 lies in $\mathbf{O}(n)$ by Lemma 3, and is thus $-I$ by Lemmas 1 and 2; this shows that $B^2 = -I$. Now every element of Δ is of the form A^u or of the form BA^u; as $BA^uB^{-1} = A^{-u}$ and $(\beta g)^{-1} = -\beta g^{-1}$, $\beta g = A^u$ would give us $BgB^{-1} = -g^{-1}$, contradicting Lemma 4. Thus $\beta g = BA^u$ for some integer u. For any integer v, $BA^v \in \mathbf{O}(n)$ would give $\beta g BA^v = A^{v-u} \in \beta \cdot \mathbf{O}(n)$, contradicting Lemma 4. Thus $\Delta \cap \mathbf{O}(n)$ is the set of all A^v, and $\Delta \cap \beta \cdot \mathbf{O}(n)$ is the set of all BA^v. In particular, $B \in \Delta \cap \beta \cdot \mathbf{O}(n)$, so $B = \beta k$ for some $k \in \mathbf{O}(n)$. Now

$$ABA^{-1} = B^2 \cdot AB^{-1}A^{-1} = BA^{-2},$$

so

$$A^{2^{\alpha-3}}BA^{-2^{\alpha-3}} = BA^{-2^{\alpha-2}} = B^3 = B^{-1},$$

which says that

$$A^{2^{\alpha-3}}kA^{-2^{\alpha-3}} = -k^{-1},$$

contradicting Lemma 4. We conclude that every 2-Sylow subgroup of Γ is cyclic. This proves Lemma 6.

6.4. If a finite group of order N has every Sylow subgroup cyclic, then (**6**, p. 175) it has two generators A and B with defining relations $A^u = B^v = 1$, $BAB^{-1} = A^r$, $0 < u$, $N = uv$, $((r-1)v, u) = 1$ and $r^v \equiv 1 \pmod{u}$. The group is cyclic if and only if $r \equiv 1 \pmod{u}$, i.e., $u = 1$. In any case, u is odd (**2**, p. 130 or **3**, p. 679), and the order d of r in the multiplicative group G_u of mod u residues primes to u as a divisor of v, say $v = dw$. If a subgroup is cyclic whenever its order is the product of two primes, then (**5**, p. 204, or from **2**, p. 160) w is divisible by every prime divisor of d. By Lemmas 5 and 6, we may take this to be a description of Γ; any subgroup of Γ has a similar description.

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As u is odd, $A \in \mathbf{SO}(n)$; then $B \in \beta \cdot \mathbf{O}(n)$ because Γ meets $\beta \cdot \mathbf{O}(n)$, say $B = \beta g$. By Lemma 3, $v = 4t$ for some integer $t > 0$, so Γ has order $4ut$.

Suppose that t is even, and let Δ be the subgroup of $\mathbf{O}(n)$ generated by A and $B' = g$ (where $B = \beta g$); B' has order $4t$ because t is even and $B^{2t} = -I$; it follows that Γ and Δ have the same presentation, and are thus isomorphic. Now (**2**, p. 160 and **3**, p. 679) $q = dp$ and we have an orthonormal basis $\{e_j\}$ of $\mathbf{R}^n$ in which A has matrix diag$\{\mathbf{a}, \ldots, \mathbf{a}\}$ where each of the p blocks $\mathbf{a}$ is given by

$$\mathbf{a} = \operatorname{diag}\{R(2\pi/u), R(2\pi r/u), \ldots, R(2\pi r^{d-1}/u)\},$$

and in which B' has matrix diag$\{\mathbf{b}_1, \ldots, \mathbf{b}_p\}$, where

$$\mathbf{b}_i = \begin{pmatrix} 0 & I & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & I & 0 & \cdots & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ 0 & 0 & 0 & 0 & \cdots & 0 & I \\ \mathbf{x}_i & 0 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix} \quad \text{in } 2 \times 2 \text{ blocks}$$

being a matrix of degree $2d$, and where $\mathbf{x}_i = R(2\pi k_i/w)$ for some integer k_i prime to w.

In this case, Γ acts freely on M. For $\Gamma \cap \Delta = \Gamma \cap \mathbf{O}(n)$ acts freely on M, and, if $\gamma \in \Gamma \cap \beta \cdot \mathbf{O}(n)$ had a fixed point on M, then $\gamma^2 \in \Gamma \cap \Delta$ would have a fixed point on M, whence $\gamma^2 = I$. But γ must be of the form $\beta A^a (B')^b$ with b odd, which is impossible because

$$I = \gamma^2 = -A^a(B')^b A^a (B')^b = A^{a+ar^b}(B')^{2(t+b)},$$

whence $(B')^{2(t+b)} = I$, implying that b is an odd multiple of the even integer t.

Now suppose that t is odd, and let Φ be the subgroup of $\mathbf{O}(n)$ generated by A and by g (where $B = \beta g$). $-I = B^{2t} = \beta^{2t} g^{2t} = -g^{2t}$ shows that the order of g divides $2t$; as t is odd, it is easy to check that g has order t or $2t$, and $-I \in \Gamma \cap \mathbf{O}(n) \subset \Phi$ shows that g cannot have order t. Thus Φ has order $2tu$, and is consequently equal to its subgroup $\Gamma \cap \mathbf{O}(n)$ of order $2tu$. It follows that $\Gamma \cap \beta \cdot \mathbf{O}(n)$ is $\beta g \cdot \Phi$, and thus contains β because $g^{-1} \in \Phi$, whence $\Gamma = \Phi \cup \beta \cdot \Phi$. Given that Φ acts freely on M, the fact that Φ has no element of order 4 (hence no element of square $-I$) shows that Γ acts freely on M, much as in the preceding paragraph.

We have now proved:

THEOREM 5. *Let $\mathfrak{m}$ be the class of all odd-dimensional Riemannian manifolds which admit a Riemannian covering by the (oriented) real Grassmann manifold $\mathbf{G}_{q,2q}(\mathbf{R})$, and let $\mathbf{Z}_{u,v,r}$ denote the abstract group of order uv and with all Sylow subgroups cyclic, which has presentation $\{X, Y : X^u = Y^v = 1, YXY^{-1} = X^r\}$ where $0 < u$, $((r-1)v, u) = 1$ and $r^v \equiv 1 \pmod{u}$. Then $\mathfrak{m}$ is the class of all Riemannian manifolds which are isometric to a quotient $\mathbf{G}_{q,2q}(\mathbf{R})/\Gamma$, where:*

(1) *Γ is a finite subgroup of $\mathbf{SO}(2q)$ which acts freely on the unit sphere $\mathbf{S}^{2q-1} \subset \mathbf{R}^{2q}$; or*

(2) $\Gamma = \Phi \cup \beta \cdot \Phi$ *is isomorphic to* $\mathbf{Z}_{u,2v,r}$, *where* Φ *is a subgroup of* $\mathbf{SO}(2q)$ *which is isomorphic to a group* $\mathbf{Z}_{u,v,r}$ *satisfying*:

(a) *q is divisible by the order d of r in the multiplicative group of* mod u *residues prime to u, say $q = dp$,*

(b) *$v = dw$ where w is divisible by every prime divisor of d,*

(c) *d is odd, and $w = 2w'$ with w' odd, so that $\mathbf{Z}_{u,v,r}$ has order $2udw'$ with udw' odd; the isomorphism is a direct sum of p of the faithful irreducible $\mathbf{SO}(2d)$-representations f_k of $\mathbf{Z}_{u,v,r}$ given in 2×2 blocks by*

$$f_k(X) = \mathrm{diag}\{R(2\pi/u), R(2\pi r/u), \ldots, R(2\pi r^{d-1}/u)\},$$

$$f_k(Y) = \begin{pmatrix} 0 & I & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & I & 0 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & I \\ R(2\pi k/w) & 0 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix}$$

with k prime to w; or

(3) Γ *is isomorphic to a group* $\mathbf{Z}_{u,v,r}$ *satisfying conditions* (a) *and* (b) *above, as well as*:

(c′) *d is odd and* $w \equiv 0 \pmod 8$, *so that $\mathbf{Z}_{u,v,r}$ has order divisible by* 8; *Γ is generated by $f(X)$ and $\beta f(Y)$ where f is a direct sum of p of the faithful irreducible $\mathbf{SO}(2d)$-representations f_k of $\mathbf{Z}_{u,v,r}$ described above.*

References

1. É. Cartan, *Sur certaines formes riemanniennes remarquables des géométries à groupe fondamental simple*, Ann. Sci. Éc. Norm. Supér. *44* (1927), 345–467.
2. G. Vincent. *Les groupes linéaires finis sans points fixes*, Comm. Math. Helv. *20* (1947), 117–171.
3. J. A. Wolf, *The manifolds covered by a Riemannian homogeneous manifold*, Amer. J. Math. *82* (1960), 661–688.
4. ——— *Locally symmetric homogeneous spaces*, to appear in Comm. Math. Helv.
5. H. Zassenhaus, *Über endliche Fastkörper*, Hamburg Abh. *11* (1935), 187–220.
6. ——— *The theory of groups* (2nd edition, New York, 1958).
7. D. Montgomery and L. Zippin, *Topological transformation groups* (New York, 1955).

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