HILBERT CUSP FORMS AND SPECIAL VALUES OF DIRICHLET SERIES OF RANKIN TYPE

by MIN HO LEE

(Received 31 July, 1996)

1. Introduction. Let K be a totally real number field of degree n over $\mathbb{Q}$, and let $\mathfrak{c}$ be an integral ideal of a maximal order $\mathcal{O}_K$ of K. Given a nonnegative integer j and a Hecke character on the group $\mathbb{A}_K^\times$ of ideles of K, let $\mathcal{S}(\mathfrak{c}, \psi)$ denote the space of Hilbert cusp forms of holomorphic type on $\mathcal{H}^n$ of weight j, level $\mathfrak{c}$ and character ψ, where $\mathcal{H}^n$ is the n-th power of the Poincaré upper half plane $\mathcal{H}$. Let $\mathbf{g}$ be an element of $\mathcal{S}_l(\mathfrak{c}, \mathbf{1})$, where $\mathbf{1}$ is the trivial character. If $\mathbf{u} \in S_k(\mathfrak{c}, \psi)$, then the product $\mathbf{gu}$ is an element of $S_{k+l}(\mathfrak{c}, \psi)$, and therefore we can consider the linear map $\Phi_{\mathbf{g}}: \mathcal{S}_k(\mathfrak{c}, \psi) \to \mathcal{S}_{k+l}(\mathfrak{c}, \psi)$ sending $\mathbf{u}$ to $\mathbf{gu}$. Let $\Phi_{\mathbf{g}}^*: \mathcal{S}_{k+l}(\mathfrak{c}, \psi) \to \mathcal{S}_k(\mathfrak{c}, \psi)$ be the adjoint of the linear map $\Phi_{\mathbf{g}}$ with respect to the Petersson inner product.

In this paper we study the Fourier coefficients of $\Phi_{\mathbf{g}}^*\mathbf{f}$ associated to a Hilbert cusp form $\mathbf{f}$ of holomorphic type in $\mathcal{S}_{k+l}(\mathfrak{c}, \psi)$. We define Dirichlet series of Rankin type associated to the Fourier coefficients of $\mathbf{g}$ and $\mathbf{f}$ and express the Fourier coefficients of $\Phi_{\mathbf{g}}^*\mathbf{f}$ in terms of special values of such Dirichlet series. Such a problem was treated by Kohnen [**1**] in the case of elliptic modular forms. In order to consider the Hilbert modular case we use holomorphic projection operators and Poincaré series of two variables used by Panchishkin [**3**] to prove the algebraicity of a certain expression.

2. Hilbert automorphic forms. In this section we review Hilbert automorphic forms using the language of adeles (see e.g. [**2**], [**3**] for details). Let K be a totally real number field of degree n over $\mathbb{Q}$, $\mathbb{A}_K$ its ring of adeles, $\mathcal{O}_K$ a maximal order, and I_K the group of fractional ideals of K. Let G be an algebraic group over $\mathbb{Q}$ such that $G(\mathbb{Q}) = GL(2, K)$. If $\hat{\mathcal{O}}_K$ is the profinite completion of $\mathcal{O}_K$, then $\hat{K} \cong \hat{\mathcal{O}}_K \times_{\mathbb{Z}} \mathbb{Q}$ is the subring of $\mathbb{A}_K$ of finite adeles and we have $\mathbb{A}_K = K_\infty \times \hat{K}$, where K_∞ is the subring of $\mathbb{A}_K$ of adeles at infinity. If $\mathbb{A}$ is the ring of adeles of $\mathbb{Q}$, then we have

$$G(\mathbb{A}) = GL(2, \mathbb{A}_K) = G_\infty \times GL(2, \hat{K}),$$

where $G_\infty = GL(2, K_\infty)$. Since K is totally real we can identify $G_\infty = GL(2, K_\infty)$ with $GL(2, \mathbb{R})^n$. Under this identification, let G_∞^+ be the subgroup of $GL(2, \mathbb{R})^n$ consisting of elements $\alpha = (\alpha_1, \ldots, \alpha_n)$ with

$$\alpha_j = \begin{pmatrix} a_j & b_j \\ c_j & d_j \end{pmatrix} \in GL(2, \mathbb{R}), \qquad \det \alpha_j > 0$$

for all $j = 1, \ldots, n$. Let $\mathcal{H}^n$ be the n-fold power of the Poincaré upper half plane

$$\mathcal{H} = \{z \in \mathbb{C} \mid \operatorname{Im} z > 0\}.$$

Then each element $\alpha \in G_\infty^+$ acts on $\mathcal{H}^n$ by

$$\alpha \, . \, z = (\alpha_1 z_1, \ldots, \alpha_n z_n) \in \mathcal{H}^n$$

https://doi.org/10.1017/S0017089500032365 Published online by Cambridge University Press

for $z = (z_1, \ldots, z_n) \in \mathcal{H}^n$, where

$$\alpha_j z_j = (a_j z_j + b_j)(c_j z_j + d_j)^{-1} \in \mathcal{H}, \qquad j = 1, \ldots, n.$$

For $k \in \mathbb{Z}$ and $z = (z_1, \ldots, z_n) \in \mathcal{H}^n$, we set

$$\mathbf{e}(z) = e^{2\pi i(z_1 + \ldots + z_n)}, \qquad \mathcal{N}(z)^k = z_1^k \ldots z_n^k.$$

Given a function $f: \mathcal{H}^n \to \mathbb{C}$ and $\alpha \in G_\infty^+$ we define the function $f|_k\alpha: \mathcal{H}^n \to \mathbb{C}$ by

$$(f|_k\alpha)(z) = \mathcal{N}(cz + d)^{-k} f(\alpha z) \mathcal{N}(\det \alpha)^{k/2}$$

for all $z \in \mathcal{H}^n$, where $cz + d = (c_1 z_1 + d_1, \ldots, c_n z_n + d_n) \in \mathcal{H}^n$.

Let $\mathfrak{c} \subset \mathcal{O}_K$ be an integral ideal, and for each place $\mathfrak{p}$ of K let $\mathfrak{c}_\mathfrak{p} = \mathfrak{c}\mathcal{O}_\mathfrak{p}$ be its $\mathfrak{p}$-part. Let $\mathfrak{d}$ be the different of K, and let $\mathfrak{d}_\mathfrak{p} = \mathfrak{d}\mathcal{O}_\mathfrak{p}$ be the associated local different at $\mathfrak{p}$. We define the open subgroup $W = W(\mathfrak{c})$ of $G(\mathbb{A})$ by

$$W = G_\infty^+ \times \prod_\mathfrak{p} W(\mathfrak{p}),$$

where

$$W(\mathfrak{p}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL(2, K_\mathfrak{p}) \mid b \in \mathfrak{d}_\mathfrak{p}^{-1}, c \in \mathfrak{d}_\mathfrak{p}\mathfrak{c}_\mathfrak{p}, a, d \in \mathcal{O}_\mathfrak{p}, ad - bc \in \mathcal{O}_\mathfrak{p}^\times \right\}.$$

If $K_+^\times$ denotes the multiplicative group of all totally positive elements of K, then the quotient $I_K/K_+^\times$ is the ideal class group of K. Let $h = |I_K/K_+^\times|$ be the class number of K, and let $\{t_1, \ldots, t_h\}$ be the set of ideles such that their images $\tilde{t}_\nu$ in $\mathcal{O}_K$ form a complete system of representatives for $I_K/K_+^\times$ and

$$(t_\nu)_\infty = 1, \qquad \tilde{t}_\nu + \mathfrak{m}_0 = \mathcal{O}_K, \qquad \mathfrak{m}_0 = \prod_{\mathfrak{q} \in S_K} \mathfrak{q}$$

for $1 \le \nu \le h$, where S is a finite set of primes and S_K is the set of primes $\mathfrak{p}$ dividing each element of S. Then we have

$$G(\mathbb{A}) = \bigcup_{\nu=1}^h G(\mathbb{Q}) x_\nu W = \bigcup_{\nu=1}^h G(\mathbb{Q}) x_\nu^{-\iota} W,$$

where $x_\nu = \begin{pmatrix} 1 & 0 \\ 0 & t_\nu \end{pmatrix}$ and ι denotes the involution of 2×2 matrices given by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^\iota = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

Definition 2.1. Let $\mathfrak{c} \subset \mathcal{O}_K$ be an integral ideal as above, and let $\psi: A_K^\times \to \mathbb{C}^\times$ be a Hecke character. A *Hilbert automorphic form* of weight k, level $\mathfrak{c}$, and character ψ is a function $\mathbf{f}: G(\mathbb{A}) \to \mathbb{C}$ satisfying the following conditions:

(i) $\mathbf{f}(s\alpha x) = \psi(s)\mathbf{f}(x)$ for all $x \in G(\mathbb{A})$, $s \in \mathbb{A}_K^\times$ and $\alpha \in G(\mathbb{Q})$;

(ii) $\mathbf{f}(xw) = \psi(w^\iota)\mathbf{f}(x)$ for $w \in W$ with $w_\infty = 1$;

(iii) $\mathbf{f}(xw(\theta)) = \mathbf{f}(x) e^{-ik(\theta_1 + \ldots + \theta_n)}$ for all $x \in G(\mathbb{A})$ and

$$w(\theta) = (w_1(\theta_1), \ldots, w_n(\theta_n)) \in W,$$

where

$$w_j(\theta_j) = \begin{pmatrix} \cos\theta_j & -\sin\theta_j \\ \sin\theta_j & \cos\theta_j \end{pmatrix}, \qquad j = 1, \ldots, n.$$

A *Hilbert cusp form* is a Hilbert automorphic form satisfying the additional condition that

$$\int_{\mathbb{A}_K/K} \mathbf{f}\left(\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} g\right) dt = 0$$

for all $g \in G(\mathbb{A})$.

DEFINITION 2.2. A *Hilbert automorphic* (resp. *cusp*) *form of holomorphic type* is a Hilbert automorphic (resp. cusp) form such that for any $x \in G(\mathbb{A})$ with $x_\infty = 1$ there exists a holomorphic function $g_x : \mathcal{H}^n \to \mathbb{C}$ with $\mathbf{f}(xy) = (g_x|_k y)(\mathbf{i})$ for all $y \in G_\infty^+$, where $\mathbf{i} = (i, \ldots, i) \in \mathcal{H}^n$. We denote by $\mathcal{M}_k(\mathfrak{c}, \psi)$ (resp. $\mathcal{S}_k(\mathfrak{c}, \psi)$) the space of such Hilbert automorphic (resp. cusp) forms of weight k, level $\mathfrak{c}$ and character ψ of holomorphic type.

For $\mathbf{f} \in \mathcal{M}_k(\mathfrak{c}, \psi)$ and $1 \leq \nu \leq h$, set $f_\nu = g x_\nu^{-\iota}$, and let

$$\begin{aligned} \Gamma_\nu = \Gamma_\nu(\mathfrak{c}) &= x_\nu W x_\nu^{-1} \cap G(\mathbb{Q}) \\ &= \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G^+(\mathbb{Q}) \mid b \in \tilde{t}_\nu^{-1}\mathfrak{d}^{-1}, c \in \tilde{t}_\nu \mathfrak{d}\mathfrak{c}, a, d \in \mathcal{O}_K, ad - bc \in \mathcal{O}_K^\times \right\}. \end{aligned}$$

Then f_ν is a Hilbert modular form of weight k and character ψ for the congruence subgroup Γ_ν. Thus it satisfies $f_\nu|_k \gamma = \psi(\gamma) f_\nu$, and has a Fourier expansion of the form

$$f_\nu(z) = \sum_\xi a_\nu(\xi)\mathbf{e}(\xi z),$$

where $0 < \xi \in \tilde{t}_\nu$ or $\xi = 0$. If $\mathcal{M}_k(\Gamma_\nu, \psi)$ denotes the space of Hilbert modular forms of weight k and character ψ for Γ_ν, then the map $\mathbf{f} \mapsto (f_1, \ldots, f_h)$ determines a canonical isomorphism

$$\mathcal{M}_k(\mathfrak{c}, \psi) \cong \bigoplus_{\nu=1}^{h} \mathcal{M}_k(\Gamma_\nu, \psi).$$

We shall identify $\mathcal{M}_k(\mathfrak{c}, \psi)$ with $\bigoplus_{\nu=1}^{h} \mathcal{M}_k(\Gamma_\nu, \psi)$ so that $\mathbf{f} = (f_1, \ldots, f_h)$. Similarly, we have

$$\mathcal{S}_k(\mathfrak{c}, \psi) = \bigoplus_{\nu=1}^{h} \mathcal{S}_k(\Gamma_\nu, \psi), \qquad \mathcal{S}_k(\Gamma_\nu, \psi) = \mathcal{S}_k(\mathfrak{c}, \psi) \cap \mathcal{M}_k(\Gamma_\nu, \psi).$$

If $\mathbf{f} = (f_1, \ldots, f_h) \in \mathcal{S}_k(\mathfrak{c}, \psi)$ and $\mathbf{g} = (g_1, \ldots, g_h) \in \mathcal{M}_k(\mathfrak{c}, \psi)$, then the Petersson inner product is defined by

$$\langle \mathbf{f}, \mathbf{g} \rangle = \sum_{\nu=1}^{h} \int_{\Gamma_\nu \backslash \mathcal{H}^n} \overline{f_\nu(z)}\, g_\nu(z) \mathcal{N}(y)^k \, d\mu(z), \tag{1}$$

where $z = (z_1, \ldots, z_n)$, $z_j = x_j + iy_j$ $(1 \leq j \leq n)$, and

$$d\mu(z) = \prod_{j=1}^{n} y_j^{-2} \, dx_j \, dy_j$$

is a G_∞^+-invariant measure on $\mathcal{H}^n$.

https://doi.org/10.1017/S0017089500032365 Published online by Cambridge University Press

DEFINITION 2.3. *A C^∞ Hilbert automorphic form* of weight k, level $\mathfrak{c}$, and character ψ is a function $F: G(\mathbb{A}) \to \mathbb{C}$ satisfying the following conditions:

(i) F satisfies the conditions (i), (ii) and (iii) in Definition 2.1.

(ii) For each $x \in G(\mathbb{A})$ with $x_\infty = 1$ there exists a C^∞ function $g_x : \mathscr{H}^n \to \mathbb{C}$ with $F(xy) = (g_x|_k y)(\mathbf{i})$ for all $y \in G_\infty$.

We shall denote by $\tilde{\mathscr{M}}_k(\mathfrak{c}, \psi)$ the space of all such C^∞ Hilbert automorphic forms.

For $1 \le \nu \le h$, let $F_\nu = g_{x_\nu^{-\iota}}$. Then F_ν is a C^∞ Hilbert modular form on $\mathscr{H}^n$ of weight k and character ψ relative to the congruence subgroup Γ_ν. In particular, it satisfies $(F_\nu|_k \gamma) = \psi(\gamma) F_\nu$ for each $\gamma \in \Gamma_\nu$ and has a Fourier expansion of the form

$$F_\nu(z) = \sum_{\xi \in \tilde{t}_\nu} a_\nu(\xi, y) \mathbf{e}(\xi x),$$

where the maps $y \mapsto a_\nu(\xi, y)$ are C^∞ functions on

$$(\mathbb{R}_+)^n = \{y = (y_1, \ldots, y_n) \mid y_j > 0 \text{ for all } j\}.$$

If $\tilde{\mathscr{M}}_k(\Gamma_\nu, \psi)$ denotes the space of all C^∞ Hilbert modular forms of weight k and character ψ for Γ_ν, then we have

$$\tilde{\mathscr{M}}_k(\mathfrak{c}, \psi) \cong \bigoplus_{\nu=1}^{h} \tilde{\mathscr{M}}_k(\Gamma_\nu, \psi).$$

As in the case of $\mathscr{M}_k(\mathfrak{c}, \psi)$ and $\mathscr{S}_k(\mathfrak{c}, \psi)$, we shall identity $F \in \tilde{\mathscr{M}}_k(\mathfrak{c}, \psi)$ with its image $(F_1, \ldots, F_h)$ in $\bigoplus_{\nu=1}^{h} \tilde{\mathscr{M}}_k(\Gamma_\nu, \psi)$ under this isomorphism. The Petersson inner product in (1) can be extended to elements $\mathbf{f} \in \mathscr{S}_k(\mathfrak{c}, \psi)$ and $F \in \tilde{\mathscr{M}}_k(\mathfrak{c}, \psi)$.

THEOREM 2.4. *Let $F = (F_1, \ldots, F_h) \in \tilde{\mathscr{M}}_k(\mathfrak{c}, \psi)$ be a C^∞ Hilbert automorphic form of moderate growth such that for each ν the Fourier expansion*

$$F_\nu(z) = \sum_{\xi \in \tilde{t}_\nu} a_\nu(\xi, y) \mathbf{e}(\xi x)$$

contains only terms with totally positive $\xi \in \tilde{t}_\nu$. For $1 \le \nu \le h$, we set

$$a_\nu(\xi) = \frac{(4\pi)^{n(k-1)} (\mathcal{N}\xi)^{k-1}}{\Gamma(k-1)^n} \int_{(\mathbb{R}_+)^n} a_\nu(\xi, y) \mathbf{e}(i\xi y) y^{k-2}\, dy, \tag{2}$$

where $\Gamma(k-1)$ is the value of the gamma function

$$\Gamma(s) = \int_0^\infty e^{-y} y^{s-1}\, dy$$

at $s = k-1$. If $F^H = (F_1^H, \ldots, F_h^H)$ with

$$F_\nu^H(z) = \sum_{0 < \xi \in \tilde{t}_\nu} a_\nu(\xi) \mathbf{e}(\xi z)$$

for each ν, then F^H is a Hilbert automorphic form of holomorphic type in $\mathscr{M}_k(\mathfrak{c}, \psi)$ and

$$\langle g, F \rangle = \langle g, F^H \rangle \tag{3}$$

for all $g \in \mathscr{S}_k(\mathfrak{c}, \psi)$.

Proof. See Proposition 4.7 in [**3**, Chapter 4]. □

3. Fourier coefficients. Let $\mathbf{g} = (g_1, \ldots, g_h)$ be an element of $\mathcal{S}_l(\mathfrak{c}) = \mathcal{S}_l(\mathfrak{c}, \mathbf{1}) = \bigoplus_{\nu=1}^{h} \mathcal{S}_\lambda(\Gamma_\nu, \mathbf{1})$, where $\mathbf{1}$ denotes the trivial character, and let $\mathbf{u} = (u_1, \ldots, u_h) \in \mathcal{S}_k(\mathfrak{c}, \psi) = \bigoplus_{\nu=1}^{h} \mathcal{S}_k(\Gamma_\nu, \psi)$. Then the product $\mathbf{gu}$ is an element of $\mathcal{S}_{k+l}(\mathfrak{c}, \psi)$. Thus we can consider the linear map $\Phi_{\mathbf{g}}: \mathcal{S}_k(\mathfrak{c}, \psi) \to \mathcal{S}_{k+l}(\mathfrak{c}, \psi)$ sending $\mathbf{u} \in \mathcal{S}_k(\mathfrak{c}, \psi)$ to $\mathbf{gu} \in \mathcal{S}_{k+l}(\mathfrak{c}, \psi)$. Let

$$\Phi_{\mathbf{g}}^*: \mathcal{S}_{k+l}(\mathfrak{c}, \psi) \to \mathcal{S}_k(\mathfrak{c}, \psi)$$

be the adjoint of the linear map $\Phi_{\mathbf{g}}$ with respect to the Petersson inner product. For $1 \le \nu \le h$ let $P_{k,\nu}^{\psi}$ be the Poincaré series of two variables given by

$$P_{k,\nu}^{\psi}(z, w, s) = \sum_{\gamma \in \Gamma_\nu} \psi(\gamma) J(\gamma, z)^{-k} |J(\gamma, z)|^{-2s} (\gamma s + s)^k |\gamma s + s|^{-2s}, \tag{4}$$

for $z, w \in \mathcal{H}^n$, where $J(\gamma, z) = \det \gamma^{-1/2}(cz + d)$ for $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_\nu$. The series in (4) converges absolutely and uniformly on any compact subset of $\mathcal{H}^n \times \mathcal{H}^n$ for $k + \mathrm{Re}(2s) > 2$ (see [**3**, p. 139]).

LEMMA 3.1. *If $g_\nu \in \mathcal{S}_l(\Gamma_\nu, \mathbf{1})$ and $f_\nu \in \mathcal{S}_{k+l}(\Gamma_\nu, \psi)$, then the function*

$$f_\nu(z)\overline{g_\nu(z)}\mathcal{N}(\mathrm{Im}\, z)^l$$

is a C^∞ Hilbert modular form in $\tilde{\mathcal{M}}_k(\Gamma_\nu, \psi)$ of moderate growth.

Proof. For each $\gamma \in \Gamma_\nu$ we have

$$f_\nu(\gamma z) = \psi(\gamma) J(\gamma, z)^{k+l} f_\nu(z), \qquad \overline{g_\nu(\gamma z)} = \overline{J(\gamma, z)^l}\, \overline{g_\nu(z)}.$$

Since $\mathcal{N}(\mathrm{Im}\, \gamma z)^l = |J(\gamma, z)|^{-2l} \mathcal{N}(\mathrm{Im}\, z)^l$, it follows that

$$f_\nu(\gamma z)\overline{g_\nu(\gamma z)}\mathcal{N}(\mathrm{Im}\, \gamma z)^l = \psi(\gamma) J(\gamma, z)^k f_\nu(z)\overline{g_\nu(z)}\mathcal{N}(\mathrm{Im}\, z)^l.$$

Furthermore, from the cusp conditions for f_ν and g_ν it follows that $f(z)\overline{g(z)}\mathcal{N}(\mathrm{Im}\, z)^l$ is of moderate growth. □

PROPOSITION 3.2. *Let $\mathbf{g} \in \mathcal{S}_l(\mathfrak{c}) = \mathcal{S}_l(\mathfrak{c}, \mathbf{1})$, $\mathbf{f} \in \mathcal{S}_{k+l}(\mathfrak{c}, \psi)$, and let $F^H \in \mathcal{S}_k(\mathfrak{c}, \psi)$ be the Hilbert automorphic form of holomorphic type associated to the C^∞ Hilbert automorphic form*

$$F(z) = \mathbf{f}(z)\overline{\mathbf{g}(z)}\mathcal{N}(\mathrm{Im}\, z)^l.$$

Then we have $\Phi_{\mathbf{g}}^*(\mathbf{f}) = F^H$.

Proof. Let $\mathbf{f} = (f_1, \ldots, f_h) \in \mathcal{S}_{k+l}(\mathfrak{c}, \psi)$, $\mathbf{g} = (g_1, \ldots, g_h) \in \mathcal{S}_l(\mathfrak{c})$, and let $P_{k,\nu}^{\psi}$ be the Poincaré series given by (4). Then for $1 \le \nu \le h$ we have

$$\begin{aligned}
\langle P_{k,\nu}^{\psi}(-\bar{z}, w, s), (\Phi_{\mathbf{g}}^* f_\nu)(w)\rangle &= \langle \Phi_{\mathbf{g}} P_{k,\nu}^{\psi}(-\bar{z}, w, s), f_\nu(w)\rangle \\
&= \langle g_\nu(w) P_{k,\nu}^{\psi}(-\bar{z}, w, s), f_\nu(w)\rangle \\
&= \int_{D_\nu} \overline{g_\nu(w) P_{k,\nu}^{\psi}(-\bar{z}, w, s)} f_\nu(w) \mathcal{N}(v)^{k+l}\, d\mu(w) \\
&= \int_{D_\nu} \overline{P_{k,\nu}^{\psi}(-\bar{z}, w, s)} (f_\nu(w)\overline{g_\nu(w)}\mathcal{N}(v)^l) \mathcal{N}(v)^k\, d\mu(w) \\
&= \langle P_{k,\nu}^{\psi}(-\bar{z}, w, s), F_\nu(w)\rangle,
\end{aligned}$$

where $w=(w_1,\ldots,w_n)$, $w_j=u_j+iv_j$ and $d\mu(w)=\prod_{j=1}^{n} v_j^{-2}\,du_j\,dv_j$. If $F^H=(F_1^H,\ldots,F_h^H)$, then by (3) we obtain

$$\langle P_{k,v}^{\psi}(-\bar{z},w,s),\,(\Phi_{\mathbf{g}}^{*}f_v)(w)\rangle=\langle P_{k,v}^{\psi}(-\bar{z},w,s),\,F_v^H(w)\rangle.$$

Thus it follows from [**3**, p. 139] that

$$c(k,s)\Phi_{\mathbf{g}}^{*}f_v(z)=c(k,s)F_v^H(z)$$

for $1\le v\le h$, where

$$c(k,s)=2^{n(2-s-k)}i^{-nk}\pi^n(k+s-1)^{-n}.$$

Hence we have

$$\Phi_{\mathbf{g}}^{*}f_v(z)=F_v^H(z)$$

for all v, and therefore the proposition follows. □

By Proposition 3.2, given $\mathbf{f}\in\mathcal{S}_{k+l}(\mathfrak{c},\psi)$, we have

$$\Phi_{\mathbf{g}}^{*}\mathbf{f}=F^H=(F_1^H,\ldots,F_h^H)\in\mathcal{S}_k(\mathfrak{c},\psi),$$

and each $F_v^H\in\mathcal{S}_k(\Gamma_v,\psi)$ has a Fourier expansion of the form

$$F_v^H(z)=\sum_{0<\xi\in\tilde{t}_v}a_v(\xi)\mathbf{e}(\xi z). \tag{5}$$

Let $\mathbf{f}=(f_1,\ldots,f_h)$ and $\mathbf{g}=(g_1,\ldots,g_h)$ be as in Proposition 3.2 with

$$f_v(z)=\sum_{0<\xi\in\tilde{t}_v}A_v(\xi)\mathbf{e}(\xi x),\qquad g_v(z)=\sum_{0<\xi\in\tilde{t}_v}B_v(\xi)\mathbf{e}(\xi x) \tag{6}$$

for $1\le v\le h$. Then for $\xi\in\tilde{t}_v$ we define the Dirichlet series $L_v(\mathbf{f},\mathbf{g},\xi,s)$ of Rankin type by

$$L_v(\mathbf{f},\mathbf{g},\xi,s)=\sum_{0<\eta\in\tilde{t}_v}\frac{A_v(\xi+\eta)\overline{B_v(\eta)}}{\mathcal{N}(\xi+\eta)^s}. \tag{7}$$

THEOREM 3.3. *Let $\mathbf{f}\in\mathcal{S}_{k+l}(\mathfrak{c},\psi)$ and $\mathbf{g}\in\mathcal{S}_l(\mathfrak{c})$ be as above, and let $a_v(\xi)$ be the Fourier coefficients of the component F_v^H of $\Phi_{\mathbf{g}}^{*}\mathbf{f}$ given in* (5). *Then we have*

$$a_v(\xi)=\frac{\Gamma(k+l-1)^n(\mathcal{N}\xi)^{k-1}}{(4\pi)^{nl}\Gamma(k-1)^n}L_v(\mathbf{f},\mathbf{g},\xi,k+l-1)$$

for $1\le v\le h$ and $0<\xi\in\tilde{t}_v$.

Proof. By (2) we have

$$a_v(\xi)=C\int_{(\mathbb{R}_+)^n}a_v(\xi,y)\mathbf{e}(i\xi y)y^{k-2}\,dy,$$

where

$$C = \frac{(4\pi)^{n(k-1)}(\mathcal{N}\xi)^{k-1}}{\Gamma(k-1)^n}. \tag{8}$$

Since $F_\nu(z) = f_\nu(z)\overline{g_\nu(z)}\mathcal{N}(\operatorname{Im} z)^l$, if A_ν and B_ν are as in (6), we have

$$\begin{aligned} F_\nu(z) &= \sum_\nu a_\nu(\xi, y)\mathbf{e}(\xi x) \\ &= \sum_{\xi,\eta} A_\nu(\mu)\overline{B_\nu(\eta)}\mathbf{e}((\mu-\eta)x)\mathbf{e}(i(\mu+\eta)y)\mathcal{N}y^l. \end{aligned}$$

Using $\xi = \mu - \eta$ or $\mu = \xi + \eta$ we obtain

$$a_\nu(\xi, y) = \sum_\eta A_\nu(\xi+\eta)\overline{B_\nu(\eta)}\mathbf{e}(i(\xi+2\eta)y)\mathcal{N}y^l.$$

Hence it follows that

$$\begin{aligned} a_\nu(\xi) &= C\int_{(\mathbb{R}_+)^n}\sum_\eta A_\nu(\xi+\eta)\overline{B_\nu(\eta)}\mathbf{e}(2i(\xi+\eta)y)\mathcal{N}y^{k+l-2}\,dy \\ &= C\sum_\eta A_\nu(\xi+\eta)\overline{B_\nu(\eta)}\int_{(\mathbb{R}_+)^n} e^{-4\pi\sum_{j=1}^n(\xi_j+\eta_j)y_j}\mathcal{N}y^{k+l-2}\,dy \\ &= C\sum_\eta A_\nu(\xi+\eta)\overline{B_\nu(\eta)}\prod_{j=1}^n\int_0^\infty e^{-4\pi(\xi_0+\eta_j)y_j}y_j^{k+l-2}\,dy_j. \end{aligned}$$

Thus, using $v_j = 4\pi(\xi_j+\eta_j)y_j$ for $1 \le j \le n$, we have

$$\begin{aligned} a_\nu(\xi) &= C\sum_\eta A_\nu(\xi+\eta)\overline{B_\nu(\eta)}\prod_{j=1}^n\left(\int_0^\infty e^{-v_j}v_j^{k+l-2}\,dv_j\right)\prod_{j=1}^n(4\pi(\xi_j+\eta_j))^{-k-l+1} \\ &= C\sum_\eta A_\nu(\xi+\eta)\overline{B_\nu(\eta)}\Gamma(k+l-1)^n(4\pi)^{n(-k-l+1)}\mathcal{N}(\xi+\eta)^{-k-l+1}. \end{aligned}$$

Hence the theorem follows from this and the relations (7) and (8). □

REFERENCES

1. W. Kohnen, Cusp forms and special values of certain Dirichlet series, *Math. Z.* **207** (1991), 657–660.

2. Y. Manin, Non-archimedean integration and Jacquet-Langlands p-adic L-functions, *Russian Math. Surveys* **31** (1976), 5–57.

3. A. Panchishkin, *Non-archimedean L-functions of Siegel and Hilbert modular forms*, Springer-Verlag, New York, 1991.

DEPARTMENT OF MATHEMATICS
UNIVERSITY OF NORTHERN IOWA
CEDAR FALLS, IOWA 50614
U.S.A.
e-mail: lee@math.uni.edu