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A CHARACTERIZATION OF TWO WEIGHT NORM INEQUALITIES FOR ONE-SIDED OPERATORS OF FRACTIONAL TYPE

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ABSTRACT. In this paper we give a characterization of the pairs of weights (ω, v) such that T maps $L^p(v)$ into $L^q(\omega)$, where T is a general one-sided operator that includes as a particular case the Weyl fractional integral. As an application we solve the following problem: given a weight v, when is there a nontrivial weight ω such that T maps $L^p(v)$ into $L^q(\omega)$?

1. **Introduction.** In [M], B. Muckenhoupt raised the question of characterizing when the weighted norm inequality

$$\left(\int_{\mathbb{R}^m} |Tf(x)|^q \omega(x)\,dx\right)^{1/q} \le C\left(\int_{\mathbb{R}^n} |f(x)|^p v(x)\,dx\right)^{1/p} \tag{1.1}$$

holds, where T is any classical operator. We are interested in the case $m = n = 1$ and T a one-sided operator. By a one-sided operator we mean an operator T acting on measurable functions f such that the values of $Tf(x)$ depend only on the values of f either in (x, ∞) or in $(-\infty, x)$.

For f locally integrable on $\mathbb{R}$, the one-sided Hardy-Littlewood maximal functions are

$$M^+f(x) = \sup_{h>0} \frac{1}{h}\int_x^{x+h} |f(y)|\,dy \quad \text{and} \quad M^-f(x) = \sup_{h>0} \frac{1}{h}\int_{x-h}^{x} |f(y)|\,dy.$$

In [S1], Eric Sawyer characterized for $1 < p < \infty$, $p = q$, the weights ω satisfying (1.1) for $T = M^+$ with $\omega = v$, as those weights ω satisfying the A_p^+ condition:

$$(A_p^+) \qquad \left(\frac{1}{h}\int_{a-h}^{a} \omega(x)\,dx\right)\left(\frac{1}{h}\int_a^{a+h} \omega^{\frac{-1}{p-1}}(x)\,dx\right)^{p-1} \le C, \quad \text{for all } a \in \mathbb{R} \text{ and } h > 0.$$

For $T = M^-$ the weights are characterized by the A_p^- condition:

$$(A_p^-) \qquad \left(\frac{1}{h}\int_{a}^{a+h} \omega(x)\,dx\right)\left(\frac{1}{h}\int_{a-h}^{a} \omega^{\frac{-1}{p-1}}(x)\,dx\right)^{p-1} \le C, \quad \text{for all } a \in \mathbb{R} \text{ and } h > 0.$$

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In the same paper he proves that for $1 < p < \infty$ the pairs of weights (ω, v) satisfying (1.1) for $T = M^+$ are those satisfying the S_p^+ condition

$$(S_p^+) \qquad \int_I \big(M^+(\chi_I v^{\frac{-1}{p-1}})\big)^p \omega \le C \int_I v^{\frac{-1}{p-1}} < \infty,$$

for all intervals $I = (a, b)$ such that $\int_{-\infty}^a \omega > 0$. The corresponding result is obtained for $T = M^-$ changing S_p^+ by the natural S_p^- condition.

For $0 < \alpha < 1$ the Weyl fractional integral W_α and the Riemann-Liouville fractional integral R_α are defined, for locally integrable functions on $\mathbb{R}$, by

$$W_\alpha f(x) = \int_x^\infty \frac{f(y)}{(y-x)^{1-\alpha}}\,dy \quad \text{and} \quad R_\alpha f(x) = \int_{-\infty}^x \frac{f(y)}{(x-y)^{1-\alpha}}\,dy$$

and for $0 \le \alpha < 1$, the fractional one-sided Hardy-Littlewood maximal functions M_α^+ and M_α^- are defined by

$$M_\alpha^+ f(x) = \sup_{h>0} h^{\alpha-1} \int_x^{x+h} |f(y)|\,dy \quad \text{and} \quad M_\alpha^- f(x) = \sup_{h>0} h^{\alpha-1} \int_{x-h}^x |f(y)|\,dy.$$

Andersen and Sawyer [AS] showed that, under the assumptions $1 < p < \frac{1}{\alpha}$ and $\frac{1}{q} = \frac{1}{p} - \alpha$, the inequality (1.1) holds with $\omega = v$ for $T = M_\alpha^+$ or $T = W_\alpha$ $(\alpha > 0)$ if and only if

$$(A_{p,q}^+) \qquad \left(\frac{1}{h}\int_{a-h}^a \omega(x)\,dx\right)^{1/q} \left(\frac{1}{h}\int_a^{a+h} \omega^{\frac{-1}{p-1}}(x)\,dx\right)^{1/p'} \le C, \quad \text{for all } a \in \mathbb{R}, h > 0,$$

and for $T = M_\alpha^-$ or $T = R_\alpha$ $(\alpha > 0)$ if and only if

$$(A_{p,q}^-) \qquad \left(\frac{1}{h}\int_a^{a+h} \omega(x)\,dx\right)^{1/q} \left(\frac{1}{h}\int_{a-h}^a \omega^{\frac{-1}{p-1}}(x)\,dx\right)^{1/p'} \le C, \quad \text{for all } a \in \mathbb{R}, h > 0,$$

where p' is the conjugate exponent of p. To prove this, they used complex interpolation of analytic families of operators. A "geometric" type proof was given by Martín-Reyes and de la Torre in [MT]. They also solved the case of different weights for the fractional one-sided Hardy-Littlewood maximal functions, for $1 < p \le q$. More precisely, they showed that the inequality (1.1) holds for $1 < p \le q$ and $T = M_\alpha^+$ if, and only if,

$(S_{p,q,\alpha}^+)$ there exists C such that for every interval I with $\sigma(I)$ finite

$$\left(\int_I \big(M_\alpha^+(\sigma\chi_I)\big)^q \omega\right)^{1/q} \le C\big(\sigma(I)\big)^{1/p},$$

where $\sigma = v^{1-p'}$ and $\sigma(I) = \int_I \sigma$.

For the Weyl fractional integral and for $1 < p \le q < \infty$ or $1 = p < q < \infty$ the pairs of weights for which the weak type inequality associated with (1.1) holds have been characterized ([LT]) as those pairs of weights (ω, v) satisfying

$$\int_I \big(R_\alpha(\chi_I\omega)\big)^{p'} v^{1-p'} \le C\left(\int_I \omega\right)^{p'/q'}, \quad \text{if } 1 < p \le q < \infty,$$

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or

$$\|R_\alpha(\chi_I\omega)v^{-1}\|_{L^\infty(v)} \le C\Big(\int_I \omega\Big)^{1/q'}, \quad \text{if } p = 1 < q < \infty.$$

(For $p < q$ this problem is solved in [LT] for a more general operator). However, as far as the author knows, there is not a characterization of the strong type inequality (1.1) with $T = W_\alpha$. In this paper we solve this problem for $1 < p \le q < \infty$. Actually, we characterize the pairs of weights (ω, v) for which (1.1) holds for a more general operator T defined by

$$Tf(x) = \int_x^\infty K(y-x)f(y)\,dy \tag{1.2}$$

where K is a positive measurable function, lower semicontinuous, with support in $(0,\infty)$, nonincreasing in $(0,\infty)$, with $\lim_{x\to\infty} K(x) = 0$ and satisfying $K(x) \le CK(2x)$, $x \in (0,\infty)$. (Observe that if $K(x) = x^{\alpha-1}\chi_{(0,\infty)}(x)$ then $T = W_\alpha$). This result is in Theorem 1. In the proof of this theorem we follow the ideas in [S2], [SW] and [SWZ] but we also need the characterization of the good weights (ω, v) for a one-sided dyadic maximal operator associated with K and defined by

$$M^+_{K,d}f(x) = \sup_{I\in A_x} K(|I|)\int_I |f(y)|\,dy \tag{1.3}$$

where $A_x = \{I = [a,b) : I \text{ is dyadic and } 0 \le a - x < b - a\}$. This characterization appears in Theorem 2.

As an application of these results, we solve the following problem: given a weight v, when is there a nontrivial weight ω, such that (1.1) holds for T defined by (1.2) or for $M^+_{K,d}$? The answer to these problems are contained in Theorems 3 and 4.

We end this section with some notation. Throughout the paper the letter I will denote an interval in $\mathbb{R}$, $|I|$ will denote the Lebesgue measure of I. If λ is a positive real number, then λI will denote the interval with the same center as I and with $|\lambda I| = \lambda|I|$ and if g is a positive measurable function and E is a measurable set, then $g(E) = \int_E g$. If $I = [a,b)$, $I^\star$ will be the interval $[b, 2b-a)$. A weight will be a nonnegative measurable function. The letter C will always mean a positive constant not necessarily the same at each occurrence and if $1 < p < \infty$ then p' will denote the number such that $p + p' = pp'$.

2. **Statement of the results.**

THEOREM 1. *Suppose that* $1 < p \le q < \infty$, ω *and* v *are two weights and*

$$Tf(x) = \int_x^\infty K(y-x)f(y)\,dy,$$

where K is a positive measurable function, lower semicontinuous, with support in $(0,\infty)$, *nonincreasing in* $(0,\infty)$, *with* $\lim_{x\to\infty} K(x) = 0$ *and satisfying* $K(x) \le CK(2x)$, $x \in (0,\infty)$. *Then the weighted inequality*

$$\Big(\int_{\mathbb{R}} |Tf|^q\omega\Big)^{1/q} \le C\Big(\int_{\mathbb{R}} |f|^p v\Big)^{1/p} \tag{2.1}$$

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holds for some constant C if, and only if, the following two conditions hold:
(2.2) *There exists C such that for every interval* $I = [a, b)$ *with* $\int_{(-\infty,a)} \omega > 0$,

$$\left(\int_{\mathbb{R}} \big(T(\chi_I\sigma)\big)^q \omega\right)^{1/q} \leq C\big(\sigma(I)\big)^{1/p} < \infty$$

and
(2.3) *there exists C such that for every interval* $I = [a, b)$ *with* $\int_{[b,\infty)} \sigma > 0$,

$$\left(\int_{\mathbb{R}} \big(T^\star(\chi_I\omega)\big)^{p'} \sigma\right)^{1/p'} \leq C\big(\omega(I)\big)^{1/q'} < \infty,$$

where $\sigma = v^{1-p'}$ *and* $T^\star$ *denotes the adjoint operator of* T, $T^\star g(x) = \int_{-\infty}^{x} K(x-y)\, g(y)\, dy$.

THEOREM 2. *Let K be as in Theorem 1. Then for weights* ω, v *and* $1 < p \leq q$, *the following two conditions are equivalent:*

(2.4) There exists C such that for every $f \geq 0$

$$\left(\int (M^+_{K,d} f)^q \omega\right)^{1/q} \leq C\left(\int f^p v\right)^{1/p}.$$

(2.5) There exists C such that for every dyadic interval $I = [a, b)$ with $\int_{(-\infty,b)} \omega > 0$,

$$\int_{I^\star} \sigma < \infty \quad \text{and} \quad \left(\int_{I\cup I^\star} \big(M^+_{K,d}(\sigma\chi_{I^\star})\big)^q \omega\right)^{1/q} \leq C\left(\int_{I^\star} \sigma\right)^{1/p}.$$

This theorem is an easy variant of Theorem 2.6 in [MT]. The proof is exactly as in [MT]. Thus we omit it.

THEOREM 3. *Let* $1 < p \leq q < \infty$ *and let K be as in Theorem 1. Suppose that there exists* $q_0 > \frac{q}{p}$ *such that* $K(x) \leq Cx^{-1/q_0}$, *for all* $x \in (0, \infty)$. *Let v be a weight,* $0 \leq v(x) \leq \infty$, *such that v is not identically infinity in any interval of the form* (c, ∞). *Then, there exists* ω *not identically zero such that the inequality*

$$\left(\int_{\mathbb{R}} |Tf|^q \omega\right)^{1/q} \leq C\left(\int_{\mathbb{R}} |f|^p v\right)^{1/p} \tag{2.6}$$

holds for some constant C and for all $f \in L^p(v)$, *if, and only if, there exists* $a \in \mathbb{R}$ *such that for all* $b > a$, *we have*

$$\int_a^b \sigma > 0 \quad \textit{and} \quad \int_b^\infty K(y-a)^{p'} \sigma(y)\, dy < \infty. \tag{2.7}$$

THEOREM 4. *Under the same assumptions of Theorem 3 we have that there exists* ω *not identically zero such that the inequality*

$$\left(\int_{\mathbb{R}} |M^+_{K,d} f|^q \omega\right)^{1/q} \leq C\left(\int_{\mathbb{R}} |f|^p v\right)^{1/p} \tag{2.8}$$

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holds for some constant C and for all $f \in L^p(v)$, if, and only if, there exists a dyadic interval I_0 with $0 < \int_{I_0^\star} \sigma$ and such that

$$\sup_{\{I \text{ dyadic}: I_0 \subset I\}} K(|I|)\Big(\int_{I^\star} \sigma\Big)^{1/p'} < \infty. \tag{2.9}$$

REMARKS.

(1) Observe that for $f \geq 0$, we have $M^+_{K,d} f(x) \leq CTf(x)$. It follows that condition (2.2) implies that $M^+_{K,d}$ is bounded from $L^p(v)$ to $L^q(\omega)$.

(2) If $K(x) \leq CK(x/2)$ for some $C < 1$ then $M^+_{K,d}$ is pointwise equivalent to the following maximal operator

$$M^+_K f(x) = \sup_{c>x} K(c-x) \int_x^c |f(y)|\, dy.$$

Observe that this condition holds if $K(x) = x^{\alpha-1}\chi_{(0,\infty)}(x)$, *i.e.*, the kernel for the Weyl operator. In this case M^+_K is M^+_α (for this case, see [MT]).

(3) Of course, one can change the orientation of the real line and obtain Theorems 1 and 3 for $T^\star$ and Theorems 2 and 4 for $M^-_{K,d}$.

(4) By duality we also can solve the following problem: given ω not identically zero, when there exists v not identically infinity such that (2.6) holds?

(5) We ask for v not identically infinity in any interval of the form (c,∞) in Theorems 3 and 4 because if there exists c such that $v = \infty$ a.e. in (c,∞), then it suffices to take $\omega = \chi_{(c,\infty)}$ to have (2.6) and (2.8).

(6) Theorem 1 of [S2] can be easily obtained as a consequence of Theorem 1.

(7) Theorem 3 is also valid for $p > 1$, $0 < q < p$ and assuming $q_0 > 1$. This follows using Hölder's inequality and the case $p = q$. Putting together Theorem 3 and this remark we observe that we have generalized Theorem 3 (b) in [AS] since we extend the range of p and q and we consider more general operators.

3. **Proof of Theorem 1.** Assume that (2.1) holds. Then so does its dual inequality

$$\Big(\int |T^\star g|^{p'}\sigma\Big)^{1/p'} \leq C\Big(\int |g|^{q'}\omega^{1-q'}\Big)^{1/q'}. \tag{3.1}$$

Let $I = [a,b)$ be such that $\int_{(-\infty,a)} \omega > 0$. Then there exists a bounded interval $J \subset (-\infty,a)$ such that $\int_J \omega > 0$. We first prove that $\sigma(I) < \infty$. Taking $g = \omega^{1/q}\chi_J$ in (3.1) we have that

$$\Big(\int |T^\star(\omega^{1/q}\chi_J)|^{p'}\sigma\Big)^{1/p'} \leq C|J|^{1/q'} < \infty$$

and for all $x \in I$, $T^\star(\omega^{1/q}\chi_J)(x) > T^\star(\omega^{1/q}\chi_J)(b) > 0$. Therefore, $\sigma(I) < \infty$. To finish the proof of (2.2) it suffices to take $f = \chi_I\sigma$ in (2.1).

Now let $I = [a,b)$ such that $\int_{[b,\infty)} \sigma > 0$ and consider a bounded interval $J \subset [b,\infty)$ such that $\int_J \sigma > 0$. Then (2.3) follows by taking $f = \sigma^{1/p'}\chi_I$ in (2.1) and $g = \chi_I\omega$ in (3.1).

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To prove the converse, we suppose that $f \in L^p(v)$ is nonnegative, bounded with compact support and such that $f\sigma^{-1}$ is bounded. For each $k \in \mathbb{Z}$, the set $\Omega_k = \{x : Tf(x) > 2^k\}$ is open since K is lower semicontinuous and the fact that $\lim_{x\to\infty} K(x) = 0$ gives that the connected components of Ω_k are of finite length. Then, as in [S2] with the correction pointed out in [SW] and [SWZ], we have

$$
(3.2)\quad
\begin{aligned}
&\text{(i)}\ \ \Omega_k = \cup_j I_j^k,\ I_j^k \text{ dyadic and } I_j^k \cap I_i^k = \emptyset \text{ for } i \neq j,\\
&\text{(ii)}\ \ 3I_j^k \subset \Omega_k \text{ and } 9I_j^k \cap \Omega_k^c \neq \emptyset \text{ for all } k, j,\\
&\text{(iii)}\ \ \sum_j \chi_{3I_j^k} \leq C\chi_{\Omega_k} \text{ for all } k,\\
&\text{(iv)}\ \ \text{the number of intervals } I_s^k \text{ intersecting a fixed interval } 3I_j^k \text{ is at most } C,\\
&\text{(v)}\ \ I_j^k \underset{\neq}{\subset} I_i^l \text{ implies } k > l.
\end{aligned}
$$

There are two types of intervals among the I_j^k's. In order to classify them we consider the right endpoint c of the connected component of Ω_k which contains I_j^k. If $9I_j^k \cap \Omega_k^c \cap (c, \infty) \neq \emptyset$, we denote I_j^k by J_j^k, otherwise, we denote I_j^k by L_j^k.

For fixed J_j^k, let b and c be the right endpoint of $3J_j^k$ and the connected component of Ω_k which contains J_j^k, respectively. Then if $x \in J_j^k$, we have

$$T(f\chi_{(3J_j^k)^c})(x) = \int_b^c K(y-x)f(y)\chi_{(3J_j^k)^c}(y)\,dy + \int_c^\infty K(y-x)f(y)\chi_{(3J_j^k)^c}(y)\,dy.$$

Since K is nonincreasing and $c \notin \Omega_k$ it follows that

$$\int_c^\infty K(y-x)f(y)\chi_{(3J_j^k)^c}(y)\,dy \leq \int_c^\infty K(y-c)f(y)\,dy = Tf(c) \leq 2^k.$$

On the other hand, it is not very difficult to prove that the assumption on K, $K(x) \leq CK(2x)$ for $x > 0$ and property (ii) in (3.2) give that

$$\int_b^c K(y-x)f(y)\chi_{(3J_j^k)^c}(y)\,dy \leq CM^+_{K,d}f(x).$$

To prove this inequality we only have to observe that the interval (b, c) is contained in the union of at most two dyadic intervals of length comparable to $|J_j^k|$ and belonging to A_x. Therefore, for $x \in J_j^k$, we have

$$T(f\chi_{(3J_j^k)^c})(x) \leq CM^+_{K,d}f(x) + 2^k. \tag{3.3}$$

This is the reason why we need to study this dyadic maximal operator.

Let us consider now an interval L_j^k. Let a be the left endpoint of the connected component of Ω_k which contains L_j^k and $[b, c) = 3L_j^k$. For $x \in L_j^k$, we have

$$Tf(x) = \int_x^c K(y-x)f(y)\,dy + \int_c^\infty K(y-a)f(y)\frac{K(y-x)}{K(y-a)}\,dy.$$

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If $y > c$ then $y-a = (y-x)+(x-a) \leq (y-x)+9|L_j^k| \leq (y-x)+9(y-x) = 10(y-x)$. Then $K(y-x) \leq C^4K\big(2^4(y-x)\big) \leq C^4K(y-a)$, by the growth condition of K and the fact that K is nonincreasing. Therefore

$$\begin{aligned} Tf(x) &\leq \int_x^c K(y-x)f(y)\,dy + C^4\int_c^\infty K(y-a)f(y)\,dy \\ &\leq T(f\chi_{(3L_j^k)})(x) + C^4Tf(a) \leq T(f\chi_{(3L_j^k)})(x) + C^4 2^k, \end{aligned}$$

since $a \notin \Omega_k$. Choose an integer $m \geq 3$ such that $2^{m-2} > C^4$. Define $G_j^k = L_j^k \cap (\Omega_{k+m-1} - \Omega_{k+m})$. Then, for $x \in G_j^k$, we have

$$T(f\chi_{(3L_j^k)})(x) \geq Tf(x) - C^4 2^k > 2^{k+m-1} - 2^{k+m-2} \geq 2^k,$$

and so,

$$1 \leq \frac{1}{2^k}T(f\chi_{(3L_j^k)})(x), \quad \text{for } x \in G_j^k. \tag{3.4}$$

Let us consider again inequality (3.3). Define $A_j^k = \{x \in J_j^k : CM_{K,d}^+f(x) \leq 2^k\}$, where C is the constant appearing in (3.3), $B_j^k = J_j^k - A_j^k$ and let $D_j^k = A_j^k \cap (\Omega_{k+m-1} - \Omega_{k+m})$ and $F_j^k = B_j^k \cap (\Omega_{k+m-1} - \Omega_{k+m})$. Then,

$$Tf(x) \leq 2^{k+m} \text{ and } 2^k < CM_{K,d}^+f(x), \quad \text{for all } x \in F_j^k. \tag{3.5}$$

If $x \in D_j^k$ we have

$$\begin{aligned} 2^{k+m-1} < Tf(x) &= T(f\chi_{(3J_j^k)})(x) + T(f\chi_{(3J_j^k)^c})(x) \leq T(f\chi_{(3J_j^k)})(x) \\ &\quad + CM_{K,d}^+f(x) + 2^k \leq T(f\chi_{(3J_j^k)})(x) + 2^{k+1} \end{aligned}$$

and so

$$T(f\chi_{(3J_j^k)})(x) > 2^{k+m-1} - 2^{k+1} \geq 2^{k+2} - 2^{k+1} > 2^k.$$

Thus,

$$1 \leq \frac{1}{2^k}T(f\chi_{(3J_j^k)})(x), \quad \text{for } x \in D_j^k. \tag{3.6}$$

We now estimate the left side of (2.1) by

$$\begin{aligned} \int_{\mathbb{R}}\big(Tf(x)\big)^q\omega(x)\,dx &= \sum_{k\in\mathbb{Z}}\int_{\Omega_{k+m-1}-\Omega_{k+m}}\big(Tf(x)\big)^q\omega(x)\,dx \\ &\leq \sum_{k,j}\int_{D_j^k}\big(Tf(x)\big)^q\omega(x)\,dx \\ &\quad + \sum_{k,j}\int_{F_j^k}\big(Tf(x)\big)^q\omega(x)\,dx \\ &\quad + \sum_{k,j}\int_{G_j^k}\big(Tf(x)\big)^q\omega(x)\,dx = \text{(I)} + \text{(II)} + \text{(III)}. \end{aligned} \tag{3.7}$$

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We first estimate the term (II). Using (3.5), the fact that the F_j^k are disjoint on k and j and remark (1), we have

$$(3.8)\qquad \begin{aligned}(\mathrm{II}) &\leq \sum_{k,j} 2^{mq} \int_{F_j^k} 2^{kq}\omega(x)\,dx \leq C\sum_{k,j}\int_{F_j^k}\big(M_{K,d}^+ f(x)\big)^q \omega(x)\,dx \\ &\leq C\int_{\mathbb{R}}(M_{K,d}^+ f(x))^q\omega(x)\,dx \leq C\Big(\int_{\mathbb{R}} f^p v\Big)^{q/p}.\end{aligned}$$

To estimate the terms (I) and (III), we observe that (3.4) and (3.6) allow us to treat (I) and (III) jointly. If we denote J_j^k or L_j^k by I_j^k and D_j^k or G_j^k by E_j^k, the inequalities (3.4) and (3.6) can be unified as

$$1 \leq \frac{1}{2^k} T(f\chi_{(3I_j^k)})(x), \qquad \text{for } x \in E_j^k.$$

Then

$$(3.9)\qquad (\mathrm{I}) + (\mathrm{III}) \leq \sum_{k,j}\int_{E_j^k}\big(Tf(x)\big)^q\omega(x)\,dx \leq C\sum_{k,j} 2^{kq}\omega(E_j^k).$$

Now, using duality,

$$(3.10)\qquad \begin{aligned}\omega(E_j^k) &\leq \frac{1}{2^k}\int_{E_j^k} T(f\chi_{(3I_j^k)})(x)\omega(x)\,dx = \frac{1}{2^k}\int_{3I_j^k} f(x)T^\star(\chi_{E_j^k}\omega)(x)\,dx \\ &= \frac{1}{2^k}\Big(\int_{3I_j^k-\Omega_{k+m}} f(x)T^\star(\chi_{E_j^k}\omega)(x)\,dx + \int_{3I_j^k\cap\Omega_{k+m}} f(x)T^\star(\chi_{E_j^k}\omega)(x)\,dx\Big) \\ &= \frac{1}{2^k}(\sigma_j^k + \tau_j^k).\end{aligned}$$

Define, as in [S2], the following sets:

$$\begin{aligned} E &= \big\{(k,j) : \omega(E_j^k) \leq \beta\omega(I_j^k)\big\}, \\ F &= \big\{(k,j) : \omega(E_j^k) > \beta\omega(I_j^k) \text{ and } \sigma_j^k > \tau_j^k\big\}, \\ G &= \big\{(k,j) : \omega(E_j^k) > \beta\omega(I_j^k) \text{ and } \sigma_j^k \leq \tau_j^k\big\},\end{aligned}$$

where β satisfies $0 < \beta < 1$ and it will be chosen at the end of the proof. Then, taking into account (3.9) and (3.10) we can write

$$(3.11)\qquad \begin{aligned}(\mathrm{I}) + (\mathrm{III}) &\leq C\Big(\sum_{(k,j)\in E} + \sum_{(k,j)\in F} + \sum_{(k,j)\in G}\Big) 2^{kq}\omega(E_j^k) \\ &= (\mathrm{IV}) + (\mathrm{V}) + (\mathrm{VI}).\end{aligned}$$

Observe that we only have to consider those (k,j) for which $\omega(E_j^k) \neq 0$. If there exist (k,j) and $(k+m,i)$ such that $I_j^k = I_i^{k+m}$, then $\omega(E_j^k) = 0$ because $E_j^k \subset I_j^k \cap (\Omega_{k+m-1} - \Omega_{k+m})$, thus we do not consider this (k,j). Therefore, fixed two intervals I_j^k and I_i^{k+m}, or they are disjoint or $I_i^{k+m} \underset{\neq}{\subset} I_j^k$.

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To estimate the sum over the set E, we use the fact that the I_j^k are disjoint in j and Fubini's theorem. Then

(3.12)
$$\begin{aligned}
\text{(IV)} &\le C\beta \sum_{(k,j)\in E} 2^{kq}\omega(I_j^k) \\
&\le C\beta \sum_k 2^{kq}\omega\Big(\big\{x : Tf(x) > 2^k\big\}\Big) \\
&= C\beta \sum_k \sum_{i=k}^{\infty} 2^{kq}\omega\Big(\big\{x : 2^i < Tf(x) \le 2^{i+1}\big\}\Big) \\
&\le C\beta \sum_k \sum_{i=k}^{\infty} 2^{kq}2^{-iq} \int_{\{x:2^i<Tf(x)\le 2^{i+1}\}} \big(Tf(x)\big)^q \omega(x)\,dx \\
&= C\beta \sum_i \sum_{k=-\infty}^{i} 2^{kq}2^{-iq} \int_{\{x:2^i<Tf(x)\le 2^{i+1}\}} \big(Tf(x)\big)^q \omega(x)\,dx \\
&= C\beta \sum_i \frac{2^q}{2^q-1} \int_{\{x\,:\,2^i<Tf(x)\le 2^{i+1}\}} \big(Tf(x)\big)^q \omega(x)\,dx \\
&= C\beta \int_{\mathbb{R}} \big(Tf(x)\big)^q \omega(x)\,dx.
\end{aligned}$$

We now estimate (V). Using inequality (3.10), the definition of F, Hölder's inequality and condition (2.3) we get

(3.13)
$$\begin{aligned}
\text{(V)} &= C\sum_{(k,j)\in F} 2^{kq}\omega(E_j^k) = C\sum_{(k,j)\in F} \omega(E_j^k)\left(\frac{\sigma_j^k+\tau_j^k}{\omega(E_j^k)}\right)^q \\
&\le C\beta^{-q} \sum_{(k,j)\in F} \omega(E_j^k)\frac{(\sigma_j^k)^q}{\omega(I_j^k)^q} \\
&= C\beta^{-q} \sum_{(k,j)\in F} \frac{\omega(E_j^k)}{\omega(I_j^k)^q}\left(\int_{3I_j^k-\Omega_{k+m}} fT^\star(\chi_{E_j^k}\omega)\right)^q \\
&\le C\beta^{-q} \sum_{(k,j)\in F} \frac{\omega(E_j^k)}{\omega(I_j^k)^q}\left(\int_{3I_j^k-\Omega_{k+m}} f^p v\right)^{q/p}\left(\int_{3I_j^k-\Omega_{k+m}} \big(T^\star(\chi_{I_j^k}\omega)\big)^{p'}\sigma\right)^{q/p'} \\
&\le C\beta^{-q} \sum_{(k,j)} \frac{\omega(E_j^k)}{\omega(I_j^k)}\left(\int_{3I_j^k-\Omega_{k+m}} f^p v\right)^{q/p} \le C\beta^{-q}\left(\sum_{(k,j)}\int_{3I_j^k-\Omega_{k+m}} f^p v\right)^{q/p} \\
&\le C\beta^{-q}\left(\sum_k \int_{\Omega_k-\Omega_{k+m}} f^p v\right)^{q/p} \le C\beta^{-q}\left(\int_{\mathbb{R}} f^p v\right)^{q/p},
\end{aligned}$$

where we have also used that $E_j^k \subset I_j^k$, the facts that the intervals of the form $3I_j^k$ are "almost" disjoint (parts (iii) and (iv) of (3.2)), that they are all contained in Ω_k and that $1 \le q/p$. Observe that we can use the condition (2.3) because if $I_j^k = [a_j^k, b_j^k)$, then $\int_{b_j^k}^{\infty} \sigma > 0$, otherwise $\operatorname{sop} f \subset (-\infty, b_j^k]$ and taking $x \in 3I_j^k$, $x > b_j^k$ we have $Tf(x) = 0$ but $3I_j^k \subset \Omega_k$ ((3.2), (ii)) which is a contradiction.

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We are going now to estimate the sum over the set G in (3.11). In order to do this we estimate

$$\tau_j^k = \int_{3I_j^k\cap\Omega_{k+m}} fT^\star(\chi_{E_j^k}\omega).$$

Let $H_j^k = \{i : I_i^{k+m}\cap 3I_j^k \neq \emptyset\}$. Then $3I_j^k\cap\Omega_{k+m} \subset \cup_{i\in H_j^k} I_i^{k+m}$. Fix I_i^{k+m} and let a be the left end-point of the interval $3I_i^{k+m}$. If $y \notin 3I_i^{k+m}$ and $y \le a$, then

$$\sup_{x\in I_i^{k+m}} (x-y) \le 2 \inf_{x\in I_i^{k+m}} (x-y),$$

which implies, by the growth condition imposed on K and the fact that K is nonincreasing, that

$$\begin{aligned}\sup_{x\in I_i^{k+m}} K(x-y) &= K\Big(\inf_{x\in I_i^{k+m}}(x-y)\Big) \le CK\Big(2\inf_{x\in I_i^{k+m}}(x-y)\Big)\\ &\le CK\Big(\sup_{x\in I_i^{k+m}}(x-y)\Big) = C\inf_{x\in I_i^{k+m}} K(x-y).\end{aligned}$$

Since $3I_i^{k+m}\subset\Omega_{k+m}$ and $E_j^k\cap\Omega_{k+m}=\emptyset$, we have that $3I_i^{k+m}\cap E_j^k=\emptyset$. It follows that for all $x\in I_i^{k+m}$

$$T^\star(\chi_{E_j^k}\omega)(x) = \int_{-\infty}^a K(x-y)\chi_{E_j^k}(y)\omega(y)\,dy,$$

and thus

$$\sup_{x\in I_i^{k+m}} T^\star(\chi_{E_j^k}\omega)(x) \le C\inf_{x\in I_i^{k+m}} T^\star(\chi_{E_j^k}\omega)(x). \tag{3.14}$$

Using this we can write the following:

$$\begin{aligned}\tau_j^k &= \int_{3I_j^k\cap\Omega_{k+m}} f(x)T^\star(\chi_{E_j^k}\omega)(x)\,dx\\ &\le \sum_{i\in H_j^k}\int_{I_i^{k+m}} f(x)T^\star(\chi_{E_j^k}\omega)(x)\,dx\\ &\le \sum_{i\in H_j^k}\int_{I_i^{k+m}} f(x)\sup_{x\in I_i^{k+m}} T^\star(\chi_{E_j^k}\omega)(x)\,dx\\ &\le C\sum_{i\in H_j^k}\inf_{x\in I_i^{k+m}} T^\star(\chi_{E_j^k}\omega)(x)\int_{I_i^{k+m}} f(x)\,dx\\ &\le C\sum_{i\in H_j^k}\int_{I_i^{k+m}} T^\star(\chi_{E_j^k}\omega)(x)\sigma(x)\,dx\Big(\sigma(I_i^{k+m})\Big)^{-1}\int_{I_i^{k+m}} f(x)\,dx.\end{aligned} \tag{3.15}$$

Observe that if $\sigma(I_i^{k+m}) = 0$ then $\int_{I_i^{k+m}} f(x)\,dx = 0$ since $f\in L^p(v)$ and therefore, from now on, in the last term we are summing over those i's such that $\sigma(I_i^{k+m})>0$.

Let $C_j^k = \big(\sigma(I_j^k)\big)^{-1}\int_{I_j^k} f(x)\,dx$ where the quotient is understood to be zero if $\sigma(I_j^k)=0$. Then, for all $x\in I_j^k$ we have

$$C_j^k = \big(\sigma(I_j^k)\big)^{-1}\int_{I_j^k} f\sigma^{-1}\sigma \le M_\sigma(f\sigma^{-1})(x),$$

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where, if μ is a positive Borel measure, $M_\mu f(x) = \sup_{x\in I}\big(\mu(I)\big)^{-1}\int_I |f|\,d\mu$ (and the quotient is understood to be zero if $\mu(I) = 0$). Let $N_j^k = \{s : I_s^k \cap 3I_j^k \neq \emptyset\}$. Notice that the cardinality of N_j^k is at most C by (3.2), (iv).

In the inequality (3.15) it appears the integral over I_i^{k+m}, with $i \in H_j^k$. Let $s \in N_j^k$. Then I_s^k and I_i^{k+m} are disjoint or $I_i^{k+m} \underset{\neq}{\subset} I_s^k$ by (3.2), (v) and the comment after (3.11). Then

$$
\begin{aligned}
\tau_j^k &\leq C \sum_{i\in H_j^k} C_i^{k+m} \int_{I_i^{k+m}} T^\star(\chi_{E_j^k}\omega)(x)\sigma(x)\,dx \\
&\leq C \sum_{s\in N_j^k}\Big[\sum_{i\in H_j^k : I_i^{k+m}\subset I_s^k} C_i^{k+m} \int_{I_i^{k+m}} T^\star(\chi_{E_j^k}\omega)(x)\sigma(x)\,dx\Big].
\end{aligned} \tag{3.16}
$$

We remind that we are estimating

$$
(\mathrm{VI}) = C \sum_{(k,j)\in G} 2^{kq}\omega(E_j^k).
$$

Let N and M be integers such that $0 \leq M < m$. Define

$$
G_{N,M} = \big\{(k,j) \in G : \omega(E_j^k) \neq 0, k \geq N \text{ and } k \equiv M \pmod m\big\}.
$$

We now claim that

$$
\sum_{\{(k,j)\in G_{N,M}\}} 2^{kq}\omega(E_j^k) \leq C\Big(\int f^p v\Big)^{q/p}, \tag{3.17}
$$

with constant C that not depends on N and M.

Fix N and M and consider the "principal" intervals as in [MW] defined as follows: $\Gamma_0 = \{(k,j) \in G_{N,M} : I_j^k \text{ is maximal}\}$. If Γ_n, has been defined, let Γ_{n+1} consist of those $(k,j) \in G_{N,M}$ for which there is $(t,u) \in \Gamma_n$ with $I_j^k \subset I_u^t$, $C_j^k > 2C_u^t$ and $C_i^l \leq 2C_u^t$ for those I_i^l such that $I_j^k \underset{\neq}{\subset} I_i^l \subset I_u^t$. Let $\Gamma = \cup_{n=0}^\infty \Gamma_n$. For each $(k,j) \in G_{N,M}$ let $P(I_j^k)$ be the smallest interval I_u^t containing I_j^k and such that $(t,u) \in \Gamma$. Observe that the map P is well defined because no interval I_j^k may occur as one of the I_i^{k+m} (since $\omega(E_j^k) \neq 0$). Observe that $P(I_j^k) = I_u^t$ implies $C_j^k \leq 2C_u^t$.

Using inequality (3.16) we estimate the first term of (3.17) as follows:

$$
\begin{aligned}
&\sum_{(k,j)\in G_{N,M}} 2^{kq}\omega(E_j^k) \\
&\qquad\leq \sum_{(k,j)\in G_{N,M}} \omega(E_j^k)\frac{(2\tau_j^k)^q}{\big(\omega(E_j^k)\big)^q} \\
&\qquad\leq C\beta^{-q} \sum_{(k,j)\in G_{N,M}} \frac{\omega(E_j^k)}{\big(\omega(I_j^k)\big)^q}(\tau_j^k)^q
\end{aligned} \tag{3.18}
$$

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$$\le C\beta^{-q}\sum_{(k,j)\in G_{N,M}}\sum_{s\in N_j^k}\frac{\omega(E_j^k)}{\big(\omega(I_j^k)\big)^q}\Bigg[\sum_{\{i:I_i^{k+m}\subset I_s^k \text{ and } (k+m,i)\notin\Gamma\}}\bigg(\int_{I_i^{k+m}}T^\star(\chi_{E_j^k}\omega)\sigma\bigg)C_i^{k+m}\Bigg]^q$$
$$+\,C\beta^{-q}\sum_{(k,j)\in G_{N,M}}\sum_{s\in N_j^k}\frac{\omega(E_j^k)}{\big(\omega(I_j^k)\big)^q}\Bigg[\sum_{\{i\in H_j^k:(k+m,i)\in\Gamma\}}\bigg(\int_{I_i^{k+m}}T^\star(\chi_{E_j^k}\omega)\sigma\bigg)C_i^{k+m}\Bigg]^q$$
$$=(\mathrm{VII})+(\mathrm{VIII}).$$

It appears on (VII) the sum over the set $\{i : I_i^{k+m}\subset I_s^k \text{ and } (k+m,i)\notin\Gamma\}$; notice that if $I_i^{k+m}\subset I_s^k$ and $(k+m,i)\notin\Gamma$ then $P(I_i^{k+m})=P(I_s^k)$. To estimate (VII) we first observe that for a fixed $(t,u)\in\Gamma$ we have

(3.19)
$$\sum_{(k,j)\in G_{N,M}}\sum_{\{s\in N_j^k:P(I_s^k)=I_u^t\}}\frac{\omega(E_j^k)}{\big(\omega(I_j^k)\big)^q}\Bigg[\sum_{\{i:I_i^{k+m}\subset I_s^k \text{ and } (k+m,i)\notin\Gamma\}}C_i^{k+m}\int_{I_i^{k+m}}T^\star(\chi_{E_j^k}\omega)\sigma\Bigg]^q$$
$$\le C\sum_{(k,j)\in G_{N,M}}\sum_{\{s\in N_j^k:P(I_s^k)=I_u^t\}}\omega(E_j^k)\bigg[\frac{1}{\omega(I_j^k)}\int_{I_s^k}T^\star(\chi_{I_j^k}\omega)\sigma\bigg]^q(C_u^t)^q.$$

In the last inequality we have used the following: the I_i^{k+m} are disjoint on i and they are all contained in I_s^k; $P(I_i^{k+m})=P(I_s^k)=I_u^t$, thus $C_i^{k+m}\le 2C_u^t$ and $E_j^k\subset I_j^k$. Let use now that $I_s^k\subset I_u^t$, duality, the fact that the cardinality of N_j^k is at most C, the fact that the E_j^k are disjoint on k and j and that for all $x\in E_j^k\subset I_j^k$ we have that $\big(\omega(I_j^k)\big)^{-1}\int_{I_j^k}T(\chi_{I_u^t}\sigma)\omega\le M_\omega\big(T(\chi_{I_u^t}\sigma)\big)(x)$, to estimate right-hand side of (3.19) with the following:

(3.20)
$$C(C_u^t)^q\sum_{(k,j)\in G_{N,M}}\sum_{\{s\in N_j^k:P(I_s^k)=I_u^t\}}\omega(E_j^k)\bigg[\frac{1}{\omega(I_j^k)}\int_{I_j^k}T(\chi_{I_u^t}\sigma)\omega\bigg]^q$$
$$\le C(C_u^t)^q\sum_{(k,j)\in G_{N,M}}\int_{E_j^k}\Big(M_\omega\big(T(\chi_{I_u^t}\sigma)\big)\Big)^q\omega$$
$$\le C(C_u^t)^q\int_{\mathbb{R}}\Big(M_\omega\big(T(\chi_{I_u^t}\sigma)\big)\Big)^q\omega.$$

Finally, we use the fact that M_ω is bounded from $L^q(\omega)$ into $L^q(\omega)$ for all $q>1$ and we apply condition (2.2) to get that the last term of (3.20) is bounded by

$$C(C_u^t)^q\int_{\mathbb{R}}\big(T(\chi_{I_u^t}\sigma)\big)^q\omega\le C(C_u^t)^q\bigg(\int_{I_u^t}\sigma\bigg)^{q/p}.$$

Combining this with (3.19) and (3.20) and summing over $(t,u)\in\Gamma$ we obtain

(3.21)
$$(\mathrm{VII})\le C\beta^{-q}\sum_{(t,u)\in\Gamma}\bigg(\int_{I_u^t}\sigma\bigg)^{q/p}(C_u^t)^q$$
$$\le C\beta^{-q}\bigg(\sum_{(t,u)\in\Gamma}(C_u^t)^p\int_{I_u^t}\sigma\bigg)^{q/p}.$$

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We now consider (VIII). Let us fix $(k,j) \in G_{N,M}$. It follows from Hölder's inequality, Jensen's inequality and condition (2.3) that

(3.22)
$$
\begin{aligned}
&\frac{\omega(E_j^k)}{\omega(I_j^k)^q}\Big[\sum_{\{i\in H_j^k:(k+m,i)\in\Gamma\}} \sigma(I_i^{k+m})^{-1/p}\sigma(I_i^{k+m})^{1/p}\Big(\int_{I_i^{k+m}} T^\star(\chi_{E_j^k}\omega)\sigma\Big)C_i^{k+m}\Big]^q\\
&\quad\le \frac{\omega(E_j^k)}{\omega(I_j^k)^q}\Big[\sum_{i\in H_j^k}\sigma(I_i^{k+m})^{-p'/p}\Big(\int_{I_i^{k+m}} T^\star(\chi_{E_j^k}\omega)\sigma\Big)^{p'}\Big]^{q/p'}\\
&\qquad\times\Big[\sum_{\{i\in H_j^k:(k+m,i)\in\Gamma\}}\sigma(I_i^{k+m})(C_i^{k+m})^p\Big]^{q/p}\\
&\quad= \frac{\omega(E_j^k)}{\omega(I_j^k)^q}\Big[\sum_{i\in H_j^k}\sigma(I_i^{k+m})^{\frac{-p'}{p}+p'}\Big(\sigma(I_i^{k+m})^{-1}\int_{I_i^{k+m}} T^\star(\chi_{E_j^k}\omega)\sigma\Big)^{p'}\Big]^{q/p'}\\
&\qquad\times\Big[\sum_{\{i\in H_j^k:(k+m,i)\in\Gamma\}}\sigma(I_i^{k+m})(C_i^{k+m})^p\Big]^{q/p}\\
&\quad\le \frac{\omega(E_j^k)}{\omega(I_j^k)^q}\Big(\sum_{i\in H_j^k}\int_{I_i^{k+m}}\big(T^\star(\chi_{E_j^k}\omega)\big)^{p'}\sigma\Big)^{q/p'}\\
&\qquad\times\Big[\sum_{\{i\in H_j^k:(k+m,i)\in\Gamma\}}\sigma(I_i^{k+m})(C_i^{k+m})^p\Big]^{q/p}\\
&\quad\le \frac{\omega(E_j^k)}{\omega(I_j^k)^q}\Big(\int_{\mathbb{R}}\big(T^\star(\chi_{I_j^k}\omega)\big)^{p'}\sigma\Big)^{q/p'}\Big[\sum_{\{i\in H_j^k:(k+m,i)\in\Gamma\}}\sigma(I_i^{k+m})(C_i^{k+m})^p\Big]^{q/p}\\
&\quad\le C\frac{\omega(E_j^k)}{\omega(I_j^k)^q}\omega(I_j^k)^{q/q'}\Big[\sum_{\{i\in H_j^k:(k+m,i)\in\Gamma\}}\sigma(I_i^{k+m})(C_i^{k+m})^p\Big]^{q/p}\\
&\quad\le C\Big[\sum_{\{i\in H_j^k:(k+m,i)\in\Gamma\}}\sigma(I_i^{k+m})(C_i^{k+m})^p\Big]^{q/p}.
\end{aligned}
$$

Taking into account that $p \le q$ we obtain

$$
(\mathrm{VIII}) \le C\beta^{-q}\Big[\sum_{\{(k,j)\in G_{N,M},\, i\in H_j^k:(k+m,i)\in\Gamma\}}\sigma(I_i^{k+m})(C_i^{k+m})^p\Big]^{q/p}.
$$

We claim now that the last sum can be changed by a sum over $(t,u) \in \Gamma$. In fact, for fixed $(k+m,i)$, the number of index j such that $I_i^{k+m} \cap 3I_j^k \ne \emptyset$ is at most C (by (3.2), (iii) and (iv)). In fact, since $I_i^{k+m} \subset \Omega_{k+m} \subset \Omega_k$, there exists s such that $I_i^{k+m} \subset I_s^k$ and the number of index j such that $3I_j^k \cap I_s^k \ne \emptyset$ is at most C. Therefore

$$
(\mathrm{VIII}) \le C\beta^{-q}\Big(\sum_{(t,u)\in\Gamma}\sigma(I_u^t)(C_u^t)^p\Big)^{q/p}. \tag{3.23}
$$

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Combining (3.21) and (3.23) we get

$$\text{(VII)} + \text{(VIII)} \leq C\beta^{-q}\Big(\sum_{(t,u)\in\Gamma}\sigma(I_u^t)(C_u^t)^p\Big)^{q/p} \leq C\beta^{-q}\Big(\int_{\mathbb{R}}\Big(\sum_{(t,u)\in\Gamma}(C_u^t)^p\chi_{I_u^t}(x)\Big)\sigma(x)\,dx\Big)^{q/p}. \tag{3.24}$$

Observe that for fixed x

$$\sum_{(t,u)\in\Gamma}(C_u^t)^p\chi_{I_u^t}(x) = (C_{u_0}^{t_0})^p + (C_{u_1}^{t_1})^p + \cdots,$$

where

$$x \in \cdots I_{u_2}^{t_2} \subset I_{u_1}^{t_1} \subset I_{u_0}^{t_0}, \text{ with } (t_0,u_0),(t_1,u_1),(t_2,u_2),\ldots \in \Gamma$$

and

$$C_{u_1}^{t_1} > 2C_{u_0}^{t_0}, \quad C_{u_2}^{t_2} > 2C_{u_1}^{t_1} > 2^2C_{u_0}^{t_0}, \quad \ldots.$$

Each partial sum can be bounded as follows:

$$(C_{u_0}^{t_0})^p + (C_{u_1}^{t_1})^p + \cdots + (C_{u_s}^{t_s})^p \leq (C_{u_s}^{t_s})^p\frac{2^p}{2^p-1} \leq \frac{2^p}{2^p-1}\sup_{\{I_u^t : x\in I_u^t,(t,u)\in\Gamma\}}(C_u^t)^p \leq C\big(M_\sigma(f\sigma)\big)^p(x). \tag{3.25}$$

Therefore, using that M_σ is of strong type (p,p) respect to the measure $\sigma(x)\,dx$, we have

$$\text{(VII)} + \text{(VIII)} \leq C\beta^{-q}\Big(\int_{\mathbb{R}}(M_\sigma(f\sigma))^p\sigma\Big)^{q/p} \leq C\beta^{-q}\Big(\int_{\mathbb{R}}f^p\sigma^p\sigma\Big)^{q/p} = C\beta^{-q}\Big(\int_{\mathbb{R}}f^p v\Big)^{q/p}. \tag{3.26}$$

Combining now inequalities (3.18) and (3.26) we get inequality (3.17) with a constant C independent of N and M. Then, from (3.7), (3.8), (3.11), (3.12), (3.13), (3.18) and (3.26) we get

$$\int_{\mathbb{R}}(Tf)^q\omega \leq C\beta\int_{\mathbb{R}}(Tf)^q\omega + C\beta^{-q}\Big(\int_{\mathbb{R}}f^p v\Big)^{q/p}. \tag{3.27}$$

Choose β small enough to have $C\beta < 1/2$. Observe that the conditions imposed on f implies that

$$\int_{\mathbb{R}}(Tf)^q\omega < \infty.$$

Then, we can substract $C\beta\int_{\mathbb{R}}(Tf)^q\omega$ in both members of inequality (3.27) to get

$$\int_{\mathbb{R}}(Tf)^q\omega \leq C\Big(\int_{\mathbb{R}}f^p v\Big)^{q/p},$$

for all $f \geq 0$, bounded, with compact support and such that $f\sigma^{-1}$ is bounded. This finishes the proof of Theorem 1.

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4. **Proofs of Theorem 3 and Theorem 4.**

PROOF OF THEOREM 3. First suppose that there exists ω not identically zero such that (2.6) holds. Then there is an interval $I_0 = [a_0, b_0)$ such that $\omega(I_0) > 0$. If we denote by A the set $\{x > b_0 : v(x) < \infty\}$, then $|A| > 0$, since v is not identically infinity a.e. in (b_0, ∞).

For fixed $N \in \mathbb{N}$ we consider $\sigma_N(x) = \min\{\sigma(x), N\}$. Then $\sigma_N \in L^1_{loc}(b_0, \infty)$, thus, Lebesgue differentiation theorem gives that

$$\sigma_N(x) = \lim_{h \to 0^+} \frac{1}{h} \int_x^{x+h} \sigma_N \quad \text{a.e. } x \in (b_0, \infty).$$

Since $|A| > 0$, there exists $a \in A$ such that

$$\sigma_N(a) = \lim_{h \to 0^+} \frac{1}{h} \int_a^{a+h} \sigma_N.$$

Taking into account that $\sigma_N(a) > 0$ we have that $\lim_{h \to 0^+} \frac{1}{h} \int_a^{a+h} \sigma_N > 0$. This implies that $\int_a^{a+h} \sigma_N > 0$ for all $h > 0$ and therefore $\int_a^b \sigma > 0$ for all $b > a$.

We are going to prove now that

$$\int_b^\infty K(y-a)^{p'} \sigma(y)\, dy < \infty, \quad \text{for all } b > a.$$

Suppose that $\int_b^\infty K(y-a)^{p'} \sigma(y)\, dy = \infty$. Then $v^{-1}(y) K(y-a) \chi_{(b,\infty)}(y) \notin L^{p'}(v)$ and therefore there is $g \geq 0$, $g \in L^p(v)$, such that $\int_b^\infty g(y) K(y-a)\, dy = \infty$. For each $x \in I_0$ we have

$$Tg(x) = \int_x^\infty K(y-x) g(y)\, dy \geq \int_b^\infty g(y) K(y-a) \frac{K(y-x)}{K(y-a)}\, dy.$$

Let us dominate $\frac{K(y-x)}{K(y-a)}$ from below for $y \in (b, \infty)$. Let $c = a + (a - a_0)$. If $y \in (c, \infty)$, then $y - x \leq 2(y-a)$ and thus $K(y-a) \leq CK\big(2(y-a)\big) \leq CK(y-x)$. This implies that

$$\frac{1}{C} \leq \frac{K(y-x)}{K(y-a)}$$

for $y \in (c, \infty)$. If $c \leq b$ this inequality holds for all $y > b$ and in this case we would have obtained the estimation that we need. However if $c > b$ we still have to dominate $\frac{K(y-x)}{K(y-a)}$ from below for the numbers $y \in (b, c)$. In this case, *i.e.*, $c > b$ and $y \in (b, c)$, we have $y - x \leq c - a_0$ and $y - a \geq b - a$, thus

$$\frac{K(c-a_0)}{K(b-a)} \leq \frac{K(y-x)}{K(y-a)}.$$

Therefore, in both cases, we have obtained that there exists a positive constant C such that

$$C \leq \frac{K(y-x)}{K(y-a)}, \quad \text{for all } y > b.$$

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As a consequence we obtain

$$Tg(x) \geq C \int_b^\infty g(y)K(y-a)\,dy = \infty$$

for all $x \in I_0$. By (2.6) and the fact that $\omega(I_0) > 0$ this inequality implies that

$$\infty = \int_{I_0} |Tg(x)|^q \omega(x)\,dx \leq C\Big(\int g^p(x)v(x)\,dx\Big)^{q/p},$$

which is a contradiction since $g \in L^p(v)$.

Conversely, suppose that there exists $a \in \mathbb{R}$ such that (2.7) holds for all $b > a$. Then we can find an interval $I_0 = [a, b)$ such that $\sigma(I_0) > 0$ and $\sigma(I_0^\star) > 0$. Fix I_0 and set $\omega = \chi_{I_0}\big(T(\sigma\chi_{I_0\cup I_0^\star})\big)^{-q/p'}$. Observe that $T(\sigma\chi_{I_0\cup I_0^\star})(x)$ is strictly positive in I_0 since $\sigma(I_0^\star) > 0$. To see that ω is nontrivial we are going to prove that $T(\sigma\chi_{I_0\cup I_0^\star})(x) < \infty$ a.e. $x \in I_0$.

Let m be such that $a < m < b$ and let c be the right endpoint of $I_0^\star$. Then if $x \in [m, b)$

$$T(\sigma\chi_{I_0\cup I_0^\star})(x) = T(\sigma\chi_{[m,c)})(x).$$

The assumption on K, $K(x) \leq Cx^{-1/q_0}$, gives that T is dominated by the Weyl fractional integral W_α with $\alpha = 1-q_0^{-1}$. Therefore T is of weak type $(1, q_0)$. This and condition (2.7) gives, for all $\lambda > 0$, the following:

$$\begin{aligned}
\Big|\Big\{x \in [m, b) : T(\sigma\chi_{I_0\cup I_0^\star})(x) > \lambda\Big\}\Big|
&= \Big|\Big\{x \in [m, b) : T(\sigma\chi_{[m,c)})(x) > \lambda\Big\}\Big| \\
&\leq C\lambda^{-q_0}\bigg(\int_m^c \sigma(y)K(y-a)^{p'}K(y-a)^{-p'}\,dy\bigg)^{q_0} \\
&\leq C\lambda^{-q_0}K(c-a)^{-p'q_0}\bigg(\int_m^\infty \sigma(y)K(y-a)^{p'}\,dy\bigg)^{q_0} \leq C(m)\lambda^{-q_0},
\end{aligned}$$

where $C(m)$ is a constant that depends on m. Letting λ go to infinity we have that

$$\Big|\Big\{x \in [m, b) : T(\sigma\chi_{I_0\cup I_0^\star})(x) = \infty\Big\}\Big| = 0.$$

This argument is valid for all $m \in (a, b)$, therefore $T(\sigma\chi_{I_0\cup I_0^\star})(x) < \infty$ a.e. $x \in I_0$.

In order to prove (2.6) for the weight ω, it suffices by Theorem 1 to establish that (2.2) and (2.3) hold.

We first prove (2.2). Let $I = [d, e)$ be such that $\int_{(-\infty,d)} \omega > 0$. Then $d > a$ since the support of ω is I_0. We begin by proving that $\sigma(I) < \infty$. This follows from (2.7) and the following inequality:

$$\sigma(I) = \int_d^e \sigma(y)K(y-a)^{p'}K(y-a)^{-p'}\,dy \leq K(e-a)^{-p'}\int_d^e \sigma(y)K(y-a)^{p'}\,dy.$$

Let $f_1 = \sigma\chi_{I\cap(I_0\cup I_0^\star)}$ and $f_2 = \sigma\chi_{I-(I_0\cup I_0^\star)}$. Then $\sigma\chi_I = f_1 + f_2$ and

$$\Big(\int_{\mathbb{R}}(T(\sigma\chi_I))^q\omega\Big)^{1/q} \leq \Big(\int_{\mathbb{R}}(Tf_1)^q\omega\Big)^{1/q} + \Big(\int_{\mathbb{R}}(Tf_2)^q\omega\Big)^{1/q}. \tag{4.1}$$

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Since T is of weak type $(1, q_0)$ we obtain

$$\int_{\mathbb{R}} (Tf_1)^q \omega = \int_{I_0} \Big(T(\sigma\chi_{I\cap(I_0\cup I_0^\star)})\Big)^q \Big(T(\sigma\chi_{I_0\cup I_0^\star})\Big)^{-q/p'} \tag{4.2}$$
$$\leq \int_{I_0} \Big(T(\sigma\chi_{I\cap(I_0\cup I_0^\star)})\Big)^{q/p}$$
$$\leq \int_0^\infty \frac{q}{p}\lambda^{\frac{q}{p}-1} \min\Big\{|I_0|, C\Big(\frac{\sigma(I)}{\lambda}\Big)^{q_0}\Big\}\, d\lambda.$$

Now we write the integral over $(0,\infty)$ as the sum of the integral over $\big(0, C|I_0|^{-1/q_0}\sigma(I)\big)$ and the integral over $[C|I_0|^{-1/q_0}\sigma(I),\infty)$, where C is the constant appearing in (4.2). In the first integral the minimum is $|I_0|$, while in the second integral the minimum is $C\big(\sigma(I)\big)^{q_0}\lambda^{-q_0}$. Then using that $\frac{q}{p} - q_0 < 0$, we obtain that

$$\int_{\mathbb{R}} (Tf_1)^q \omega \leq C\big(\sigma(I)\big)^{q/p}, \tag{4.3}$$

where C depends only on p, q, q_0, I_0.

To handle Tf_2, we observe that it suffices to consider only the intervals $I = [d,e)$ such that $e > c$ where $I_0^\star = [b,c)$. Let $y > c$. Then for all $x \in I_0$ we have that $\frac{1}{2}(y-a) \leq y - x \leq y - a$. Using Hölder's inequality with the measure σ we obtain for all $x \in I_0$

$$Tf_2(x) \leq \int_c^\infty \sigma(y)\chi_I(y)K(y-x)\,dy \leq C\int_c^\infty \sigma(y)\chi_I(y)K(y-a)\,dy$$
$$\leq C\Big(\int_c^\infty K(y-a)^{p'}\sigma(y)\,dy\Big)^{1/p'}\big(\sigma(I)\big)^{1/p} \leq C\big(\sigma(I)\big)^{1/p} < \infty,$$

where we have used that $\int_c^\infty K(y-a)^{p'}\sigma(y)\,dy$ is a finite constant by (2.5). Consequently

$$\int_{\mathbb{R}} (Tf_2)^q \omega \leq C\omega(I_0)\big(\sigma(I)\big)^{q/p} = C\big(\sigma(I)\big)^{q/p}, \tag{4.4}$$

since $\omega(I_0) < \infty$ (observe that ω is bounded with compact support, in fact $\omega(x) \leq \big(K(2|I_0|)\sigma(I_0^\star)\big)^{-q/p'}$). This finishes the proof of (2.2).

Now, we are going to prove (2.3). Let $I = [d,e)$ be such that $\int_{[e,\infty)}\sigma > 0$. Then $\big(\omega(I)\big)^{1/q'} < \infty$ since ω is bounded with compact support. Let $\sigma = f_1 + f_2$ where $f_1 = \sigma\chi_{I_0\cup I_0^\star}$ and $f_2 = \sigma\chi_{\mathbb{R}-(I_0\cup I_0^\star)}$. By duality we have

$$\Big(\int_{\mathbb{R}} \big(T^\star(\omega\chi_I)\big)^{p'}\sigma\Big)^{1/p'} = \|T^\star(\omega\chi_I)\|_{L^{p'}(\sigma)} \tag{4.5}$$
$$= \sup_{\{g\geq 0: \|g\|_{L^p(\sigma)}=1\}} \int_{\mathbb{R}} T^\star(\omega\chi_I)g\sigma$$
$$= \sup_{\{g\geq 0: \|g\|_{L^p(\sigma)}=1\}} \int_{\mathbb{R}} \omega\chi_I T(g\sigma)$$
$$\leq \sup_{\{g\geq 0: \|g\|_{L^p(\sigma)}=1\}} \int_{I\cap I_0} \omega T(gf_1)$$
$$+ \sup_{\{g\geq 0: \|g\|_{L^p(\sigma)}=1\}} \int_{I\cap I_0} \omega T(gf_2) = \text{(I)} + \text{(II)}.$$

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Let us estimate (I). If $x \in I \cap I_0$, Hölder's inequality gives that

(4.6)
$$T(gf_1)(x) = \int_{(x,\infty)\cap(I_0\cup I_0^\star)} \sigma(y)g(y)K(y-x)\,dy$$
$$\leq \Big(\int_{(x,\infty)\cap(I_0\cup I_0^\star)} \sigma(y)K(y-x)\,dy\Big)^{1/p'} \Big(\int_{(x,\infty)\cap(I_0\cup I_0^\star)} g^p(y)\sigma(y)K(y-x)\,dy\Big)^{1/p}$$
$$= \big(T(\sigma\chi_{(I_0\cup I_0^\star)})(x)\big)^{1/p'} \big(T(g^p\sigma\chi_{(I_0\cup I_0^\star)})(x)\big)^{1/p}.$$

Now, we use Hölder's inequality to obtain

(4.7)
$$\int_{I\cap I_0} \omega T(gf_1)$$
$$\leq \Big(\int_{I\cap I_0} \omega\Big)^{1/q'} \Big(\int_{I\cap I_0} \big(T(gf_1)\big)^q \omega\Big)^{1/q}$$
$$\leq \big(\omega(I)\big)^{1/q'} \Big[\int_{I\cap I_0} \big(T(\sigma\chi_{(I_0\cup I_0^\star)})\big)^{q/p'} \big(T(\sigma\chi_{I_0\cup I_0^\star})\big)^{-q/p'} \big(T(g^p\sigma\chi_{(I_0\cup I_0^\star)})\big)^{q/p}\Big]^{1/q}$$
$$= \big(\omega(I)\big)^{1/q'} \Big(\int_{I\cap I_0} \big(T(g^p\sigma\chi_{(I_0\cup I_0^\star)})\big)^{q/p}\Big)^{1/q}.$$

The weak type $(1, q_0)$ of T and the same argument as in the proof of (4.3) give that

$$\Big(\int_{I\cap I_0} \big(T(g^p\sigma\chi_{(I_0\cup I_0^\star)})\big)^{q/p}\Big)^{1/q} \leq C\Big(\int_{(I_0\cup I_0^\star)} g^p\sigma\Big)^{1/p} \leq C. \tag{4.8}$$

Putting together the inequalities (4.6), (4.7) and (4.8) we obtain (I)$\leq C\big(\omega(I)\big)^{1/q'}$.

We now estimate (II). Let $x \in I \cap I_0$, then the growth condition imposed on K gives that

(4.9)
$$T(gf_2)(x) = \int_{(x,\infty)\cap\big(\mathbb{R}-(I_0\cup I_0^\star)\big)} \sigma(y)g(y)K(y-x)\,dy$$
$$\leq \Big(\int_{(x,\infty)\cap\big(\mathbb{R}-(I_0\cup I_0^\star)\big)} \sigma(y)K(y-x)^{p'}\,dy\Big)^{1/p'} \Big(\int g^p\sigma\Big)^{1/p}$$
$$\leq C\Big(\int_c^\infty \sigma(y)K(y-a)^{p'}\,dy\Big)^{1/p'} = C.$$

As a consequence

(4.10)
$$\int_{I\cap I_0} \omega T(gf_2) \leq \Big(\int_{I\cap I_0} \omega\Big)^{1/q'} \Big(\int_{I\cap I_0} C^q\omega\Big)^{1/q}$$
$$\leq C\big(\omega(I)\big)^{1/q'} \omega(I_0)^{1/q} \leq C\big(\omega(I)\big)^{1/q'}.$$

Then (II)$\leq C\big(\omega(I)\big)^{1/q'}$ and so (2.3) holds.

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PROOF OF THEOREM 4. We first assume that there exists ω not identically zero such that (2.8) holds. Let $I_1 = [a_1, b_1)$ be a dyadic interval such that $\omega(I_1) > 0$. As in the case of T, the fact that v is not identically infinity in (b_1, ∞) yields that there is $a > b_1$ such that $\int_a^b \sigma > 0$ for all $b > a$.

Let I_0 be dyadic with $I_1 \subset I_0$ and $a \in I_0^\star$. This dyadic interval satisfies that $\omega(I_0) > 0$ and $\sigma(I_0^\star) > 0$. We claim that if I is a dyadic interval with $I_0 \subset I$ then $\sigma(I^\star) < \infty$. We are going to prove this by contradiction.

Suppose that $\sigma(I^\star) = \infty$. Then $v^{-1}\chi_{I^\star} \notin L^{p'}(v)$ and thus there is $g \geq 0$, $g \in L^p(v)$, such that $\int g v^{-1}\chi_{I^\star} v = \int_{I^\star} g = \infty$. Let $x \in I$, then $I^\star \in A_x$ and

$$M_{K,d}^+ g(x) \geq K(|I|) \int_{I^\star} |g(t)|\, dt = \infty.$$

But, since $I_1 \subset I$, this implies that

$$\infty = \Big(\int_I (M_{K,d}^+ g)^q \omega \Big)^{1/q} \leq \Big(\int_{\mathbb{R}} (M_{K,d}^+ g)^q \omega \Big)^{1/q} \leq C \Big(\int_{\mathbb{R}} |g|^p v \Big)^{1/p} < \infty.$$

This is a contradiction. Therefore $\sigma(I^\star) < \infty$.

On the other hand, if $\sigma(I^\star) > 0$, we have

$$K(|I|)\sigma(I^\star)\big(\omega(I)\big)^{1/q} \leq \Big(\int_I (M_{K,d}^+(\sigma\chi_{I^\star}))^q \omega \Big)^{1/q} \leq C\big(\sigma(I^\star)\big)^{1/p} < \infty,$$

which implies that $\omega(I) < \infty$. Since $\omega(I_0) \leq \omega(I)$, we obtain

$$K(|I|)\sigma(I^\star)\big(\omega(I_0)\big)^{1/q} \leq C\big(\sigma(I^\star)\big)^{1/p}$$

and then, taking into account that $0 < \sigma(I^\star) < \infty$ and $0 < \omega(I_0) < \infty$, this inequality yields that

$$K(|I|)\Big(\int_{I^\star} \sigma \Big)^{1/p'} \leq C\big(\omega(I_0)\big)^{-1/q} = C.$$

Consequently

$$\begin{aligned} &\sup_{\{I \text{ dyadic}, I \supset I_0\}} K(|I|)\Big(\int_{I^\star} \sigma \Big)^{1/p'} \\ &= \sup_{\{I \text{ dyadic}, I \supset I_0 \text{ and } \sigma(I^\star) > 0\}} K(|I|)\Big(\int_{I^\star} \sigma \Big)^{1/p'} < \infty. \end{aligned}$$

Conversely, assume that (2.9) holds. Let J_1 be the left half part of $I_0^\star$. If $x \in J_1$ then

$$M_{K,d}^+(\sigma\chi_{I_0^\star})(x) \geq K(|J_1|) \int_{J_1^\star} \sigma.$$

If $\int_{J_1^\star} \sigma > 0$, we take $\omega = \chi_{J_1}\big(M_{K,d}^+(\sigma\chi_{I_0^\star})\big)^{-q/p'}$. If $\int_{J_1^\star} \sigma = 0$, we consider the left half part of J_1 and call it J_2. Then, for $x \in J_2$, we have

$$M_{K,d}^+(\sigma\chi_{I_0^\star})(x) \geq K(|J_2|) \int_{J_2^\star} \sigma.$$

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If $\int_{J_2^\star}\sigma > 0$, we take $\omega = \chi_{J_2}\big(M^+_{K,d}(\sigma\chi_{I_0^\star})\big)^{-q/p'}$. If $\int_{J_2^\star}\sigma = 0$, we consider J_3, *etc*. This process can not continue indefinitely, because $\int_{I_0^\star}\sigma > 0$ and $\cup_{i=1}^{\infty} J_i^\star = I_0^\star$. Then there is a dyadic interval J strictly contained in $I_0^\star$, with the same left endpoint that $I_0^\star$ and such that $\int_{J^\star}\sigma > 0$. Fix J and set $\omega = \chi_J\big(M^+_{K,d}(\sigma\chi_{I_0^\star})\big)^{-q/p'}$. Observe that for all $x \in J$, $M^+_{K,d}(\sigma\chi_{I_0^\star})(x) > 0$. Furthermore, by (2.9), $\int_{I_0^\star}\sigma < \infty$. This and the fact that $M^+_{K,d}$ is of weak type $(1, q_0)$ (because $M^+_{K,d}|f| \le CT|f|$) give that $M^+_{K,d}(\sigma\chi_{I_0^\star})(x) < \infty$ a.e. $x \in J$. Then ω is nontrivial and it is bounded with compact support.

To prove (2.8) we use Theorem 2. We are going to show that for every dyadic interval $I = [a, b)$ with $\int_{(-\infty,b)}\omega > 0$, one has that

$$\int_{I^\star}\sigma < \infty \quad \text{and} \quad \Big(\int_{I\cup I^\star}\big(M^+_{K,d}(\sigma\chi_{I^\star})\big)^q\omega\Big)^{1/q} \le C\Big(\int_{I^\star}\sigma\Big)^{1/p}.$$

Let $I = [a, b)$ dyadic with $\int_{(-\infty,b)}\omega > 0$. To prove that $\int_{I^\star}\sigma < \infty$ we are going to see that there exists a dyadic interval Q such that $I_0 \subset Q$ and $I^\star \subset Q^\star$. Once we have proved this, we have that $\int_{I^\star}\sigma < \infty$ by (2.9). In order to prove the existence of Q, we observe that we have the following three cases: $I_0 \subset I$, $I \subset I_0$, and $I_0 \cap I = \emptyset$. In the first case we choose $Q = I$. The second case is impossible because $\int_{(-\infty,b)}\omega > 0$ and the support of ω is J. In the third case we have to work harder. First we observe that I is on the right of I_0. If $I_0 \subset (-\infty, 0)$, and $I \subset [0,\infty)$, then it is obvious that there exists Q with the required property. If $I_0 \subset (-\infty, 0)$ and $I \subset (-\infty, 0)$ or $I_0 \subset [0,\infty)$ and $I \subset [0,\infty)$ then there is a dyadic interval H such that $I_0, I \subset H$. Let H be the smallest one with this property and let $H_1, H_2 \subset H$ be the dyadic intervals with $|H_1| = \frac{1}{2}|H| = |H_2|$. Then necessarily $I_0 \subset H_1$ and $I \subset H_2$. Since $H_1^\star = H_2$ we have that $I^\star \subset H_1^\star$ or $I^\star \subset H^\star$. If $I^\star \subset H_1^\star$, we choose $Q = H_1$ and if $I^\star \subset H^\star$, we choose $Q = H$.

In order to prove that

$$\Big(\int_{I\cup I^\star}\big(M^+_{K,d}(\sigma\chi_{I^\star})\big)^q\omega\Big)^{1/q} \le C\Big(\int_{I^\star}\sigma\Big)^{1/p},$$

it is clear that we only have to consider I with $(I \cup I^\star) \cap J \ne \emptyset$. Let $f_1 = \sigma\chi_{I^\star\cap I_0^\star}$ and $f_2 = \sigma\chi_{(I^\star - I_0^\star)}$. It suffices to prove the above inequality with $\sigma\chi_{I^\star}$ replaced by f_1 and f_2. Using that $M^+_{K,d}$ is of weak type $(1, q_0)$ and arguing as we did with T in the proof of Theorem 3, we obtain

$$\begin{aligned}\int_{I\cup I^\star}(M^+_{K,d}f_1)^q\omega &\le \int_{(I\cup I^\star)\cap J}\big(M^+_{K,d}(\sigma\chi_{I^\star\cap I_0^\star})\big)^{q/p} \\ &\le C|J|^{1-\frac{q}{q_0 p}}\big(\sigma(I^\star)\big)^{q/p} = C\big(\sigma(I^\star)\big)^{q/p},\end{aligned} \tag{4.11}$$

where C depends only on p, q, q_0 and J.

Let us estimate now $\int_{I\cup I^\star}(M^+_{K,d}f_2)^q\omega$.

If $I^\star \subset I_0^\star$ then $f_2 = 0$ and there is nothing to prove. If $I^\star \not\subseteq I_0^\star$ then $I^\star \cap (\mathbb{R} - I_0^\star) \ne \emptyset$. Since we are considering that $(I \cup I^\star) \cap J \ne \emptyset$, we have that $I \cap J \ne \emptyset$ or $I^\star \cap J \ne \emptyset$. If $I^\star \cap J \ne \emptyset$ then $I^\star \cap I_0^\star \ne \emptyset$, but this implies that $I_0^\star \subset I^\star$ which is a contradiction with

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the fact that $\int_{(-\infty,b)}\omega > 0$. Thus, necessarily $I^\star \cap J = \emptyset$ and $I \cap J \neq \emptyset$. We have two possibilities, $I \subset J$ or $J \subsetneq I$. Observe that $I \subset J$ leads to $I^\star \subset I_0^\star$ which is a contradiction. Then we have that $J \subsetneq I$. If $I \not\supseteq I_0^\star$ then $I^\star \cap I_0^\star \neq \emptyset$ and since $I^\star \cap (\mathbb{R} - I_0^\star) \neq \emptyset$ we obtain that $I_0^\star \subset I^\star$ which is again a contradiction. Therefore $J \subsetneq I_0^\star \subset I$.

Recall that we are estimating $\int_{I\cup I^\star}(M^+_{K,d}f_2)^q\omega$. Let $x \in J$ and let $\tilde{I}$ be such that $\tilde{I}^\star \in A_x$ and $\tilde{I}^\star \cap I^\star \neq \emptyset$. Then we can find a dyadic interval H such that $I_0 \subset H$, $I^\star \cap \tilde{I}^\star \subset H^\star$ and such that $|H| = |I^\star \cap \tilde{I}^\star|$ or $|H| = 2|I^\star \cap \tilde{I}^\star|$. Thus, by condition (2.7),

$$\begin{aligned} K(|\tilde{I}|)\int_{\tilde{I}^\star}\sigma\chi_{(I^\star - I_0^\star)} = K(|\tilde{I}|)\int_{\tilde{I}^\star\cap I^\star}\sigma \le K(|\tilde{I}|)\Big(\int_{\tilde{I}^\star\cap I^\star}\sigma\Big)^{1/p'}\Big(\int_{I^\star}\sigma\Big)^{1/p} \\ \le CK(|H|)\Big(\int_{H^\star}\sigma\Big)^{1/p'}\Big(\int_{I^\star}\sigma\Big)^{1/p} \le C\Big(\int_{I^\star}\sigma\Big)^{1/p}. \end{aligned}$$

It follows that

$$\Big(\int_{I\cup I^\star}(M^+_{K,d}f_2)^q\omega\Big)^{1/q} \le C\big(\omega(J)\big)^{1/q}\big(\sigma(I^\star)\big)^{1/p} = C\big(\sigma(I^\star)\big)^{1/p}.$$

This finishes the proof of Theorem 4.

FINAL REMARKS.

(1) It is possible to change the integrals over $\mathbb{R}$ in conditions (2.2) and (2.3) of Theorem 1, by integrals over I. We can do it by the following result.

THEOREM 5. *Let $1 < p \le q < \infty$ or $p = 1 < q < \infty$. Let K, T and $T^\star$ be as in Theorem 1.*

(1) If $1 < p \le q < \infty$ the following conditions are equivalent

(a) There exists C such that for all $f \in L^p(v)$ and all $\lambda > 0$,

$$\omega\Big(\big\{x : |Tf(x)| > \lambda\big\}\Big) \le C\Big(\frac{1}{\lambda^p}\int |f|^p v\Big)^{q/p}.$$

(b) There exists C such that for every interval $I = [a,b)$ with $\int_{[b,\infty)}\sigma > 0$,

$$\Big(\int_{\mathbb{R}}\big(T^\star(\chi_I\omega)\big)^{p'}\sigma\Big)^{1/p'} \le C\big(\omega(I)\big)^{1/q'} < \infty.$$

(c) There exists C such that for every interval $I = [a,b)$ with $\int_{[b,\infty)}\sigma > 0$,

$$\Big(\int_{I}\big(T^\star(\chi_I\omega)\big)^{p'}\sigma\Big)^{1/p'} \le C\big(\omega(I)\big)^{1/q'} < \infty.$$

(2) If $p = 1 < q < \infty$ then (a) *is equivalent to*

(d) There exists C such that for every bounded interval I

$$\|T^\star(\chi_I\omega)v^{-1}\|_{L^\infty(v)} \le C\big(\omega(I)\big)^{1/q'} < \infty.$$

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PROOF OF THEOREM 5. *We first prove that (a) ⇒ (b). Using duality and (a) we have*

$$\begin{aligned}\Big(\int_{\mathbb{R}}\big(T^\star(\chi_I\omega)\big)^{p'}\sigma\Big)^{1/p'} &= \|T^\star(\chi_I\omega)v^{-1}\|_{L^{p'}(v)}\\ &= \sup_{\{g\geq 0:\|g\|_{L^p(v)}=1\}}\int_{\mathbb{R}} T^\star(\chi_I\omega)g\\ &= \sup_{\{g\geq 0:\|g\|_{L^p(v)}=1\}}\int_I Tg\,\omega\\ &= \sup_{\{g\geq 0:\|g\|_{L^p(v)}=1\}}\int_0^\infty \omega\Big(\big\{x\in I : Tg(x)>\lambda\big\}\Big)\,d\lambda\\ &\leq C\int_0^\infty \min\big(\omega(I), C\lambda^{-q}\big)\,d\lambda = C\omega(I)^{1/q'}.\end{aligned}$$

It is obvious that (b) ⇒ (c).

To prove that (c) ⇒ (a) observe that this is a generalization of Theorem 2 in [LT]. The proof follows the same pattern, changing the kernel $\frac{1}{x^{1-\alpha}}$ by $K(x)$, the only exception being the point where we have to prove that $A_t = \sup_{0<\lambda<t}\lambda^q\omega\Big(\big\{x : Tf(x)>\lambda\big\}\Big)$ is finite. We are going to prove this.

As in [LT] it is enough to consider the case of small t and we may assume that f is nonnegative and bounded with compact support $[a,b]\subset(-\infty,\beta)$, where $\beta=\inf\{x : \int_{[x,\infty)}\sigma=0\}$. Therefore, $\int_{[b,\infty)}\sigma>0$ and $\omega(a,b)<\infty$ by condition (c). Then, as in [LT] we only have to prove that

$$\sup_{0<\lambda<t}\lambda^q\omega\Big(\big\{x<a : Tf(x)>\lambda\big\}\Big)<\infty.$$

Observe that $x<a$ and $Tf(x)>\lambda$ imply that $\lambda<K(a-x)\int_a^b f$. Let

$$B_\lambda=\Big\{y : K(y)>\frac{\lambda}{\int_a^b f}\Big\}.$$

Since K is nonincreasing and lower semicontinuous, B_λ is an open interval, $B_\lambda=(0,s)$. Since $\lim_{x\to\infty}K(x)=0$, s can not be infinity. On the other hand, $K(s)=\lambda(\int_a^b f)^{-1}$, since K is lower semicontinuous. Therefore, $x<a$ and $Tf(x)>\lambda$ imply that $a-x\in B_\lambda$ and then $x\in(a-s,a)$.

Choose t small enough to have that if $\lambda<t$ then $s>b-a$. Then

$$\lambda^q\omega\Big(\big\{x<a : Tf(x)>\lambda\big\}\Big)\leq\lambda^q\int_{a-s}^a\omega=K(s)^q\Big(\int_a^b f\Big)^q\int_{a-s}^a\omega.$$

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If $p > 1$ we may use Hölder's inequality and get

$$\begin{aligned}\lambda^q\omega\big(\{x < a : Tf(x) > \lambda\}\big) &\le \left(\int_a^b f^p v\right)^{q/p}\left(\int_a^b \sigma\right)^{q/p'} K(s)^q \int_{a-s}^a \omega\\ &= \left(\int_a^b f^p v\right)^{q/p}\left(\int_a^b \sigma(y)K(s)^{p'}\,dy\right)^{q/p'} \int_{a-s}^a \omega\\ &\le C\left(\int_a^b f^p v\right)^{q/p}\left(\int_a^b \sigma(y)K(y-a+s)^{p'}\,dy\right)^{q/p'} \int_{a-s}^a \omega\\ &\le C\left(\int_a^b f^p v\right)^{q/p} < \infty.\end{aligned}$$

We have used that $s > b-a$ implies $y-a+s < 2s$, the growth condition of K and the fact that (c) implies that there exists C such that

$$\left(\int_a^b \sigma(y)K(y-a+s)^{p'}\,dy\right)^{q/p'} \int_{a-s}^a \omega \le C.$$

(Claim (1.3)⇒(2.1) in [LT]). If $p = 1$ we follow the same proof as in [LT].

(2) Changing the orientation of the real line we obtain the last theorem for $T^\star$. Therefore, for $1 < p \le q < \infty$ the operator T is bounded from $L^p(v)$ to $L^q(\omega)$, if, and only if, it is of weak type (p,q) with respect to the measures (v,ω) and $T^\star$ is of weak type (q',p') with respect to the measures $(\omega^{1-q'}, v^{1-p'})$.

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