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A CHARACTERIZATION OF THE ZASSENHAUS GROUPS

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Introduction

A doubly transitive permutation group $\mathfrak{G}$ on the set of symbols Ω is called a Zassenhaus group if $\mathfrak{G}$ satisfies the following condition: the identity is the only element leaving three distinct symbols fixed.

The Zassenhaus groups were classified by H. Zassenhaus [14], W. Feit [3], N. Ito [7], and M. Suzuki [9]. There have been several characterizations of the Zassenhaus groups. Namely M. Suzuki [10] has proved that if a non abelian simple group $\mathfrak{G}$ has a non-trivial partition then $\mathfrak{G}$ is isomorphic with one of the groups $\mathrm{PSL}(2, q)$ or $\mathbf{Sz}(2^n)$. Since each of the groups $\mathrm{PSL}(2, q)$, $\mathbf{Sz}(2^n)$ has a non-trivial partition, a theorem of Suzuki characterizes them.

In this paper we shall characterize the Zassenhaus groups as permutation groups by a property of the centralizer of their involutions.

Let $\mathfrak{G}$ be a finite permutation group on a set of n symbols $\Omega = \{1, 2, \cdots n\}$. For every $i (0 \leq i \leq n)$, we define a subset $\mathfrak{C}_i$ of $\mathfrak{G}$ in the following way:

$$\mathfrak{C}_i = \{G \in \mathfrak{G} \mid G \text{ leaves exactly } i \text{ distinct symbols fixed}\}.$$

Clearly each $\mathfrak{C}_i$ is a union of some conjugate classes of $\mathfrak{G}$. In particular $\mathfrak{C}_n = \{1\}$. A subset $\mathfrak{C}_i$ may be empty for some i. We shall set a following condition:

(c_i) there exists an involution $I^{(i)} \in \mathfrak{C}_i$ such that the centralizer $\mathfrak{C}_{\mathfrak{G}}(I^{(i)})$ of $I^{(i)}$ in $\mathfrak{G}$ is contained in $\mathfrak{C}_i \cup \{1\}$.

It is easy to see that every conjugate element J of $I^{(i)}$ has the same property as $I^{(i)}$. As a matter of fact, the linear fractional groups $\mathrm{PSL}(2, q)$ and Suzuki's simple groups $\mathbf{Sz}(2^m)$ satisfy one of the conditions (c_0), (c_1) or

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(c_2). More strongly the above mentioned simple groups satisfy the following condition (a_i) for $i = 0, 1$ and 2:

(a_i) for every element A of $\mathfrak{C}_i$, the centralizer $\mathfrak{C}_{\mathfrak{G}}(A)$ is contained in $\mathfrak{C}_i \cup \{1\}$.

Other than PSL$(2, q)$ and **Sz**(2^m), the Mathieu group $\mathfrak{M}_{22}$ of degree 22 satisfies the condition (a_1). If we consider the Mathieu group $\mathfrak{M}_{11}$ as a permutation group of degree 12, then $\mathfrak{M}_{11}$ satisfies (a_2). It is interesting to investigate the structure of $\mathfrak{G}$ satisfying the condition (a_i) for some i. It seems, however, difficult to treat.

Now we state our result.

THEOREM. *Let $\mathfrak{G}$ be a doubly transitive permutation group on Ω. Let us assume that $\mathfrak{G}$ satisfies the condition (c_i) for some i. Then $\mathfrak{G}$ is isomorphic with one of the groups PSL$(2, q)$ or* ***Sz***(2^m), *or $\mathfrak{G}$ has a regular normal subgroup.*

Remark. There exists a non solvable exactly doubly transitive group satisfying (c_1) (see Zassenhaus [15]). Therefore the last statement of the theorem is necessary even if we assume that G is non solvable.

The proof of the above theorem is divided into two cases;

case (1): $i = 0$ or 1,

case (2): $i \geq 2$.

In case (1) our aim is to prove that the stabilizer $\mathfrak{H}$ of a symbol 1 has a normal subgroup $\mathfrak{K}$ which is regular on $\Omega - \{1\}$. After it is proved, the elementary argument shows that $\mathfrak{G}$ is a Zassenhaus group. In case (2) we shall apply an interesting work of N. Iwahori [8] who has investigated the structure of groups of positive type. In later section we shall recall his definitions and results. Using a result of N. Iwahori we shall prove that a Sylow 2-subgroup $\mathfrak{S}$ of $\mathfrak{G}$ is a dihedral group and the centralizer $\mathfrak{C}_{\mathfrak{G}}(I)$ of a central involution I of $\mathfrak{S}$ has an abelian normal 2-complement. By a theorem of D. Gorenstein-J. Walter [6], we can easily prove our theorem.

Our notation is mostly standard. Denote by $(\mathfrak{G}, \Omega)$ a permutation group on a set Ω of n symbols $\{1, 2, \cdots, n\}$. If a subgroup $\mathfrak{A}$ of $\mathfrak{G}$ acts on a subset Δ of Ω, we denote a permutation group induced by $\mathfrak{A}$ on Δ by $(\mathfrak{A}^{\Delta}, \Delta)$ or simply by $\mathfrak{A}^{\Delta}$. $\mathfrak{A}^{\Delta}$ is a homomorphic image of $\mathfrak{A}$. The normalizer or the centralizer of a subset $\mathfrak{X}$ of $\mathfrak{G}$ is denoted by $\mathfrak{N}_{\mathfrak{G}}(\mathfrak{X})$ or $\mathfrak{C}_{\mathfrak{G}}(\mathfrak{X})$ respectively, or simply by $\mathfrak{N}(\mathfrak{X})$, $\mathfrak{C}(\mathfrak{X})$ if no confusion seems to occur. The image of a symbol j by the action of an element G of $\mathfrak{G}$ is denoted by j^G. $|\mathfrak{M}|$ is the cardinality of a certain set $\mathfrak{M}$. All groups considered are finite.

Proof of Theorem

1. Preliminary Lemmas

First we shall prove two lemmas.

LEMMA 1. *Let $\mathfrak{G}$ be a permutation group satisfying the condition (c_i) for some i. If all the involutions of $\mathfrak{G}$ are contained in a single conjugate class, then involutions are only elements which have transpositions in their cycle decompositions.*

Proof. Let A be an element of $\mathfrak{G}$ whose cycle decomposition contains a transposition:

$$A = (a, b) \cdot \cdot \cdot \cdot \cdot .$$

Then A is a 2-singular element. Therefore A is commutative with a certain involution I which is conjugate to $I^{(i)}$ by assumption. If A^2 is not the identity element of $\mathfrak{G}$, then A^2 is commutative with I and A^2 leaves at least $i+2$ symbols invariant. This is impossible. This follows the lemma.

LEMMA 2. *Let $\mathfrak{G}$ be a doubly transitive permutation group satisfying the condition (c_i) for some i. If all the involutions of $\mathfrak{G}$ are contained in a single conjugate class, then the order of the centralizer $\mathfrak{C}_{\mathfrak{G}}(I)$ of any involution I is equal to $n-i$.*

Proof. Let $\beta(G)$ denote the number of transpositions in the cycle decomposition of an element G of $\mathfrak{G}$. Then by a theorem of G. Frobenius [5] we get a following equality:

$$\sum_{G \in \mathfrak{G}} \beta(G) = |\mathfrak{G}|/2.$$

By Lemma 1, $\beta(G) > 0$ if and only if G is an involution of $\mathfrak{G}$. Hence

$$\beta(I) \cdot |\mathfrak{G}|/|\mathfrak{C}(I)| = |\mathfrak{G}|/2.$$

On the other hand, since an involution I has $n-i/2$ transpositions we get easily

$$|\mathfrak{C}(I)| = n-i.$$

2. Case (1): $i=0$ or $i=1$.

Let $\mathfrak{G}$ be a non-solvable doubly transitive group on Ω satisfying the condition (c_i) for $i=0$ or $i=1$. Assume that $\mathfrak{G}$ has no regular normal subgroup. Denote by $\mathfrak{H}$ the stabilizer of the symbol 1 and by $\mathfrak{K}$ the stabilizer of the symbols 1 and 2. Let J be an involution of $\mathfrak{G}$ which is conjugate to $I^{(i)}$ where $i=0$ or 1. By the double transitivity of $\mathfrak{G}$ we can choose J such that a cyclic decomposition of J is $(12)\cdots$. J is contained in the normalizer $\mathfrak{N}_{\mathfrak{G}}(\mathfrak{K})$ of $\mathfrak{K}$ in $\mathfrak{G}$. Therefore J induces an automorphism of order 2 on $\mathfrak{K}$. By the condition (c_0) or (c_1), J has no fixed element in $\mathfrak{K}$. Hence $\mathfrak{K}$ is an abelian group of odd order. J inverts every element of $\mathfrak{K}$.

Lemma 3. *If $i=0$ or 1, then all the involutions of $\mathfrak{G}$ are contained in a single conjugate class.*

Proof. Let J_1 and J_2 be two involutions of $\mathfrak{G}$. By the double transitivity of $\mathfrak{G}$, there exists an element A such that

$$J_1=(ab)\cdots\cdots.$$
$$J_2^A=(ab)\cdots\cdots.$$

Hence the element $B=J_1J_2^A$ is contained in a suitable conjugate subgroup $\mathfrak{K}^G$ of $\mathfrak{K}$. Therefore the order of B is odd. This implies J_1 and J_2^A are conjugate to each other in $\mathfrak{K}^G$. Thus we have proved our lemma.

If $i=1$, then by Lemma 3 every involution has the same property as $I^{(1)}$. Therefore we can choose an involution I which is conjugate to $I^{(1)}$ and leaves the symbol 1 fixed.

Lemma 4. *If $i=1$, then $\mathfrak{H}=\mathfrak{C}_{\mathfrak{G}}(I)\mathfrak{K}$. Furthermore, every involution of $\mathfrak{H}$ is written in a form I^K where K is an element of $\mathfrak{K}$.*

Proof. Let I_1 and I_2 be two involutions of $\mathfrak{H}$. Then by Lemma 3 $I_1^G=I_2$, $G\in\mathfrak{G}$. Therefore $1^{G^{-1}I_1G}=1^{I_2}=1$. Hence I_1 leaves the symbol $1^{G^{-1}}$ fixed. Since $I_1\in\mathfrak{C}_1$, $1^{G^{-1}}=1$. Hence $G\in\mathfrak{H}$. In particular $\mathfrak{C}_{\mathfrak{G}}(I)\subset\mathfrak{H}$. By Lemma 2 and Lemma 3, we have $|\mathfrak{C}(I)|=n-1$. Since the order of $\mathfrak{H}$ is $(n-1)\cdot|\mathfrak{K}|$ and $\mathfrak{C}(I)\cap\mathfrak{K}=\{1\}$ by the condition (c_1), we conclude $\mathfrak{H}=\mathfrak{C}(I)\cdot\mathfrak{K}$. Thus we have proved our lemma.

Lemma 5. *If $i=0$ or 1, then $[\mathfrak{N}_{\mathfrak{G}}(\mathfrak{K}):\mathfrak{K}]=2$.*

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Proof. Let Δ be a set of symbols of Ω which are left fixed individually by every element of $\mathfrak{K}$. By a theorem of Witt [13], $\mathfrak{N}(\mathfrak{K})/\mathfrak{K}$ is considered as a doubly transitive permutation group on Δ. We can easily prove that this permutation group is exactly doubly transitive. Therefore we can conclude that $|\Delta| = q^s$ where q is a prime number. Assume $q = 2$. Then a Sylow 2-subgroup of $\mathfrak{N}(\mathfrak{K})$ is an elementary abelian 2-group of order 2^s. Since every involution of $\mathfrak{N}(\mathfrak{K})$ inverts every element of $\mathfrak{K}$, we conclude $s = 1$. This implies $[\mathfrak{N}(\mathfrak{K}) : \mathfrak{K}] = 2$. Next assume that q is odd. Since $|\mathfrak{H} \cap \mathfrak{N}(\mathfrak{K})/\mathfrak{K}| = q^s - 1$. There exists an involution I_1 of $\mathfrak{H}$ which acts on $\mathfrak{K}$. Clearly $i = 1$ and $n =$ odd in this case. Since $\mathfrak{K}$ is an abelian group, all the involutions of $\mathfrak{H}$ act on $\mathfrak{K}$ by Lemma 4. Therefore if a Sylow 2-subgroup of $\mathfrak{H}$ has at least two involutions, then there exists an involution I_2 which acts trivially on $\mathfrak{K}$, which is impossible by the condition (c_1). Thus a Sylow 2-subgroup of $\mathfrak{H}$ has only one involution. Since n is odd, a Sylow 2-subgroup of $\mathfrak{G}$ is isomorphic to that of $\mathfrak{H}$ and has only one involution. Hence a Sylow 2-subgroup of $\mathfrak{G}$ is either cyclic or generalized quaternion group. Therefore $\mathfrak{G}$ has a regular normal subgroup (Burnside [2], Brauer-Suzuki [1], Feit-Thompson [4]). This is impossible. Thus we have proved our lemma.

LEMMA 6. *If $i = 0$ or 1, then $\mathfrak{K}$ has a normal complement $\mathfrak{L}$ in $\mathfrak{H}$. Namely $\mathfrak{H} = \mathfrak{L} \cdot \mathfrak{K}$, $\mathfrak{L} \cap \mathfrak{K} = 1$.*

Proof. By Burnside's splitting theorem, it suffices to show that $\mathfrak{N}_{\mathfrak{H}}(\mathfrak{K}_p) = \mathfrak{C}_{\mathfrak{H}}(\mathfrak{K}_p) = \mathfrak{K}$ for every Sylow p-subgroup $\mathfrak{K}_p$ of $\mathfrak{K}$. For, if so, $\mathfrak{K}_p$ is a Sylow p-subgroup of $\mathfrak{H}$ and it has a normal complement $\mathfrak{L}_p$ in $\mathfrak{H}$. Put $\bigcap_{p \mid |\mathfrak{K}|} \mathfrak{L}_p = \mathfrak{L}$. Clearly $\mathfrak{L}$ is a normal complement of $\mathfrak{K}$ in $\mathfrak{H}$. Let Δ be a set of symbols of Ω which are left fixed individually by every element of $\mathfrak{K}_p$. Let us assume that $|\Delta| \geq 3$. By a theorem of Witt $\mathfrak{N}_{\mathfrak{G}}(\mathfrak{K}_p)^{\Delta}$ is a doubly transitive group on Δ. Since $\mathfrak{C}_{\mathfrak{G}}(\mathfrak{K}_p)$ contains $\mathfrak{K}$ and $\mathfrak{K}$ leaves just two symbols 1, 2 invariant by Lemma 5, $\mathfrak{C}(\mathfrak{K}_p)^{\Delta}$ is a non-trivial normal subgroup of $\mathfrak{N}(\mathfrak{K}_p)^{\Delta}$ of odd order. By the double transitivity of $\mathfrak{N}(\mathfrak{K}_p)^{\Delta}$, $\mathfrak{C}(\mathfrak{K}_p)^{\Delta}$ is transitive. Hence $|\Delta|$ is odd. Since $\Delta^J = \Delta$, an involution J keeps at least one symbol unchanged. Hence $n =$ odd and $i = 1$. The order of the group $\mathfrak{H} \cap \mathfrak{N}(\mathfrak{K}_p)$ is divisible by $|\Delta| - 1$. Therefore $\mathfrak{H}$ has an involution which acts on $\mathfrak{K}_p$. Hence all the involutions of $\mathfrak{H}$ act on $\mathfrak{K}_p$ by Lemma 4. This implies that a Sylow 2-subgroup of $\mathfrak{G}$ has only one involution. Hence $\mathfrak{G}$ has a regular normal subgroup. This is not the case. Hence $|\Delta| = 2$. Hence $\mathfrak{N}(\mathfrak{K}_p) = \langle J, \mathfrak{K} \rangle$. Therefore $\mathfrak{N}_{\mathfrak{H}}(\mathfrak{K}_p) = \mathfrak{C}_{\mathfrak{H}}(\mathfrak{K}_p) = \mathfrak{K}$. This yields our lemma.

Proposition 1. *Let $\mathfrak{G}$ be a doubly transitive permutation group satisfying the condition (c_i) for $i = 0$ or 1. Then $\mathfrak{G}$ is isomorphic with one of the groups $PSL(2, q)$ or $Sz(2^m)$, or $\mathfrak{G}$ has a regular normal subgroup.*

Proof. Assume that $\mathfrak{G}$ has no regular normal subgroup. By Lemma 6, $\mathfrak{H}$ has a normal subgroup $\mathfrak{L}$ of order $n-1$ which is regular on $\Omega - \{1\}$. Therefore $\mathfrak{G}$ admits a decomposition:

$$\mathfrak{G} = \mathfrak{H} + \mathfrak{H} J \mathfrak{L}.$$

Every element of $\mathfrak{G} - \mathfrak{H}$ is uniquely expressed in a form $L'KJL$ where $L', L \in \mathfrak{L}$, $K \in \mathfrak{K}$. Next we shall show that $\mathfrak{K}$ is a T.I. set in $\mathfrak{G}$. Since $\mathfrak{K}$ is an abelian subgroup, it suffices to show that the centralizer of any non-identity element of $\mathfrak{K}$ is equal to $\mathfrak{K}$. Let an element $K_1 \in \mathfrak{K}$ is commutative with an element of $\mathfrak{G} - \mathfrak{H}$. Assume $K_1 L' K J L = L' K J L K_1$ where $K_1, K \in \mathfrak{K}$, $L', L \in \mathfrak{L}$. Then $L'^{K_1^{-1}} K_1 K J L = L' K K_1^{-1} J L^{K_1}$. By the uniqueness of expression of an element of $\mathfrak{G} - \mathfrak{H}$ we get $K_1 K = K K_1^{-1}$. This implies $K_1 = 1$, since $\mathfrak{K}$ is an abelian group of odd order. If K_1 is commutative with an element L of $\mathfrak{L}$, then K_1^J is commutative with $L^J \in \mathfrak{G} - \mathfrak{H}$. This is impossible by the above fact. Therefore $\mathfrak{K}$ is a T.I. set in $\mathfrak{G}$. Let us assume that an element $A \neq 1$ of $\mathfrak{K}$ keeps at least three distinct symbols, say 1, 2, 3, unchanged. Then $A \in \mathfrak{K} \cap \mathfrak{K}^H$ where $1^H = 1, 2^H = 3$. Therefore $\mathfrak{K} = \mathfrak{K}^H$ and $\mathfrak{K}$ keeps 1, 2, 3 invariant. By Lemma 3, this is impossible. Therefore $\mathfrak{G}$ is a Zassenhaus group. Since $\mathfrak{G}$ has only one class of involutions and the order of any involution of $\mathfrak{G}$ is $|\Omega|$ or $|\Omega| - 1$, we get easily our proposition.

3. Case (2): $i \geq 2$.

First we shall recall a result of N. Iwahori [8].

Let $\mathfrak{G}$ be a permutation group on $\mathfrak{M}$. We call $\mathfrak{M}$ a $\mathfrak{G}$-space. Define a subset $\mathfrak{M}_G (G \in \mathfrak{G})$ of $\mathfrak{M}$ as follows.

$$\mathfrak{M}_G = \{m \in \mathfrak{M} \mid m^G = m\}.$$

Definition 1. A permutation groups $\mathfrak{G}$ on $\mathfrak{M}$ is of type k if the following two conditions are satisfied;

(i) $|\mathfrak{M}_G| = k$, for every non identity element $G \in \mathfrak{G}$,

(ii) $\bigcap_{G \in \mathfrak{G}} \mathfrak{M}_G = \phi$, where ϕ denotes the empty set.

N. Iwahori's main result is the following theorem.

THEOREM. *If $\mathfrak{G}$ admits a $\mathfrak{G}$-space $\mathfrak{M}$ of type 2, then $\mathfrak{G}$ is isomorphic to one of the following groups:*

(i) A_4: *the alternating group of degree* 4,

(ii) S_4: *the symmetric group of degree* 4,

(iii) $\mathfrak{A}_5$: *the alternating group of degree* 5 *or*

(iv) *a generalized dihedral group with dihedral Sylow* 2*-subgroups.*

Here a generalized dihedral group is defined as follows. Let $\mathfrak{A}$ be an abelian group and τ be an automorphism of $\mathfrak{A}$ such that if $A \in \mathfrak{A}$, then $A^\tau = A^{-1}$, where A^τ denotes the image of A by τ. Under these conditions, holomorph of $\mathfrak{A}$ by τ is called a generalized dihedral group.

In order to prove his theorem, N. Iwahori has proved several lemmas. We shall quote one of them here.

LEMMA 7 (*Lemma* 1. 3 *in* [8]). *Let $\mathfrak{G}$ be a finite group and $\mathfrak{M}$ a $\mathfrak{G}$-space of type $k > 0$. Let A and B be elements in $\mathfrak{G} - \{1\}$ of orders a and b respectively. Assume that*

(i) $AB = BA$, *and*

(ii) $a \neq b$ *or* $a = b \neq$ *prime.*

Then $\mathfrak{M}_A = \mathfrak{M}_B$.

Now we shall apply his argument to our case. Let $\mathfrak{G}$ be a non solvable doubly transitive group on Ω satisfying the condition (c_i) for $i \geq 2$. As in section 2, let us denote the stabilizer of the symbol 1 by $\mathfrak{H}$ and the stabilizer of two symbols 1 and 2 by $\mathfrak{K}$. J is an involution of $\mathfrak{G}$ which is conjugate to $I^{(i)}$. We can choose J such that a cyclic decomposition of J is $(12)\cdots$. In the rest of this paper we shall use the notation I instead of $I^{(i)}$.

LEMMA 8. *The centralizer $\mathfrak{C}(I)$ of I admits a $\mathfrak{C}(I)$-space of positive type.*

Proof. We may assume that I leaves i symbols, say $1, 2, \cdots, i$ invariant. If $\mathfrak{C}(I)$ does not admit a $\mathfrak{C}(I)$-space of positive type, then by the condition (c_i) every element $A \neq 1$ of $\mathfrak{C}(I)$ leaves just i symbols $1, 2, \cdots, i$ invariant.

Clearly every conjugate subgroup of $\mathfrak{C}(I)$ does not also admit a $\mathfrak{C}(I)$-space of positive type. Therefore if $\mathfrak{C}(I^G) \cap \mathfrak{C}(I) > \{1\}$ then every element of $\mathfrak{C}(I^G)$ leaves just i symbols $1, 2, \cdots, i$, invariant. Let $\mathfrak{K}_2$ be a Sylow 2-subgroup of $\mathfrak{K}$ which is non-trivial by the condition (c_i) $(i \geq 2)$. Since J acts on $\mathfrak{K}$, we may assume $\mathfrak{K}_2^J = \mathfrak{K}_2$. Put $\mathfrak{S} = \langle J, \mathfrak{K}_2 \rangle$. Then there exists an involution I_1 of $\mathfrak{K}_2$ which is conjugate to I and $\mathfrak{C}(J) \cap \mathfrak{C}(I_1) \supset \mathfrak{Z}(\mathfrak{S}) > \{1\}$. Thus every element of $\mathfrak{C}(J)$ leaves 1, 2, invariant. In particular J leaves $1, 2$ invariant. This is impossible, since J has a cyclic decomposition $(12) \cdots$. Thus we have proved our lemma.

LEMMA 9. *$\mathfrak{C}(I)$ is an elementary abelian 2-group or a generalized dihedral group.*

Proof. Since $\mathfrak{C}(I)$ admits a $\mathfrak{C}(I)$-space of positive type, we may apply Lemma 7. Assume that $\mathfrak{C}(I)$ is not an elementary abelian 2-group. Let $\mathfrak{N}$ be a (normal) subgroup of $\mathfrak{C}(I)$ which is generated by all non-involutions of $\mathfrak{C}(I)$. By Lemma 7, every element of $\mathfrak{N}$ leaves $1, 2, \cdots i$ fixed. This implies that $\mathfrak{N}$ is a proper subgroup of $\mathfrak{C}(I)$. If A is an element of $\mathfrak{C}(I) - \mathfrak{N}$, then $A^2 = 1$. Therefore $(AN)^2 = 1$ for $N \in \mathfrak{N}$. Hence $A^{-1}NA = N^{-1}$. Hence $\mathfrak{N}$ is an abelian subgroup of $\mathfrak{C}(I)$. If B is another element of $\mathfrak{C}(I) - \mathfrak{N}$, then $B^2 = 1$ and the element AB centralizes $\mathfrak{N}$. Hence $A \equiv B \pmod{\mathfrak{N}}$. This implies that $[\mathfrak{C}(I) : \mathfrak{N}] = 2$. This follows our lemma.

LEMMA 10. *If $\mathfrak{C}(I)$ is not an elementary abelian 2-group, then $\mathfrak{C}(I)$ admits a $\mathfrak{C}(I)$-space of type 2.*

Proof. Let Γ be a subset of $\{1, 2, \cdots, i\}$ consisting of elements left fixed by every element of $\mathfrak{C}(I)$. Put $\Delta = \{1, 2, \cdots, i\} - \Gamma$. Since $\mathfrak{C}(I)$ admits a $\mathfrak{C}(I)$-space of positive type, we have $|\Delta| = k \geq 1$. Let r be the number of orbits of $\mathfrak{C}(I)$ on $\Omega - \Gamma = \mathfrak{M}$. Then

$$r|\mathfrak{C}(I)| = |\mathfrak{M}| + k(|\mathfrak{C}(I)| - 1)$$

(Wielandt [13] p. 8 Ex. 3. 10).
Hence

$$|\mathfrak{C}(I)| = \frac{|\mathfrak{M}| - k}{r - k} = \frac{n - (i - k) - k}{r - k} = \frac{n - i}{r - k} \leqq n - i.$$

On the other hand, using a equality of Frobenius $\sum_{G \in \mathfrak{G}} \beta(G) = |\mathfrak{G}|/2$ we get

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$$\frac{n-i}{2} \cdot \frac{|\mathfrak{G}|}{|\mathfrak{C}(I)|} \leq |\mathfrak{G}|/2 .$$

Hence $|\mathfrak{C}(I)| \geq n-i$. Hence $|\mathfrak{C}(I)| = n-i$, $r = k+1$ and $|\mathfrak{M}| = |\mathfrak{C}(I)| + k$. Since $\mathfrak{C}(I)$ has a normal subgroup $\mathfrak{N}$ of index 2 which leaves all the symbols of Δ fixed, Δ decomposes into $k/2$ orbits of $\mathfrak{C}(I)$. Since by the condition (c_i) any element of $\mathfrak{N}$ has no fixed symbols on $\mathfrak{M} - \Delta$ each of the remaining orbits of $\mathfrak{C}(I)$ of $\mathfrak{M} - \Delta$ has length at least $|\mathfrak{C}(I)|/2$ hence exactly $|\mathfrak{C}(I)|/2$. Therefore we have the following equality.

$$\frac{k}{2} + 2 = r = k + 1.$$

Hence $k = 2$. Thus we have proved our lemma.

LEMMA 11. *All the involutions of $\mathfrak{G}$ are contained in a single conjugate class.*

Proof. In the proof of Lemma 10, we have proved the equality $|\mathfrak{C}(I)| = n - i$. This relation also holds when $\mathfrak{C}(I)$ is an elementary abelian 2-group, because in proving the equality $|\mathfrak{C}(I)| = n - i$ we have used only the fact that $|\mathfrak{C}(I)|$ admits a $\mathfrak{C}(I)$-space of positive type. Using a equality $\sum \beta(G) = \frac{1}{2} |\mathfrak{G}|$, we can easily prove that there exists no involution which is not conjugate to I.

PROPOSITION 2. *Let $\mathfrak{G}$ be a doubly transitive permutation group satisfying the condition (c_i) for $i \geq 2$. Then $i = 2$ and $\mathfrak{G}$ is isomorphic to one of the groups $PSL(2, q)$ where q is a power of a certain odd prime, or $\mathfrak{G}$ has a regular normal subgroup.*

Proof. If $\mathfrak{C}(I)$ is an elementary abelian 2-group, then by Lemma 11, $\mathfrak{G}$ is a (CIT)-group (Suzuki [11]). If $\mathfrak{G}$ has a non trivial solvable normal subgroup, then $\mathfrak{G}$ has a regular normal subgroup $\mathfrak{N}$. Assume that $\mathfrak{G}$ has no regular normal subgroup. By Theorem 5 of Suzuki [11] and the main theorem of Suzuki [9], $\mathfrak{G}$ is isomorphic to one of the following groups: $LF(2, 2^\alpha)$, $Sz(2^\beta)$, $PSL(2, q)$, $PSL(3, 4)$ or M_9 (This is a group of order $9 \cdot 8 \cdot 7 = 720$, which is the projective group of one variable over the near-filed of 9 elements; Zassenhaus [14]). Since $\mathfrak{G}$ is a (CIT) group, in the above mentioned groups only $PSL(2, 2^\alpha)$ has elementary abelian 2-Sylow subgroups. If $PSL(2, 4) = PSL(2, 5)$ is considered as permutation group of degree 6, $PSL(2, 5)$ satisfies the condition (c_2). If $2^\alpha > 4$, the group $PSL(2, 2^\alpha)$ does not

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satisfy the condition (c_i) for $i \geq 2$. Therefore $\mathfrak{G} \cong \mathrm{PSL}(2,5)$. Next let us assume that $\mathfrak{C}(I)$ is not an elementary abelian 2-group. By Lemma 10 and by a theorem of N. Iwahori, $\mathfrak{C}(I)$ is a generalized dihedral group with dihedral Sylow 2-subgroups. Since I is a central involution of a certain Sylow 2-subgroup $\mathfrak{T}$ of $\mathfrak{G}$ by Lemma 11, $\mathfrak{T}$ is a dihedral group. Since $\mathfrak{C}(I)$ has a abelian normal 2-complement by a theorem of D. Gorenstein-J. Walter [6], $\mathfrak{G}$ is isomorphic to one of the following groups: $\mathrm{PSL}(2,q)$, $\mathrm{PGL}(2,q)$ where q is a power of an odd prime, or $\mathfrak{A}_7$ the alternating group of degree 7. Here we used the fact that $\mathfrak{G}$ has not a solvable normal subgroup and that a group of odd order is solvable (W. Feit-J. Thompson [4]). On the other hand the group $\mathrm{PGL}(2,q)$ (q is odd) has two conjugate classes consisting of involutions. The group $\mathfrak{A}_7$ does not satisfy (c_i), because $\mathfrak{A}_7$ has one class of involutions and a involution $(12)(34)$ is commutative with $(1324)(56)$. Hence $\mathfrak{G} \cong \mathrm{PSL}(2,q)$ (q is odd).

Combining Proposition 1 and Proposition 2 we have our main theorem stated in the introduction.

Remark. Recently M. Suzuki [12] has proved the following result.

THEOREM. *Let $\mathfrak{G}$ be a finite group. Suppose that $\mathfrak{G}$ contains a subgroup $\mathfrak{H}$ which satisfies the following two conditions:*

(1) *$\mathfrak{H}$ is a generalized dihedral group, and*

(2) *$\mathfrak{H} = \mathfrak{C}_{\mathfrak{G}}(J)$ for any involution J of the center of $\mathfrak{H}$.*

Then, if $\mathfrak{G}$ is not solvable, $\mathfrak{G}$ contains a normal subgroup $\mathfrak{N}$ such that the order of $\mathfrak{N}$ is either odd or twise an odd number, and that $\mathfrak{G}/\mathfrak{N} \cong PSL(2,q)$ or $PGL(2,q)$ for some prime power $q > 3$.

If we use this theorem, our proof in case (ii) become rather short.

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