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ON A CLASS OF ANALYTIC FUNCTIONS

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1. Introduction

Let $S(\alpha)$ denote the class of functions

(1) $$f(z) = z + a_2 z^2 + \cdots + a_n z^n + \cdots$$

regular and analytic in the unit disc $E = \{z : |z| < 1|\}$ and satisfying the condition

(2) $$\text{Re}\left[\frac{f(z)}{z}\right]^{\frac{1}{2}} > \frac{1}{2\alpha}, \qquad \alpha \geqq 1, \; |z| < 1 .$$

It was shown by Robertson (1936) that if $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ is univalent and starlike in E then $f(z)$ satisfies $\text{Re}\sqrt{\dfrac{f(z)}{z}} > \dfrac{1}{2}$. In this paper we determine the radius of starlikeness of functions belonging to the class $S(\alpha)$.

We also obtain coefficient estimates for functions in the class $S(\alpha)$, thus generalizing a result due to Dvorak (1967).

It was further shown by Dvorak (1967) that every function $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ regular and univalent in E satisfies the condition $\text{Re}[f(z)/z]^{\frac{1}{2}} > \frac{1}{2}$ in a circle of radius r_0 with $0.83 < r_0 < 0.84$. The exact value of r_0 has been obtained by several authors in Durren and Schober (1971), Kühnau (1971 and 1971a), Reade and Umezawa (1971). We shall find the exact value of $r_0(\alpha)$ for which the univalent function $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ satisfies the condition (2).

2. Radius of starlikeness

THEOREM 2.1. *Let* $f(z) \in S(\alpha)$. *Let* $\alpha_0 > 1$ *denote the smallest positive root of the equation*

$$32\alpha^2 - 104\alpha^2 + 98\alpha - 27 = 0 .$$

(i) *for* $1 \leqq \alpha \leqq \alpha_0$, $f(z)$ *is starlike in*

$$|z| < \left[\frac{8\sqrt{4\alpha-2} - (6\alpha+5)}{18\alpha-17}\right]^{\frac{1}{2}},$$

(ii) *for* $\alpha \geqq \alpha_0$, $f(z)$ *is starlike in*

$$|z| < \frac{\sqrt{(20\alpha^2 - 28\alpha + 9)} - (4\alpha - 3)}{2(\alpha-1)}.$$

These bounds are sharp.

PROOF. Since $f(z) \in S(\alpha)$, we can write

$$\sqrt{\frac{f(z)}{z}} = \frac{1}{2\alpha} + \left(1 - \frac{1}{2\alpha}\right) p(z) \tag{3}$$

where $p(z)$ is regular in E and satisfies the conditions $p(0) = 1$ and $\operatorname{Re} p(z) > 0$ for $z \in E$. Also we know that any such function $p(z)$ can be written in the form

$$p(z) = \frac{1 - w(z)}{1 + w(z)}, \tag{4}$$

where $w(z)$ is regular in E and satisfies the conditions $w(0) = 0$ and $|w(z)| < 1$ for $z \in E$.

(3) and (4) yield

$$\sqrt{\frac{f(z)}{z}} = \frac{\alpha + (1-\alpha)w(z)}{\alpha(1 + w(z))}. \tag{5}$$

Differentiating (5) we get

$$\frac{zf'(z)}{f(z)} = 1 - \frac{2(1 + A)zw'(z)}{(1 + w(z))\,(1 - A\,w(z))}, \tag{6}$$

where $A = 1 - 1/\alpha$, $\alpha \geqq 1$. If we let $\phi(z) = w(z)/z$, then $|\phi(z)| < 1$ and $\phi(z)$ is regular in $|z| < 1$. Hence (Nehari (1952; page 168))

$$|\phi'(z)| \leqq \frac{1 - |\phi(z)|^2}{1 - |z|^2}. \tag{7}$$

Substituting for $\phi(z)$ in terms of $w(z)$ we obtain from (7)

$$|zw'(z) - w(z)| \leqq \frac{r^2 - |w(z)|^2}{1 - r^2}, \quad r = |z|,$$

which, with (6) yields

$$(8)\quad \operatorname{Re}\left[\frac{zf'(z)}{f(z)}\right] \geqq 1-2(1+A)\left[\operatorname{Re}\frac{w(z)}{(1+w(z))(1-Aw(z))} + \frac{r^2-|w(z)|^2}{(1-r^2)|1+w(z)|\,|1-Aw(z)|}\right]$$

$$= \frac{1}{1+A}\left[3A-1+2\operatorname{Re}\left(p(z)-\frac{A}{p(z)}\right)-\frac{2(r^2|p(z)+A|^2-|1-p(z)|^2)}{(1-r^2)|p(z)|}\right]$$

where $p(z) = [1-Aw(z)]/[1+w(z)]$, $0 \leqq A \leqq 1$.

It is easy to see that the transformation $p(z) = [1-Aw(z)]/[1+w(z)]$ maps the circle $|w(z)| \leqq r$ onto the circle

$$(9)\qquad |p(z)-a| \leqq d,\ a=\frac{1+Ar^2}{1-r^2},\quad d=\frac{(1+A)r}{1-r^2},\quad r=|z|.$$

If we put $p(z) = \mathrm{R}e^{i\theta}$ and denote the right hand side of (8) by $S(R,\theta)$. Then

$$(10)\qquad S(R,\theta) = \frac{1}{1+A}\left[3A-1+2R+2\left(R-\frac{A}{R}-2a\right)\cos\theta+\frac{2(a^2-d^2)}{R}\right].$$

Now

$$\frac{\partial S}{\partial \theta} = \frac{2}{1+A}\cdot\sin\theta\cdot T(R)$$

where $T(R) = 2a + A/R - R$, $a-d \leqq R \leqq a+d$. Since $T(R)$ clearly is a monotone decreasing function of R, and since

$$T(a+d) = 2\,\frac{1+Ar^2}{1-r^2}+\frac{A(1-r)}{1+Ar}-\frac{1+Ar}{1-r}$$

$$= \left[\frac{2(1+Ar^2)}{1-r^2}-\frac{1+Ar}{1-r}\right]+\frac{A(1-r)}{1+Ar}$$

$$= \frac{1-Ar}{1+r}+\frac{A(1-r)}{1+Ar} > 0,$$

It follows that $T(R)$ remains positive for $a-d \leqq R \leqq a+d$. Therefore, the maximum of $S(R,\theta)$ inside the circle $|p(z)-a| \leqq d$ is attained for $\theta = 0$. By Putting $\theta = 0$ in (10) we obtain

$$(11)\qquad S(R,0) = \frac{1}{1+A}\left[3A-1+2\left(2R-\frac{A}{R}-2a\right)+\frac{2(a^2-d^2)}{R}\right],\quad a-d \leqq R \leqq a+d.$$

Since

$$\frac{\partial S}{\partial R} = \frac{2}{1+A}\left[2+\frac{A}{R^2}-\frac{(a^2-d^2)}{R^2}\right]$$

$$= \frac{2}{1+A}\left[2-\frac{(1-A)(1+Ar^2)}{1-r^2}\,\frac{1}{R^2}\right] = \frac{2}{1+A}\left[2-\frac{(1-A)a}{R^2}\right],$$

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We see that the absolute minimum of $S(R,0)$ in $(0,\infty)$ is attained at $R=\sqrt{((1-A)a)/2}$ and equals

$$\frac{1}{1+A}\left[3A-1+4\sqrt{2(1-A)a}-4a\right]. \tag{12}$$

It is easy to see that $R_0 < a+d$, but R_0 is not always greater than $a-d$. In such a case when $R_0 \notin [a-d, a+d]$ the minimum of $S(R,0)$ on the segment $[a-d, a+d]$ is attained at $R_1 = a-d$ and equals

$$\frac{1-(1+3A)r-Ar^2}{(1+r)(1-Ar)}. \tag{13}$$

The two minima given by (12) and (13) coincide for such values of A for which $R_0 = R_1$. We thus conclude that

$$\operatorname{Re}\left[\frac{zf'(z)}{f(z)}\right] \geqq \frac{1}{1+A}\left[3A-1-4a+4\sqrt{2(1-A)a}\right] \text{ for } R_0 \geqq R_1, \tag{14}$$

and

$$\operatorname{Re}\left[\frac{zf'(z)}{f(z)}\right] \geqq \frac{1-(1+3A)r-Ar^2}{(1+r)(1-Ar)} \text{ for } R_0 \leqq R_1. \tag{15}$$

The equality sign in (14) is attained for the function

$$f(z) = z\left[\frac{1+(1/\alpha-1)z}{1+z}\right]^2. \tag{16}$$

The equality sign in (15) is attained for the function

$$f(z) = z\left[\frac{1-1/\alpha\cos\theta\cdot z+(1/\alpha-1)z^2}{1-2\cos\theta\cdot z+z^2}\right]^2 \tag{17}$$

where $\cos\theta$ is determined from

$$\frac{1-(1+A)r\cos\theta+Ar^2}{1-2r\cos\theta+r^2} = R_0.$$

Hence the radius of starlikeness for the class $S(\alpha)$ which may be obtained from (14) and (15) is given by

$$3A-1-4a+4\sqrt{2(1-A)a} = 0, \qquad R_0 \geqq R_1, \tag{18}$$

$$1-(1+3A)r-Ar^2 = 0,\ A = 1-1/\alpha,\ R_0 \leqq R_1, \tag{19}$$

which yield

$$r_s = \left[\frac{8(4\alpha-2)-(6\alpha+5)}{18\alpha-17}\right]^{\frac{1}{2}}, \quad R_0 \geqq R_1, \tag{20}$$

and

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(21) $$r_s = \frac{\sqrt{(20\alpha^2 - 28\alpha + 9)} - (4\alpha - 3)}{2(\alpha - 1)}, \qquad R_0 \leqq R_1 .$$

The two minima given by (14) and (15) become equal to each other for such a $A(0 \leqq A < 1)$ for which

(22) $$R_0 = \left[\frac{(1-A)a}{2}\right]^{\frac{1}{2}} = a - d = R_1 .$$

Hence the values of α for which the two values of r_s given (20) and (21) become equal are obtained by eleminating r from (19) and (22). We obtain $-27A^3 - 17A^2 + 11A + 1 = 0$, and hence

(23) $$K(\alpha) = 32a^3 - 104\alpha^2 + 98\alpha - 27 = 0,$$

Since $K(1) = -1 < 0$ and $K(\infty)) = +\infty$, it follows that α_0 in the theorem lies in $(1, \infty)$.

The functions given by (16) and (17) show that the bounds are sharp.

3. Coefficient estimates

THEOREM 3.1. *If* $f(z) = z + a_2 z^2 + \cdots + a_n z^n + \cdots$ *is regular and analytic in* E *and satisfies* (2), *then*

$$|a_n| \leqq 4\left(1 - \frac{1}{2\alpha}\right)\left[n\left(1 - \frac{1}{2\alpha}\right) - \left(1 - \frac{1}{\alpha}\right)\right] \text{ for } n = 2, 3, \cdots .$$

These bounds are sharp.

PROOF. On putting $p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n$ in (3) we get

(24) $$f(z) = z\left[\frac{1}{2\alpha} + \left(1 - \frac{1}{2\alpha}\right)\left(1 + \sum_{n=1}^{\infty} p_n z^n\right)\right]^2 .$$

On substituting the power series exapnsion for $f(z)$ from (1) in (24) and then equating the coefficients of z^{2m} and z^{2m+1} we get

(25) $$a_{2m+1} = 2\left(1 - \frac{1}{2\alpha}\right)p_{2m} + \left(1 - \frac{1}{2\alpha}\right)^2\left(p_m^2 + 2 \sum p_r p_s\right), \quad r + s = 2m,$$

and

(26) $$a_{2m+2} = 2\left(1 - \frac{1}{2\alpha}\right)p_{2m+1} + 2\left(1 - \frac{1}{2\alpha}\right)^2 \sum p_r p_s, \quad r + s = 2m + 1 .$$

Since $\operatorname{Re} p(z) > 0$ for $z \in E$, we have (Nehari (1952; page 170))

(27) $$|p_n| \leqq 2 \quad \text{for } n = 1, 2, 3, \cdots .$$

From (25), (26) and (27) we easily obtain the bounds

$$(28)\qquad \begin{aligned} |a_n| &\leqq 4\left(1-\frac{1}{2\alpha}\right)\left(n-1-\frac{n-2}{2\alpha}\right) \\ &= 4\left(1-\frac{1}{2\alpha}\right)\left[n\left(1-\frac{1}{2\alpha}\right)-\left(1-\frac{1}{\alpha}\right)\right], \quad n=2,3,\cdots. \end{aligned}$$

The bounds are attained by the extremal function

$$f(z) = z\left[\frac{1}{2\alpha}+\left(1-\frac{1}{2\alpha}\right)\frac{1+z}{1-z}\right]^2.$$

This completes the proof of the theorem.

REMARK 1. On putting $\alpha = 1$ in Theorem 3.1. we get

$$|a_n| \leqq n, \quad n = 2,3,\cdots,$$

which is a result obtained by Dvorak (1967).

4. An inequality for univalent functions

THEOREM 4.1. *Let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ be regular and univalent in E. Then $f(z)$ satisfies (2) for $|z| < r_0(\alpha)$ where $r_0(\alpha)$ is the smallest positive root of the equation*

$$(29)\quad \left[S^{-1}\left(\frac{1}{2}\log\frac{1+r}{1-r}\right)\right]^2+\left[E^{-1}\left(\frac{\sqrt{1-r^2}}{4\alpha}\log\frac{1+r}{1-r}\right)\right]^2=\left[\frac{1}{2}\log\frac{1+r}{1-r}\right]^2,$$

where $S^{-1}(x)$ and $E^{-1}(x)$ are the inverses of $S(x) = x/\sin x$ and $E(x) = xe^{-x}$ respectively. The result is sharp.

PROOF. Condition (2) is equivalent to the inequality

$$(30)\qquad \left|\sqrt{\frac{z}{f(z)}}-\alpha\right| < \alpha.$$

Since $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ is regular and univalent in E, we have Gulusin (1947; page 113)

$$(31)\qquad \left|\log\sqrt{\frac{z}{f(z)}}-\frac{1}{2}\log(1-|z|^2)\right| \leqq \frac{1}{2}\log\left(\frac{1+|z|}{1-|z|}\right).$$

Putting $W = \log\sqrt{z/f(z)}$, $A = \frac{1}{2}\log(1-|z|^2)$, $B = \frac{1}{2}\log(1+|z|)/(1-|z|)$, $W_1 = e^W = Re^{i\phi}$ in (30) and (31) we obtain

$$(32)\qquad R < 2\alpha\cos\phi$$

and

$$(\log R - A)^2 + \phi^2 < B^2, \tag{33}$$

respectively.

If $|z| = r$ is small, it is evident that the region defined by (33) lies in the region (32). As r increases, the boundary of (33) touches the boundary of (32) before r reaches 1. At such a point of contact we must have

$$\log R = \log(2\alpha\cos\phi) = (A + \sqrt{B^2 - \phi^2} \tag{34}$$

and

$$\frac{dR}{d\phi} = -2\alpha\sin\phi = -\frac{\phi}{\sqrt{B^2-\phi^2}}\exp(A + \sqrt{B^2 - \phi^2}). \tag{35}$$

On eliminating ϕ from (34) and (35) we get

$$\tfrac{1}{2}Be^A = \alpha\sqrt{B^2 - \phi^2}\,\exp[-\sqrt{B^2 - \phi^2}]. \tag{36}$$

From (35) and 36) we obtain

$$\frac{\phi}{\sin\phi} = B. \tag{37}$$

If we denote by $E^{-1}(x)$ and $S^{-1}(x)$ the inverses of $E(x) = xe^{-x}$ and $S(x) = x/\sin x$ respectively, then (36) and (37) yield (29).

The result is sharp because the inequality (31) is sharp.

THEOREM 4.2. *Let $g(z) = z + a_3z^3 + \cdots$ be analytic, univalent and odd in E. Then* $\mathrm{Re}[(g(z))/z] > 1/2\alpha$ for $|z| < r_1(\alpha)$, *where $r_1(\alpha)$ is the smallest positive root of the equation*

$$\left(S^{-1}\left[\frac{1}{2}\log\frac{1+\sqrt{r}}{1-\sqrt{r}}\right]\right)^2 + \left[E^{-1}\left(\frac{\sqrt{1-r}}{4\alpha}\log\frac{1+\sqrt{r}}{1-\sqrt{r}}\right)\right]^2 = \left[\frac{1}{2}\log\frac{1+\sqrt{r}}{1-\sqrt{r}}\right]^2.$$

The result is sharp.

PROOF. If we take $f(z^2) = (g(z))^2$, then $f(z)$ is analytic and univalent in E and we then apply Theorem 4.1 to obtain the above theorem.

REMARK. On putting $\alpha = 1$ in Theorems 4.1 and 4.2. We obtain Theorem C and D proved by Reade and Umezawa (1971). This shows our theorems generalize the results obtained earlier by Reade and Umezawa (1971) and Duren and Schober (1971).

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