Proceedings of the Edinburgh Mathematical Society (2008) **51**, 11–26 ©
DOI:10.1017/S0013091505001690 Printed in the United Kingdom

INEQUALITIES OF THE HILBERT TYPE IN $\mathbb{R}^n$ WITH NON-CONJUGATE EXPONENTS

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(Received 22 November 2005)

Abstract In this paper we state and prove a new general Hilbert-type inequality in $\mathbb{R}^n$ with $k \geqslant 2$ non-conjugate exponents. Using Selberg's integral formula, this result is then applied to obtain explicit upper bounds for the doubly weighted Hardy–Littlewood–Sobolev inequality and some further Hilbert-type inequalities for k non-negative functions and non-conjugate exponents.

Keywords: inequalities; Hilbert's inequality; doubly weighted Hardy–Littlewood–Sobolev inequality; Selberg's integral formula; gamma function; beta function

2000 *Mathematics subject classification:* Primary 26D10; 26D15

1. Introduction

Let p and q be real parameters such that

$$p > 1, \quad q > 1, \quad \frac{1}{p} + \frac{1}{q} \geqslant 1, \tag{1.1}$$

and let p' and q' respectively be their conjugate exponents, that is,

$$\frac{1}{p} + \frac{1}{p'} = 1 \quad \text{and} \quad \frac{1}{q} + \frac{1}{q'} = 1.$$

Furthermore, define

$$\lambda = \frac{1}{p'} + \frac{1}{q'} \tag{1.2}$$

and note that $0 < \lambda \leqslant 1$ for all p and q as in (1.1). In particular, $\lambda = 1$ holds if and only if $q = p'$, that is, only when p and q are mutually conjugate. Otherwise, we have $0 < \lambda < 1$, and in such cases p and q will be referred to as non-conjugate exponents.

Considering p, q and λ as in (1.1) and (1.2), Hardy *et al.* [**3**] proved that there exists a constant $C_{p,q}$, dependent only on the parameters p and q, such that the following Hilbert-type inequality holds for all non-negative functions $f \in L^p(\mathbb{R}_+)$ and $g \in L^q(\mathbb{R}_+)$:

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{(x+y)^\lambda}\,\mathrm{d}x\,\mathrm{d}y \leqslant C_{p,q}\|f\|_{L^p(\mathbb{R}_+)}\|g\|_{L^q(\mathbb{R}_+)}. \tag{1.3}$$

However, the original proof did not provide any information about the value of the best possible constant $C_{p,q}$. That drawback was improved by Levin [**4**], who obtained an explicit upper bound for $C_{p,q}$:

$$C_{p,q} \leqslant \left(\pi \operatorname{cosec} \frac{\pi}{\lambda p'}\right)^\lambda. \tag{1.4}$$

This was an interesting result since the right-hand side of (1.4) reduces to the previously known sharp constant $\pi \operatorname{cosec}(\pi/p')$ when the exponents p and q are conjugate.

A simpler proof of (1.4), based on a single application of Hölder's inequality, was given later by Bonsall [**1**]. Moreover, in the same paper, with the same assumptions on p, q, λ, f and g, he proved another interesting Hilbert-type inequality,

$$\begin{aligned}\int_0^\infty \int_0^\infty &\frac{f(x)g(y)}{(x+y)^\lambda}\,\mathrm{d}x\,\mathrm{d}y \\ &\leqslant B^\lambda\left(\frac{1}{p'},\frac{1}{q'}\right)\|f\|_{L^p(\mathbb{R}_+)}^{p/q'}\|g\|_{L^q(\mathbb{R}_+)}^{q/p'}\left[\int_0^\infty\int_0^\infty \frac{x^{1/p'}y^{1/q'}}{(x+y)^\lambda} f^p(x)g^q(y)\,\mathrm{d}x\,\mathrm{d}y\right]^{1-\lambda},\end{aligned} \tag{1.5}$$

with the best possible constant $B^\lambda(1/p', 1/q')$, where B is the usual beta function. The idea used in the proof of (1.5) has to some extent guided us in the research we present here.

During the following decades, the Hilbert-type inequalities were discussed by several authors, who either re-proved them using various techniques, or applied and generalized them in many different ways. Here, we emphasize one of the most important such results, the so-called doubly weighted Hardy–Littlewood–Sobolev inequality of Stein and Weiss [**9**],

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{f(x)g(y)}{|x|^\alpha|x-y|^s|y|^\beta}\,\mathrm{d}x\,\mathrm{d}y \leqslant C_{\alpha,\beta,p,q,n}\|f\|_{L^p(\mathbb{R}^n)}\|g\|_{L^q(\mathbb{R}^n)}, \tag{1.6}$$

which holds for $n \in \mathbb{N}$, $p, q > 1$ such that $1/p + 1/q > 1$, λ as in (1.2), $0 \leqslant \alpha < n/p'$, $0 \leqslant \beta < n/q'$, $s = n\lambda - \alpha - \beta$, and all non-negative functions $f \in L^p(\mathbb{R}^n)$ and $g \in L^q(\mathbb{R}^n)$. In [**5**], Lieb proved the existence of optimizers for (1.6), that is, functions f and g which, when inserted into (1.6), give equality with the smallest possible constant $C_{\alpha,\beta,p,q,n}$. Moreover, for $p = q$ and $\alpha = \beta = 0$, the constant and maximizing functions were explicitly computed in [**5**]. In particular, Lieb obtained

$$C_{0,0,p,p,n} = \pi^{n/p'}\frac{\Gamma(\frac{1}{2}n - n/p')}{\Gamma(n/p)}\left[\frac{\Gamma(\frac{1}{2}n)}{\Gamma(n)}\right]^{2/p'-1},$$

where Γ is the gamma function. Unfortunately, neither $C_{\alpha,\beta,p,q,n}$ nor the optimizers are known for any other case of the parameters in (1.6). It was shown only (see, for example, [**6**]) that for the classical Hardy–Littlewood–Sobolev inequality, that is, for (1.6) with $\alpha=\beta=0$ and $s=n\lambda$, the estimate

$$C_{0,0,p,q,n} \leqslant \frac{(p')^\lambda+(q')^\lambda}{(1-\lambda)pq}\left(\frac{\lambda}{n}|\mathbb{S}^{n-1}|\right)^\lambda \tag{1.7}$$

holds, where

$$|\mathbb{S}^{n-1}|=\frac{2\pi^{n/2}}{\Gamma(\frac{1}{2}n)}$$

is the area of the unit sphere $\mathbb{S}^{n-1}$ in $\mathbb{R}^n$. For further details, see [**5**, **6**].

On the other hand, here we also refer to [**10**], in which a general Hilbert-type inequality was obtained for $k\geqslant 2$ conjugate exponents, that is, real parameters $p_1,\dots,p_k>1$, such that

$$\sum_{i=1}^{k}\frac{1}{p_i}=1.$$

Namely, let $\mu_1,\dots,\mu_k$ be positive σ-finite measures on Ω and let $K:\Omega^k\to\mathbb{R}$ and $\phi_{ij}:\Omega\to\mathbb{R}$, $i,j=1,\dots,k$, be non-negative measurable functions. If $\prod_{i,j=1}^{k}\phi_{ij}(x_j)=1$ a.e. on Ω^k, then the inequality

$$\int_{\Omega^k}K(x_1,\dots,x_k)\prod_{i=1}^{k}f_i(x_i)\,\mathrm{d}\mu_1(x_1)\cdots\mathrm{d}\mu_k(x_k) \leqslant \prod_{i=1}^{k}\biggl(\int_{\Omega}F_i(x_i)(\phi_{ii}f_i)^{p_i}(x_i)\,\mathrm{d}\mu_i(x_i)\biggr)^{1/p_i} \tag{1.8}$$

holds for all non-negative measurable functions $f_1,\dots,f_k:\Omega\to\mathbb{R}$, where

$$F_i(x_i)=\int_{\Omega^{k-1}}K(x_1,\dots,x_k)\prod_{j\neq i}\phi_{ij}^{p_i}(x_j)\,\mathrm{d}\mu_1(x_1)\cdots\mathrm{d}\mu_{i-1}(x_{i-1})\,\mathrm{d}\mu_{i+1}(x_{i+1})\cdots\mathrm{d}\mu_k(x_k),$$

for $i=1,\dots,k$. If there exists $l\in\{1,\dots,k\}$ such that $p_l<0$ and $p_i>0$ for $i\neq l$, then the sign of the inequality in (1.8) is reversed.

In this paper we extend (1.8) to a more general case with $k\geqslant 2$ non-conjugate exponents, introduced in the following section, and point out that relations (1.5) and (1.8) are only special cases of our result. The technique we establish appears to be fruitful since, by choosing particular parameters in the general Hilbert-type inequality obtained and applying the well-known Selberg integral formula (see, for example, [**8**]), we obtain explicit upper bounds for the general case of the doubly weighted Hardy–Littlewood–Sobolev inequality (1.6). Moreover, using a similar approach, we derive some further generalizations of (1.3), that is, Hilbert-type inequalities for k non-negative functions defined on $\mathbb{R}^n$ and non-conjugate exponents.

1.1. Conventions

Throughout this paper, let r' be the conjugate exponent to a positive real number $r \neq 1$, that is,

$$\frac{1}{r} + \frac{1}{r'} = 1 \quad \text{or} \quad r' = \frac{r}{r-1}.$$

The Euclidean norm of the vector $x \in \mathbb{R}^n$ will be denoted by $|x|$. Furthermore, empty sums are assumed to equal zero, all measures are assumed to be positive and σ-finite and all functions are assumed to be non-negative and measurable. Finally, we shall denote by f^* the symmetric-decreasing rearrangement of a function $f : \mathbb{R}^n \to \mathbb{R}$ vanishing at infinity.

2. A general Hilbert-type inequality

Before presenting our idea and results, we introduce the notion of general non-conjugate exponents. Let $k \in \mathbb{N}$, $k \geqslant 2$, and let real parameters $p_1, \dots, p_k$ be such that

$$p_1, \dots, p_k > 1, \quad \sum_{i=1}^{k} \frac{1}{p_i} \geqslant 1. \tag{2.1}$$

Define

$$\lambda = \frac{1}{k-1} \sum_{i=1}^{k} \frac{1}{p_i'} \quad \text{and} \quad \frac{1}{q_i} = \lambda - \frac{1}{p_i'}, \quad i = 1, \dots, k. \tag{2.2}$$

Obviously, we always have $0 < \lambda \leqslant 1$, while the equality $\lambda = 1$ holds if and only if $\sum_{i=1}^{k} 1/p_i = 1$. As we stated in § 1, in this special case the parameters $p_1, \dots, p_k$ will be called conjugate exponents. Otherwise, that is, for $\sum_{i=1}^{k} 1/p_i > 1$, they are non-conjugate. Observe that for conjugate exponents we have $q_i = p_i$, $i = 1, \dots, k$, while, for $k = 2$, $p_1 = p$ and $p_2 = q$, relations (2.1) and (2.2) respectively reduce to (1.1), (1.2), $q_1 = q'$ and $q_2 = p'$.

On the other hand, for any choice of the parameters in (2.1), it follows from (2.2) that

$$\frac{1}{q_i} + (1 - \lambda) = \frac{1}{p_i}, \quad i = 1, \dots, k, \tag{2.3}$$

and

$$\sum_{i=1}^{k} \frac{1}{q_i} + (1 - \lambda) = 1. \tag{2.4}$$

Hence, in order to apply Hölder's inequality with exponents $q_1, \dots, q_k$ and $1/(1 - \lambda)$, we require that

$$\frac{1}{q_i} > 0, \quad i = 1, \dots, k. \tag{2.5}$$

Note that for $k \geqslant 3$ conditions (2.1) and (2.2) do not automatically imply (2.5). More precisely, since (2.1) and (2.2) give only

$$\frac{1}{q_i} > \frac{2-k}{k-1} \frac{1}{p_i'}, \quad i = 1, \dots, k,$$

some of the q_i can be negative. For example, for $p_1 = 2$ and $p_2 = p_3 = \frac{20}{19}$ we have $1/q_1 = -\frac{1}{5} < 0$. Therefore, condition (2.5) is not redundant.

We are now ready to state and prove our basic result, a general Hilbert-type inequality.

Theorem 2.1. *Let $k \in \mathbb{N}$, $k \geqslant 2$, and parameters $p_1, \dots, p_k$, λ and $q_1, \dots, q_k$ be as in (2.1), (2.2) and (2.5). Let $\mu_1, \dots, \mu_k$ be positive σ-finite measures on Ω. If K is a non-negative measurable function on Ω^k, $F_1, \dots, F_k$ are positive measurable functions on Ω^k, and ϕ_{ij}, $i, j = 1, \dots, k$, are non-negative measurable functions on Ω, such that*

$$\prod_{i,j=1}^{k} \phi_{ij}(x_j) = 1 \quad \text{a.e. on } \Omega^k, \tag{2.6}$$

then the inequality

$$\begin{aligned}
&\int_{\Omega^k} K(x_1, \dots, x_k) \prod_{i=1}^{k} f_i(x_i)\, d\mu_1(x_1) \cdots d\mu_k(x_k) \\
&\quad \leqslant \prod_{i=1}^{k} \biggl[\int_{\Omega^k} (K F_i^{p_i - q_i})(x_1, \dots, x_k)(\phi_{ii} f_i)^{p_i}(x_i) \prod_{j \neq i} \phi_{ij}^{q_i}(x_j)\, d\mu_1(x_1) \cdots d\mu_k(x_k) \biggr]^{1/q_i} \\
&\qquad \times \biggl[\int_{\Omega^k} K(x_1, \dots, x_k) \prod_{i=1}^{k} F_i^{p_i}(x_1, \dots, x_k)(\phi_{ii} f_i)^{p_i}(x_i)\, d\mu_1(x_1) \cdots d\mu_k(x_k) \biggr]^{1-\lambda}
\end{aligned} \tag{2.7}$$

holds for all non-negative measurable functions $f_1, \dots, f_k$ on Ω.

Proof. Note that from (2.3) we have

$$\frac{p_i}{q_i} + p_i(1 - \lambda) = 1, \quad i = 1, \dots, k.$$

Using this and (2.4), the left-hand side of (2.7) can be written as

$$\begin{aligned}
&\int_{\Omega^k} K(x_1, \dots, x_k) \prod_{i=1}^{k} f_i(x_i)\, d\mu_1(x_1) \cdots d\mu_k(x_k) \\
&\quad = \int_{\Omega^k} K^{\sum_{i=1}^{k}(1/q_i) + 1 - \lambda}(x_1, \dots, x_k) \prod_{i=1}^{k} f_i^{(p_i/q_i) + p_i(1-\lambda)}(x_i) \\
&\qquad \times \prod_{i=1}^{k} F_i^{(p_i/q_i) - 1 + p_i(1-\lambda)}(x_1, \dots, x_k) \prod_{i=1}^{k} \phi_{ii}^{(p_i/q_i) + p_i(1-\lambda)}(x_i) \\
&\qquad \times \prod_{j \neq i} \phi_{ij}(x_j)\, d\mu_1(x_1) \cdots d\mu_k(x_k)
\end{aligned}$$

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$$= \int_{\Omega^k} \prod_{i=1}^{k} \left[(KF_i^{p_i - q_i})(x_1, \dots, x_k)(\phi_{ii} f_i)^{p_i}(x_i) \prod_{j \neq i} \phi_{ij}^{q_i}(x_j) \right]^{1/q_i}$$
$$\times \left[K(x_1, \dots, x_k) \prod_{i=1}^{k} F_i^{p_i}(x_1, \dots, x_k)(\phi_{ii} f_i)^{p_i}(x_i) \right]^{1-\lambda} \mathrm{d}\mu_1(x_1) \cdots \mathrm{d}\mu_k(x_k).$$

The inequality (2.7) now follows by using Hölder's inequality with the exponents $q_1, \dots, q_k$ and $1/(1-\lambda)$. □

Remark 2.2. Observe that, without loss of generality, condition (2.6) from the statement of Theorem 2.1 can be replaced by

$$\prod_{i=1}^{k} \phi_{ij}(x_j) = 1 \quad \text{a.e. on } \Omega,$$

for $j = 1, \dots, k$, since (2.6) implies that

$$\prod_{i=1}^{k} \phi_{ij}(x_j) = c_j = \text{const.}, \quad j = 1, \dots, k, \tag{2.8}$$

where $c_1 \cdots c_k = 1$.

Remark 2.3. Obviously, (2.7) becomes an equality if at least one of the functions involved in its left-hand side is a zero function. To discuss other non-trivial cases of equality in (2.7), we can assume without loss of generality that the functions K and f_i, $i = 1, \dots, k$, are positive. Otherwise, instead of Ω^k, we consider the set

$$S = \left\{ x = (x_1, \dots, x_k) \in \Omega^k : K(x) \prod_{i=1}^{k} f_i(x_i) > 0 \right\},$$

which has a positive measure. Now, note that the equality in (2.7) holds if and only if it holds in Hölder's inequality, that is, only if the functions

$$KF_i^{p_i - q_i}(\phi_{ii} f_i)^{p_i} \prod_{j \neq i} \phi_{ij}^{q_i}, \quad i = 1, \dots, k, \quad \text{and} \quad K \prod_{i=1}^{k} (F_i \phi_{ii} f_i)^{p_i}$$

are effectively proportional. Therefore, equality in (2.7) occurs if and only if there exist positive constants α_i, β_{ij}, $i, j = 1, \dots, k$, $j \neq i$, such that

$$KF_i^{p_i - q_i}(\phi_{ii} f_i)^{p_i} \prod_{l \neq i} \phi_{il}^{q_i} = \alpha_i K \prod_{l=1}^{k} (F_l \phi_{ll} f_l)^{p_l}, \quad i = 1, \dots, k, \tag{2.9}$$

and

$$KF_i^{p_i - q_i}(\phi_{ii} f_i)^{p_i} \prod_{l \neq i} \phi_{il}^{q_i} = \beta_{ij} KF_j^{p_j - q_j}(\phi_{jj} f_j)^{p_j} \prod_{l \neq j} \phi_{jl}^{q_j}, \quad i \neq j. \tag{2.10}$$

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Moreover, the relation (2.9) is equivalent to

$$F_i^{-q_i} = \alpha_i \prod_{l \neq i} \phi_{il}^{-q_i} (F_l \phi_{ll} f_l)^{p_l}, \quad i = 1, \dots, k. \tag{2.11}$$

In the special case when $F_i \equiv F_i(x_i)$, $i = 1, \dots, k$, the functions f_i and ϕ_{ij} from (2.10) and (2.11) can be expressed explicitly in terms of ϕ_{ii}. More precisely, from (2.11) we have

$$F_i \equiv \text{const.}, \quad i = 1, \dots, k, \tag{2.12}$$

directly, since the right-hand side of this relation depends on x_l, $l \neq i$, while the left-hand side in this setting is a function of x_i. Considering this, (2.10) becomes

$$(\phi_{ii} f_i)^{p_i} \phi_{ji}^{-q_j} = \gamma_{ij} (\phi_{jj} f_j)^{p_j} \prod_{l \neq i,j} \phi_{jl}^{q_j} \prod_{l \neq i} \phi_{il}^{-q_i}, \quad i \neq j, \tag{2.13}$$

for some positive constants γ_{ij}. Thus,

$$(\phi_{ii} f_i)^{p_i} \phi_{ji}^{-q_j} \equiv \text{const.}, \quad i = 1, \dots, k, \ j \neq i, \tag{2.14}$$

where again we have exploited the fact that the left-hand side of (2.13) depends only on x_i, while its right-hand side is a function of x_l, $l = 1, \dots, k$, $l \neq i$. The relation (2.14) further implies that $\phi_{ji}^{q_j} \phi_{li}^{-q_l} = \text{const.}$, $i = 1, \dots, k$, $j, l \neq i$, which, combined with (2.8), gives

$$\phi_{ii} \phi_{ji}^{q_j \sum_{l \neq i} 1/q_l} \equiv \text{const.}, \quad i = 1, \dots, k, \ j \neq i. \tag{2.15}$$

Since by (2.2) and (2.4) we have

$$q_j \sum_{l \neq i} \frac{1}{q_l} = q_j \left(\lambda - \frac{1}{q_i} \right) = \frac{q_j}{p_i'},$$

(2.15) can be transformed into

$$\phi_{ii}^{p_i'} \phi_{ji}^{q_j} \equiv \text{const.}, \quad i = 1, \dots, k, \ j \neq i, \tag{2.16}$$

while (2.14) becomes

$$f_i^{p_i} \equiv C_i \phi_{ii}^{-(p_i + p_i')}, \quad i = 1, \dots, k, \tag{2.17}$$

for some positive constants C_i, $i = 1, \dots, k$. Hence, if $F_i \equiv F_i(x_i)$, the conditions (2.12), (2.16) and (2.17) are necessary and sufficient for equality in (2.7).

Remark 2.4. If the parameters $p_1, \dots, p_k$ in Theorem 2.1 are such that

$$0 < p_i < 1, \quad \frac{k-1}{p_i} + 1 < \sum_{j=1}^{k} \frac{1}{p_j}, \quad i = 1, \dots, k, \tag{2.18}$$

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and λ and $q_1,\dots,q_k$ are defined by (2.2), then the sign of inequality in (2.7) is reversed. To justify this assertion, observe that the first inequality in (2.18) gives $1/p_i' < 0$, $i=1,\dots,k$, so we have $\lambda < 0$. Similarly, from the second relation in (2.18) it follows that

$$\frac{1}{q_i} = \lambda - \frac{1}{p_i'} = \frac{1}{k-1}\left(\frac{k-1}{p_i} + 1 - \sum_{j=1}^{k}\frac{1}{p_j}\right) < 0, \quad i=1,\dots,k.$$

Therefore, $q_i < 0$, $i=1,\dots,k$, and $0 < 1/(1-\lambda) < 1$, so (2.7) holds with the reversed sign of inequality as a direct consequence of Hölder's inequality (for details on the so-called reversed Hölder's inequality, see, for example, [**7**, Chapter V]). The same result is also achieved with the parameters $p_1,\dots,p_k$ satisfying

$$\sum_{i=1}^{k}\frac{1}{p_i} < 1 \quad \text{and} \quad 0 < p_l < 1, \qquad \frac{k-1}{p_i} + 1 < \sum_{j=1}^{k}\frac{1}{p_j}, \quad i \neq l, \tag{2.19}$$

for some $l \in \{1,\dots,k\}$, since from (2.19) we obtain $1/(1-\lambda) < 0$, $q_l > 0$, and $q_i < 0$, $i \neq l$.

Remark 2.5. Note that in the case of conjugate exponents ($\lambda = 1$) the inequality (2.7) becomes

$$\begin{aligned}\int_{\Omega^k} K(x_1,\dots,x_k)\prod_{i=1}^{k} f_i(x_i)\,\mathrm{d}\mu_1(x_1)\cdots\mathrm{d}\mu_k(x_k) \\ \leqslant \prod_{i=1}^{k}\left[\int_{\Omega^k} K(x_1,\dots,x_k)(\phi_{ii}f_i)^{p_i}(x_i)\prod_{j\neq i}\phi_{ij}^{p_i}(x_j)\,\mathrm{d}\mu_1(x_1)\cdots\mathrm{d}\mu_k(x_k)\right]^{1/q_i},\end{aligned}$$

whence, by Fubini's theorem, we obtain the right-hand side of (1.8). This means that Theorem 2.1 may be regarded as a generalization of the mentioned result from [**10**].

To conclude this section, we restate Theorem 2.1 for the case when $k=2$. This result is interesting in its own right, since it will be applied in the following section, where we consider a particular kernel K.

Theorem 2.6. *Let p, q and λ be as in (1.1) and (1.2). Let μ_1 and μ_2 be positive σ-finite measures on Ω. If K, F and G are non-negative measurable functions on Ω^2, and φ and ψ are non-negative measurable functions on Ω, then the inequality*

$$\begin{aligned}\int_{\Omega^2} K(x,y)f(x)g(y)\,\mathrm{d}\mu_1(x)\,\mathrm{d}\mu_2(y) \\ \leqslant \left[\int_{\Omega^2}(KF^{p-q'})(x,y)\psi^{-q'}(y)(\varphi f)^p(x)\,\mathrm{d}\mu_1(x)\,\mathrm{d}\mu_2(y)\right]^{1/q'} \\ \times\left[\int_{\Omega^2}(KG^{q-p'})(x,y)\varphi^{-p'}(x)(\psi g)^q(y)\,\mathrm{d}\mu_1(x)\,\mathrm{d}\mu_2(y)\right]^{1/p'} \\ \times\left[\int_{\Omega^2}(KF^pG^q)(x,y)(\varphi f)^p(x)(\psi g)^q(y)\,\mathrm{d}\mu_1(x)\,\mathrm{d}\mu_2(y)\right]^{1-\lambda}\end{aligned} \tag{2.20}$$

holds for all non-negative measurable functions f and g on Ω.

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Proof. The proof follows directly from Theorem 2.1 using substitutions $p_1 = p$, $p_2 = q$, $q_1 = q'$, $q_2 = p'$, $\phi_{11} = \varphi$ and $\phi_{22} = \psi$. Observe that from $\phi_{11}\phi_{21} = 1$ and $\phi_{12}\phi_{22} = 1$ we have $\phi_{21} = 1/\varphi$ and $\phi_{12} = 1/\psi$. □

Remark 2.7. If we rewrite Theorem 2.6 with $\Omega = \mathbb{R}_+$, Lebesgue measures, kernel $K(x, y) = (x + y)^{-\lambda}$ and functions $F(x, y) = G(x, y) \equiv 1$, $\varphi(x) = x^{1/pp'}$ and $\psi(y) = y^{1/qq'}$, we obtain (1.5). Hence, Theorem 2.6 can be seen as a generalization of the Bonsall result mentioned in [**1**].

3. Explicit upper bounds for the doubly weighted Hardy–Littlewood–Sobolev inequality

In this section, we consider a special case of Theorem 2.6 with $\Omega = \mathbb{R}^n$ and the kernel $K(x, y) = |x|^{-\alpha}|x - y|^{-s}|y|^{-\beta}$. More precisely, we use our general result to obtain a form of the doubly weighted Hardy–Littlewood–Sobolev inequality (1.6) with an explicit constant factor on its right-hand side. In fact, we derive explicit upper bounds for the sharp constant $C_{\alpha,\beta,p,q,n}$ for (1.6).

Our results in the following two sections will be based on Theorems 2.1 and 2.6 and the well-known Selberg integral formula

$$\int_{\mathbb{R}^{kn}} |x_k|^{\alpha_k - n}\left(\prod_{i=1}^{k-1} |x_{i+1} - x_i|^{\alpha_i - n}\right)|x_1 - y|^{\alpha_0 - n}\,\mathrm{d}x_1 \cdots \mathrm{d}x_k = \frac{\Gamma_n(\alpha_0)\cdots\Gamma_n(\alpha_k)}{\Gamma_n(\alpha_0 + \cdots + \alpha_k)}|y|^{\alpha_0 + \cdots + \alpha_k - n}, \tag{3.1}$$

for arbitrary $k, n \in \mathbb{N}$, $y \in \mathbb{R}^n$, and $0 < \alpha_0, \ldots, \alpha_k < n$ such that $0 < \sum_{i=0}^{k} \alpha_i < n$, where

$$\Gamma_n(\alpha) = \frac{\pi^{n/2}2^{\alpha}\Gamma(\frac{1}{2}\alpha)}{\Gamma(\frac{1}{2}n - \frac{1}{2}\alpha)}.$$

In [**8**], Stein derived the Selberg integral formula with two parameters using the Riesz potential (see also [**2**]). Observe that

$$\Gamma_n(n - \alpha) = \frac{(2\pi)^n}{\Gamma_n(\alpha)}, \quad 0 < \alpha < n. \tag{3.2}$$

In the next lemma we give a form of (3.1), more suitable for our computations.

Lemma 3.1. *Suppose that $k, n \in \mathbb{N}$, $0 < \beta_1, \ldots, \beta_k, s < n$ are such that $\sum_{i=1}^{k}\beta_i + s > kn$ and $y \in \mathbb{R}^n$. Then*

$$\int_{\mathbb{R}^{kn}} \frac{|x_1|^{-\beta_1}\cdots|x_k|^{-\beta_k}}{|x_1 + \cdots + x_k + y|^s}\,\mathrm{d}x_1 \cdots \mathrm{d}x_k = \frac{\Gamma_n(n - \beta_1)\cdots\Gamma_n(n - \beta_k)\Gamma_n(n - s)}{\Gamma_n((k+1)n - \beta_1 - \cdots - \beta_k - s)}|y|^{kn - \beta_1 - \cdots - \beta_k - s}. \tag{3.3}$$

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Proof. Set $\alpha_i = n-\beta_{i+1}$, $i=0,\dots,k-1$, and $\alpha_k = n-s$. Substituting first $t_1 = x_1+y$ and then $t_2 = t_1+x_2$, the left-hand side of (3.3) becomes

$$
\begin{aligned}
\int_{\mathbb{R}^{kn}} & \frac{|x_1|^{-\beta_1}\cdots|x_k|^{-\beta_k}}{|x_1+\cdots+x_k+y|^s}\,\mathrm{d}x_1\cdots\mathrm{d}x_k \\
&= \int_{\mathbb{R}^{kn}} \frac{|t_1-y|^{\alpha_0-n}|x_2|^{\alpha_1-n}\cdots|x_k|^{\alpha_{k-1}-n}}{|t_1+x_2+\cdots+x_k|^{n-\alpha_k}}\,\mathrm{d}t_1\,\mathrm{d}x_2\cdots\mathrm{d}x_k \\
&= \int_{\mathbb{R}^{kn}} \frac{|t_1-y|^{\alpha_0-n}|t_2-t_1|^{\alpha_1-n}|x_3|^{\alpha_2-n}\cdots|x_k|^{\alpha_{k-1}-n}}{|t_2+x_3+\cdots+x_k|^{n-\alpha_k}}\,\mathrm{d}t_1\,\mathrm{d}t_2\,\mathrm{d}x_3\cdots\mathrm{d}x_k. \qquad (3.4)
\end{aligned}
$$

After the sequence of similar substitutions $t_i = t_{i-1}+x_i$, $i=2,\dots,k$, the last line of (3.4) is finally equal to

$$
\begin{aligned}
\int_{\mathbb{R}^{kn}} |t_k|^{\alpha_k-n}\biggl(\prod_{i=1}^{k-1}|t_{i+1}-t_i|^{\alpha_i-n}\biggr)&|t_1-y|^{\alpha_0-n}\,\mathrm{d}t_1\cdots\mathrm{d}t_k \\
&= \frac{\Gamma_n(\alpha_0)\cdots\Gamma_n(\alpha_k)}{\Gamma_n(\alpha_0+\cdots+\alpha_k)}|y|^{\alpha_0+\cdots+\alpha_k-n} \\
&= \frac{\Gamma_n(n-\beta_1)\cdots\Gamma_n(n-\beta_k)\Gamma_n(n-s)}{\Gamma_n((k+1)n-\beta_1-\cdots-\beta_k-s)}|y|^{kn-\beta_1-\cdots-\beta_k-s},
\end{aligned}
$$

where the last two equalities are obtained by Selberg's integral formula (3.1) and by replacing α_i by the corresponding expressions including β_i. □

Since the case $k=1$ of Lemma 3.1 will be of special interest to us, we state it as a separate result.

Lemma 3.2. *Let $n\in\mathbb{N}$ and $y\in\mathbb{R}^n$. If $0<\beta,s<n$ are such that $\beta+s>n$, then*

$$\int_{\mathbb{R}^n}\frac{|x|^{-\beta}}{|x+y|^s}\,\mathrm{d}x = \frac{\Gamma_n(n-\beta)\Gamma_n(n-s)}{\Gamma_n(2n-\beta-s)}|y|^{n-\beta-s}.$$

We can now obtain the doubly weighted Hardy–Littlewood–Sobolev inequality (1.6) mentioned above. The first step is to consider the case in which the function $g\in L^q(\mathbb{R}^n)$ on its left-hand side is symmetric-decreasing, that is, $g(x)\geqslant g(y)$ whenever $|x|\leqslant|y|$. For such a function and $y\in\mathbb{R}^n$, $y\neq 0$, we have

$$
\begin{aligned}
g^q(y) &\leqslant \frac{1}{|B(|y|)|}\int_{B(|y|)} g^q(x)\,\mathrm{d}x \\
&\leqslant \frac{1}{|B(|y|)|}\int_{\mathbb{R}^n} g^q(x)\,\mathrm{d}x \\
&= \frac{n}{|\mathbb{S}^{n-1}|}|y|^{-n}\|g\|^q_{L^q(\mathbb{R}^n)}, \qquad (3.5)
\end{aligned}
$$

where $B(|y|)$ denotes the ball of radius $|y|$ in $\mathbb{R}^n$, centred at the origin, and $|B(|y|)| = |y|^n|\mathbb{S}^{n-1}|/n$ is its volume.

Theorem 3.3. *Let $n \in \mathbb{N}$, $p > 1$ and $q > 1$ such that $1/p + 1/q > 1$, and set λ as in (1.2). Let $0 \leqslant \alpha < n/p'$, $0 \leqslant \beta < n/q'$ and $s = n\lambda - \alpha - \beta$. Then the inequality*

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{f(x)g(y)}{|x|^\alpha |x-y|^s |y|^\beta}\,\mathrm{d}x\,\mathrm{d}y \leqslant \frac{(2\pi)^{2n}(|\mathbb{S}^{n-1}|/n)^{\lambda-1}}{\Gamma_n(n/p+\alpha)\Gamma_n(n/q+\beta)\Gamma_n(s)}\|f\|_{L^p(\mathbb{R}^n)}\|g\|_{L^q(\mathbb{R}^n)} \tag{3.6}$$

holds for all non-negative functions $f \in L^p(\mathbb{R}^n)$ and symmetric-decreasing functions $g \in L^q(\mathbb{R}^n)$.

Proof. Suppose that in Theorem 2.6 we have $\Omega = \mathbb{R}^n$, $K(x,y) = |x|^{-\alpha}|x-y|^{-s}|y|^{-\beta}$, $F(x,y) = G(x,y) \equiv 1$, $\varphi(x) = |x|^{n/pp'}$, $\psi(y) = |y|^{n/qq'}$ and the Lebesgue measure $\mathrm{d}x$. Then the left-hand side of (2.20) reads

$$L = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{f(x)g(y)}{|x|^\alpha |x-y|^s |y|^\beta}\,\mathrm{d}x\,\mathrm{d}y, \tag{3.7}$$

while its right-hand side is a product $R = I_1^{1/q'} I_2^{1/p'} I_3^{1-\lambda}$, where

$$I_1 = \int_{\mathbb{R}^{2n}} \frac{|x|^{n/p'}|y|^{-n/q}}{|x|^\alpha |x-y|^s |y|^\beta} f^p(x)\,\mathrm{d}x\,\mathrm{d}y,$$

$$I_2 = \int_{\mathbb{R}^{2n}} \frac{|x|^{-n/p}|y|^{n/q'}}{|x|^\alpha |x-y|^s |y|^\beta} g^q(y)\,\mathrm{d}x\,\mathrm{d}y,$$

$$I_3 = \int_{\mathbb{R}^{2n}} \frac{|x|^{n/p'}|y|^{n/q'}}{|x|^\alpha |x-y|^s |y|^\beta} f^p(x)g^q(y)\,\mathrm{d}x\,\mathrm{d}y.$$

Therefore, applying Fubini's theorem, Lemma 3.2, identity (3.2), and the fact that $\alpha + \beta + s = n\lambda$, respectively, we have

$$\begin{aligned} I_1 &= \int_{\mathbb{R}^n} |x|^{(n/p')-\alpha} f^p(x) \int_{\mathbb{R}^n} \frac{|y|^{-((n/q)+\beta)}}{|x-y|^s}\,\mathrm{d}y\,\mathrm{d}x \\ &= \int_{\mathbb{R}^n} |x|^{(n/p')-\alpha} f^p(x) \int_{\mathbb{R}^n} \frac{|z|^{-((n/q)+\beta)}}{|z+x|^s}\,\mathrm{d}z\mathrm{d}x \\ &= \frac{\Gamma_n(n-n/q-\beta)\Gamma_n(n-s)}{\Gamma_n(2n-n/q-\beta-s)} \int_{\mathbb{R}^n} |x|^{(n/p')-\alpha+n-(n/q)-\beta-s} f^p(x)\,\mathrm{d}x \\ &= \frac{(2\pi)^{2n}}{\Gamma_n(n/p+\alpha)\Gamma_n(n/q+\beta)\Gamma_n(s)}\|f\|^p_{L^p(\mathbb{R}^n)}. \end{aligned}$$

Analogously,

$$I_2 = \frac{(2\pi)^{2n}}{\Gamma_n(n/p+\alpha)\Gamma_n(n/q+\beta)\Gamma_n(s)}\|g\|^q_{L^q(\mathbb{R}^n)},$$

and, by (3.5),

$$\begin{aligned} I_3 &\leqslant \frac{n}{|\mathbb{S}^{n-1}|}\|g\|^q_{L^q(\mathbb{R}^n)} \int_{\mathbb{R}^n} |x|^{(n/p')-\alpha} f^p(x) \int_{\mathbb{R}^n} \frac{|y|^{-((n/q)+\beta)}}{|x-y|^s}\,\mathrm{d}y\,\mathrm{d}x \\ &= \frac{n}{|\mathbb{S}^{n-1}|} \cdot \frac{(2\pi)^{2n}}{\Gamma_n(n/p+\alpha)\Gamma_n(n/q+\beta)\Gamma_n(s)}\|f\|^p_{L^p(\mathbb{R}^n)}\|g\|^q_{L^q(\mathbb{R}^n)}. \end{aligned}$$

Finally, (3.6) follows by combining (3.7) and the expressions we have obtained for the integrals I_1, I_2 and I_3. □

To obtain an analogous result for arbitrary non-negative functions $f \in L^p(\mathbb{R}^n)$ and $g \in L^q(\mathbb{R}^n)$, we use the fact that for parameter $\gamma > 0$ the function $h : \mathbb{R}^n \to \mathbb{R}$, $h(x) = |x|^{-\gamma}$, is symmetric-decreasing and vanishes at infinity. Hence, $h^* = h$.

Theorem 3.4. *Let $n \in \mathbb{N}$, $p > 1$ and $q > 1$ such that $1/p + 1/q > 1$, and set λ as in (1.2). Suppose that $0 \leqslant \alpha < n/p'$, $0 \leqslant \beta < n/q'$ and $s = n\lambda - \alpha - \beta$. Then the inequality (3.6) holds for all non-negative functions $f \in L^p(\mathbb{R}^n)$ and $g \in L^q(\mathbb{R}^n)$.*

Proof. Since $x \mapsto |x|^{-\alpha}$, $x \mapsto |x|^{-s}$ and $x \mapsto |x|^{-\beta}$ are symmetric-decreasing functions vanishing at infinity, the general rearrangement inequality (see, for example, [**6**]) implies that

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{f(x)g(y)}{|x|^{\alpha}|x-y|^{s}|y|^{\beta}}\,\mathrm{d}x\,\mathrm{d}y \leqslant \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{f^*(x)g^*(y)}{|x|^{\alpha}|x-y|^{s}|y|^{\beta}}\,\mathrm{d}x\,\mathrm{d}y. \tag{3.8}$$

Clearly, by Theorem 3.3, the right-hand side of (3.8) is not greater than

$$K_{\alpha,\beta,p,q,n}\|f^*\|_{L^p(\mathbb{R}^n)}\|g^*\|_{L^q(\mathbb{R}^n)} = K_{\alpha,\beta,p,q,n}\|f\|_{L^p(\mathbb{R}^n)}\|g\|_{L^q(\mathbb{R}^n)}, \tag{3.9}$$

where $K_{\alpha,\beta,p,q,n}$ is the constant from the right-hand side of (3.6). To obtain equality in (3.9), we have used the fact that the symmetric-decreasing rearrangement is norm-preserving. □

Remark 3.5. Note that $C_{\alpha,\beta,p,q,n} \leqslant K_{\alpha,\beta,p,q,n}$, where $C_{\alpha,\beta,p,q,n}$ is the sharp constant for (1.6) and $K_{\alpha,\beta,p,q,n}$ is the constant factor involved in the right-hand side of (3.6). Hence, we obtained new explicit upper bounds for the doubly weighted Hardy–Littlewood–Sobolev inequality. In particular, for $\alpha = \beta = 0$ we have

$$K_{0,0,p,q,n} = \frac{(2\pi)^{2n}(|\mathbb{S}^{n-1}|/n)^{\lambda-1}}{\Gamma_n(n/p)\Gamma_n(n/q)\Gamma_n(n\lambda)}, \tag{3.10}$$

while for $p = q$ the constant (3.10) becomes

$$\begin{aligned} K_{0,0,p,p,n} &= \frac{(2\pi)^{2n}(|\mathbb{S}^{n-1}|/n)^{(2/p')-1}}{\Gamma_n^2(n/p)\Gamma_n(2n/p')} \\ &= \pi^{n/p'}\frac{\Gamma(\frac{1}{2}n - n/p')}{\Gamma(n/p')}\left[\frac{\Gamma(n/2p')}{\Gamma(n/2p)}\right]^2[\Gamma(\tfrac{1}{2}n+1)]^{1-(2/p')}. \end{aligned}$$

Although (1.7) provides a better estimate for $C_{0,0,p,q,n}$ than (3.10), it is important to note that our result covers all admissible choices of the parameters p, q, α and β in (1.6), so our main contribution is in extending Lieb's result given in § 1.

4. Multiple Hilbert-type inequalities

In the previous section, by applying Theorem 2.6 and Lemma 3.2, we obtained inequalities with two non-conjugate parameters. Our aim here is to establish some new specific Hilbert-type inequalities related to $k \geqslant 2$ non-conjugate parameters and k non-negative functions. To derive such results, we shall use Theorem 2.1, Lemma 3.1 and the kernel $K(x_1, \dots, x_k) = |x_1 + \cdots + x_k|^{-(k-1)n\lambda}$ on $\mathbb{R}^{kn}$. The corresponding Hilbert-type inequality is given in the following theorem.

Theorem 4.1. *Let $n \in \mathbb{N}$, $k \in \mathbb{N}$, $k \geqslant 2$, and let parameters $p_1, \dots, p_k$, λ and $q_1, \dots, q_k$ be as in (2.1), (2.2) and (2.5). If $0 < \lambda < 1/(k-1)$, then the inequality*

$$\int_{\mathbb{R}^{kn}} \frac{f_1(x_1) \cdots f_k(x_k)}{|x_1 + \cdots + x_k|^{(k-1)n\lambda}} \, \mathrm{d}x_1 \cdots \mathrm{d}x_k \leqslant \frac{(2\pi)^{kn} (|\mathbb{S}^{n-1}|/n)^{(k-1)(\lambda-1)}}{\Gamma_n((k-1)n\lambda) \prod_{i=1}^{k} \Gamma_n(n/p_i)} \|f_1\|_{L^{p_1}(\mathbb{R}^n)} \cdots \|f_k\|_{L^{p_k}(\mathbb{R}^n)} \tag{4.1}$$

holds for all non-negative functions $f_i \in L^{p_i}(\mathbb{R}^n)$, $i = 1, \dots, k$.

Proof. First, we consider a simpler special case of the functions involved in (4.1). Namely, suppose that $f_2, \dots, f_k$ are symmetric-decreasing functions. To prove our result, we rewrite Theorem 2.1 with $\Omega = \mathbb{R}^n$, $K(x_1, \dots, x_k) = |x_1 + \cdots + x_k|^{-(k-1)n\lambda}$, $F_i(x_1, \dots, x_k) \equiv 1$ and $\phi_{ij}(x_j) = |x_j|^{A_{ij}}$, where

$$A_{ij} = \begin{cases} \dfrac{n}{p_i p_i'}, & i = j, \\[2mm] -\dfrac{n}{q_i p_j}, & i \neq j, \end{cases} \tag{4.2}$$

and with Lebesgue measures $\mathrm{d}x_i$, for $i, j = 1, \dots, k$. Then the left-hand side of (2.7) becomes

$$L = \int_{\mathbb{R}^{kn}} \frac{f_1(x_1) \cdots f_k(x_k)}{|x_1 + \cdots + x_k|^{(k-1)n\lambda}} \, \mathrm{d}x_1 \cdots \mathrm{d}x_k, \tag{4.3}$$

while the right-hand side of this inequality is the product of $k+1$ factors,

$$R = I_1^{1/q_1} \cdots I_k^{1/q_k} I_{k+1}^{1-\lambda}, \tag{4.4}$$

where

$$I_i = \int_{\mathbb{R}^{kn}} \frac{|x_i|^{n/p_i'} \prod_{j \neq i} |x_j|^{-n/p_j}}{|x_1 + \cdots + x_k|^{(k-1)n\lambda}} f_i^{p_i}(x_i) \, \mathrm{d}x_1 \cdots \mathrm{d}x_k, \quad i = 1, \dots, k,$$

and

$$I_{k+1} = \int_{\mathbb{R}^{kn}} \frac{\prod_{i=1}^{k} |x_i|^{n/p_i'}}{|x_1 + \cdots + x_k|^{(k-1)n\lambda}} \prod_{i=1}^{k} f_i^{p_i}(x_i) \, \mathrm{d}x_1 \cdots \mathrm{d}x_k.$$

Before calculating these integrals, observe that, from (2.1), (2.2) and (2.5), we find that $0 < n/p_i < n$ and

$$\sum_{j\neq i}\frac{n}{p_j}+(k-1)n\lambda = n\sum_{j\neq i}\left(\frac{1}{p_j}+\lambda\right) = n\sum_{j\neq i}\left(\frac{1}{q_j}+1\right)$$
$$= (k-1)n + n\sum_{j\neq i}\frac{1}{q_j} > (k-1)n,$$

for all $i\in\{1,\dots,k\}$. Moreover, the conditions from the statement of Theorem 4.1 also imply that $0 < (k-1)n\lambda < n$. Therefore, applying Fubini's theorem, Lemma 3.1 and (3.2), for $i = 1,\dots,k$, respectively, we obtain

$$\begin{aligned} I_i &= \int_{\mathbb{R}^n} |x_i|^{n/p_i'} f_i^{p_i}(x_i) \int_{\mathbb{R}^{(k-1)n}} \frac{\prod_{j\neq i}|x_j|^{-n/p_j}}{|x_1+\cdots+x_k|^{(k-1)n\lambda}}\,\mathrm{d}x_1\cdots\mathrm{d}x_{i-1}\,\mathrm{d}x_{i+1}\cdots\mathrm{d}x_k\,\mathrm{d}x_i \\ &= \frac{\mathit{\Gamma}_n(n-(k-1)n\lambda)\prod_{j\neq i}\mathit{\Gamma}_n(n-n/p_j)}{\mathit{\Gamma}_n(kn-\sum_{j\neq i}n/p_j-(k-1)n\lambda)} \\ &\quad\times\int_{\mathbb{R}^n}|x_i|^{(n/p_i')+(k-1)n-\sum_{j\neq i}(n/p_j)-(k-1)n\lambda} f_i^{p_i}(x_i)\,\mathrm{d}x_i \\ &= \frac{(2\pi)^{kn}}{\mathit{\Gamma}_n((k-1)n\lambda)\prod_{j=1}^k\mathit{\Gamma}_n(n/p_j)}\|f_i\|^{p_i}_{L^{p_i}(\mathbb{R}^n)}. \end{aligned} \tag{4.5}$$

To estimate the last integral, I_{k+1}, in (4.4) we use the assumption that the functions $f_2,\dots,f_k$ are symmetric-decreasing. Hence, we can use (3.5), so

$$f_i^{p_i}(x_i) \leqslant \frac{n}{|\mathbb{S}^{n-1}|}|x_i|^{-n}\|f_i\|^{p_i}_{L^{p_i}(\mathbb{R}^n)}$$

holds for all $x_i\in\mathbb{R}^n$, $x_i\neq 0$. Again, according to Fubini's theorem, Lemma 3.1 and the identity (3.2), by a procedure similar to that used in (4.5) we obtain

$$\begin{aligned} I_{k+1} &\leqslant \left(\frac{n}{|\mathbb{S}^{n-1}|}\right)^{k-1}\prod_{i=2}^k\|f_i\|^{p_i}_{L^{p_i}(\mathbb{R}^n)}\int_{\mathbb{R}^{kn}}\frac{|x_1|^{n/p_1'}\prod_{i=2}^k|x_i|^{(n/p_i')-n}}{|x_1+\cdots+x_k|^{(k-1)n\lambda}}f_1^{p_1}(x_1)\,\mathrm{d}x_1\cdots\mathrm{d}x_k \\ &= \left(\frac{n}{|\mathbb{S}^{n-1}|}\right)^{k-1}\prod_{i=2}^k\|f_i\|^{p_i}_{L^{p_i}(\mathbb{R}^n)} \\ &\quad\times\int_{\mathbb{R}^n}|x_1|^{n/p_1'}f_1^{p_1}(x_1)\int_{\mathbb{R}^{(k-1)n}}\frac{\prod_{i=2}^k|x_i|^{-n/p_i}}{|x_1+\cdots+x_k|^{(k-1)n\lambda}}\,\mathrm{d}x_2\cdots\mathrm{d}x_k\,\mathrm{d}x_1 \\ &= \left(\frac{n}{|\mathbb{S}^{n-1}|}\right)^{k-1}\frac{\mathit{\Gamma}_n(n-(k-1)n\lambda)\prod_{i=2}^k\mathit{\Gamma}_n(n-n/p_i)}{\mathit{\Gamma}_n(kn-\sum_{i=2}^k(n/p_i)-(k-1)n\lambda)}\prod_{i=1}^k\|f_i\|^{p_i}_{L^{p_i}(\mathbb{R}^n)} \\ &= \left(\frac{n}{|\mathbb{S}^{n-1}|}\right)^{k-1}\frac{(2\pi)^{kn}}{\mathit{\Gamma}_n((k-1)n\lambda)\prod_{i=1}^k\mathit{\Gamma}_n(n/p_i)}\prod_{i=1}^k\|f_i\|^{p_i}_{L^{p_i}(\mathbb{R}^n)}. \end{aligned} \tag{4.6}$$

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Now we arrive at inequality (4.1) for this case by combining (4.3)–(4.6). To complete the proof, we need to consider the general case, that is, arbitrary non-negative functions $f_2, \dots, f_k$. Since the function $x \mapsto |x|^{-(k-1)n\lambda}$ is symmetric-decreasing and vanishes at infinity, by the general rearrangement inequality we have

$$\begin{aligned}\int_{\mathbb{R}^{kn}} &\frac{f_1(x_1)\cdots f_k(x_k)}{|x_1+\cdots+x_k|^{(k-1)n\lambda}}\,\mathrm{d}x_1\cdots\mathrm{d}x_k \\ &\leqslant \int_{\mathbb{R}^{kn}} \frac{f_1^*(x_1)\cdots f_k^*(x_k)}{|x_1+\cdots+x_k|^{(k-1)n\lambda}}\,\mathrm{d}x_1\cdots\mathrm{d}x_k \\ &\leqslant \frac{(2\pi)^{kn}(|\mathbb{S}^{n-1}|/n)^{(k-1)(\lambda-1)}}{\mathit{\Gamma}_n((k-1)n\lambda)\prod_{i=1}^k \mathit{\Gamma}_n(n/p_i)}\|f_1^*\|_{L^{p_1}(\mathbb{R}^n)}\cdots\|f_k^*\|_{L^{p_k}(\mathbb{R}^n)} \\ &= \frac{(2\pi)^{kn}(|\mathbb{S}^{n-1}|/n)^{(k-1)(\lambda-1)}}{\mathit{\Gamma}_n((k-1)n\lambda)\prod_{i=1}^k \mathit{\Gamma}_n(n/p_i)}\|f_1\|_{L^{p_1}(\mathbb{R}^n)}\cdots\|f_k\|_{L^{p_k}(\mathbb{R}^n)}.\end{aligned}$$

As in the proof of Theorem 3.4, here we used the fact that $f_2^*, \dots, f_k^*$ are symmetric-decreasing functions and that the mapping $f \mapsto f^*$ is norm-preserving. □

Remark 4.2. Note that the proof of Theorem 4.1 is, in fact, based on Selberg's integral formula (3.1). Some further applications of this formula can be found in [**2**], for example.

Setting $k = 2$ and $k = 3$ in Theorem 4.1, we get the following corollaries.

Corollary 4.3. *Let $n \in \mathbb{N}$, $p > 1$ and $q > 1$ be such that $1/p + 1/q > 1$, and let λ be defined by (1.2). Then*

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{f(x)g(y)}{|x+y|^{n\lambda}}\,\mathrm{d}x\,\mathrm{d}y \leqslant \frac{(2\pi)^{2n}(|\mathbb{S}^{n-1}|/n)^{\lambda-1}}{\mathit{\Gamma}_n(n\lambda)\mathit{\Gamma}_n(n/p)\mathit{\Gamma}_n(n/q)}\|f\|_{L^p(\mathbb{R}^n)}\|g\|_{L^q(\mathbb{R}^n)} \tag{4.7}$$

holds for all non-negative functions $f \in L^p(\mathbb{R}^n)$ and $g \in L^q(\mathbb{R}^n)$. In particular, if $n = 1$, then (4.7) becomes

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{f(x)g(y)}{|x+y|^{\lambda}}\,\mathrm{d}x\,\mathrm{d}y \leqslant 2^{\lambda-1}\sqrt{\pi}\frac{B(1/2p',1/2q')}{B(1/2p,1/2q)}\cdot\frac{\mathit{\Gamma}(\frac{1}{2}-\frac{1}{2}\lambda)}{\mathit{\Gamma}(1-\frac{1}{2}\lambda)}\|f\|_{L^p(\mathbb{R})}\|g\|_{L^q(\mathbb{R})}.$$

Corollary 4.4. *Let $n \in \mathbb{N}$ and parameters p_1, p_2, p_3, λ, q_1, q_2 and q_3 be as in (2.1), (2.2) and (2.5). If $0 < \lambda < \frac{1}{2}$, then*

$$\begin{aligned}\int_{\mathbb{R}^{3n}} &\frac{f(x)g(y)h(z)}{|x+y+z|^{2n\lambda}}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z \\ &\leqslant \frac{(2\pi)^{3n}(|\mathbb{S}^{n-1}|/n)^{2(\lambda-1)}}{\mathit{\Gamma}_n(2n\lambda)\mathit{\Gamma}_n(n/p_1)\mathit{\Gamma}_n(n/p_2)\mathit{\Gamma}_n(n/p_3)}\|f\|_{L^{p_1}(\mathbb{R}^n)}\|g\|_{L^{p_2}(\mathbb{R}^n)}\|h\|_{L^{p_3}(\mathbb{R}^n)}\end{aligned} \tag{4.8}$$

holds for all non-negative functions $f \in L^{p_1}(\mathbb{R}^n)$, $g \in L^{p_2}(\mathbb{R}^n)$ *and* $h \in L^{p_3}(\mathbb{R}^n)$. *In particular, if* $n = 1$, *then (4.8) reads*

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{f(x)g(y)h(z)}{|x+y+z|^{2\lambda}}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z \leqslant 2^{2(\lambda-1)}\pi\frac{\Gamma(\frac{1}{2}-\lambda)}{\Gamma(\lambda)}\prod_{i=1}^{3}\frac{\Gamma(1/2p_i')}{\Gamma(1/2p_i)}\|f\|_{L^{p_1}(\mathbb{R})}\|g\|_{L^{p_2}(\mathbb{R})}\|h\|_{L^{p_3}(\mathbb{R})}.$$

Remark 4.5. Note that $F_i \equiv 1$, $i = 1, \dots, k$, in all the presented applications of Theorem 2.1 that we considered, while Theorem 2.6 was applied with $F = G \equiv 1$. Obviously, according to the conditions from the statements of these theorems, we can use any other non-negative functions F_i and, consequently, take the infimum of the right-hand sides of the inequalities obtained over all such functions. Therefore, to conclude this paper, we mention the following open problem: can this approach give sharp Hilbert-type inequalities, that is, do there exist such functions F_i that the related inequalities are obtained with the best possible constants on their right-hand sides?

Acknowledgements. This research is supported by the Croatian Ministry of Science, Education and Sports, grant no. 058-1170889-1050.

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