ON THE NORMAL CORES OF CERTAIN SUBGROUPS OF NILPOTENT GROUPS

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Let G be a group and H a subgroup of finite index in G. Then of course H contains a G-invariant subgroup C such that G/C is finite. In attempting to establish results of a similar nature, where "finite" is replaced by, for example, "finitely generated", one notices immediately that a quite differently stated hypothesis is required. One reasonable approach would be to consider subgroups H which are "f.g. embedded" in G—indeed, the notion of a polycyclic embedding was utilised by P. Hall in [**1**].

Definition. Let H be a subgroup of the group G and let $\mathfrak{X}$ be a class of groups. Then H has a *subnormal $\mathfrak{X}$-embedding* in G, denoted $H\,\mathrm{sn}(\mathfrak{X})G$, if there is a (finite) subnormal series from H to G in which successive factors belong to $\mathfrak{X}$.

For which classes, then, does $H\,\mathrm{sn}(\mathfrak{X})G$ imply that $H/\mathrm{Core}_G\,H \in \mathfrak{X}$? As it stands, this seems too general a question, if only because the (restricted) wreath product G of two infinite cyclic groups contains a core-free, abelian subgroup H of infinite rank such that $H\,\mathrm{sn}(\mathfrak{X})G$, where $\mathfrak{X}$ is the class of polycyclic groups, say. One may hope for some progress if attention is restricted to nilpotent groups. The results presented here are not at all difficult to prove but have possibly escaped notice until now. The classes considered are those of finitely generated groups, groups with finite (Prüfer) rank and π-minimax groups, where π is some finite set of primes. We shall denote these classes by f.g., f.r. and $\mathfrak{M}_\pi$ respectively. Recall that a soluble group G is π-minimax if it has a finite subnormal series of subgroups whose factor groups are abelian and satisfy either min or max, where the set of all primes p such that C_{p^∞} appears in some factor is contained in π.
The following is proved.

Theorem. *Let G be a nilpotent group and H a subgroup of G, and let $C = \mathrm{Core}_G\,H$.*
(1) *If $H\,\mathrm{sn}(\mathrm{f.g.})G$ then G/C is* f.g.
(2) *If $H\,\mathrm{sn}(\mathrm{f.r.})G$ and G is periodic then G/C is* f.r.
(3) *If $H\,\mathrm{sn}(\mathrm{f.r.})G$ and H is isolated in G then G/C is* f.r. (*and torsionfree*).
(4) *If $H\,\mathrm{sn}(\mathfrak{M}_\pi)G$ and H is π-isolated in G then $G/C \in \mathfrak{M}_\pi$ (and is π-free).*

It will be evident from the proof of the theorem that the number of generators (resp. the rank) of G/C is bouned by a function of the nilpotency class of G and the number of generators (resp. the rank) of the factor groups arising from the subnormal series from H to G. The following two examples show that the hypothesis that H is isolated (or π-isolated) cannot be omitted from part (3) (or (4)), even in the case where G is torsionfree.

Example 1. Let $\pi = \{p_1, p_2, \ldots\}$ be an infinite set of primes. For each $i = 1, 2, \ldots$ let A_i, B_i be (multiplicative) groups, each isomorphic to the (additive) group $\mathbb{Q}_{p_i}$ of p_i-adic rationals and let G_i denote the second nilpotent product of A_i and B_i. Then the centre Z_i of G_i is $[A_i, B_i]$, which is isomorphic to $A_i \otimes_{\mathbb{Z}} B_i$ and hence to A_i. Choose nontrivial

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elements $a_i \in A_i$, $b_i \in B_i$ and let $z_i = [a_i, b_i]$. Now define G to be the direct product of the G_i, $i = 1, 2, \ldots$, and let A, B, Z be defined similarly. Write $A^* = \mathrm{Dr}\langle a_i \rangle$, $B^* = \mathrm{Dr}\langle b_i \rangle$, $Z^* = \mathrm{Dr}\langle z_i \rangle$, the products being over all i, and let $H = \langle A^*, B^* \rangle$ $(= \langle A^*, B^*, Z^* \rangle)$. Then HZ/H has rank 1 and G/HZ has rank 2 and so H sn(f.r.)G. Also, G is torsionfree (and nilpotent of class 2). Suppose $x = abz \in \mathrm{Core}_G H = C$, where $a \in A$, $b \in B$, $z \in Z$. Then $[x, A] \leqslant C$ and so $[b, A] \leqslant H$. But if $b \neq 1$ then $[b, A_i] \cong A_i$ for some i, and we have a contradiction. So $b = 1$ and, similarly, $a = 1$ and we have $C \leqslant Z$. But $H \cap Z = Z^*$ and $H/H \cap Z$ has infinite rank.

Example 2. Let p be a prime and let A, B be isomorphic to $\mathbb{Q}_p$, $\mathbb{Q}$ respectively. Let G denote the second nilpotent product of A and B and write $Z = [A, B]$. Thus $Z \cong \mathbb{Q}$. Next, let B^* be a subgroup of B such that $B^* \cong \mathbb{Z}_p$ (the group of all rationals with denominators prime to p) and write $H = \langle a, B^* \rangle$, where a is some nontrivial element of A. Then we have $H = \langle a, B^*, Z^* \rangle$, where $\mathbb{Z}_p \cong Z^* \leqslant Z$. Further, $H \lhd HZ \lhd G$ and $HZ/H \cong C_{p^\infty}$, while G/HZ is isomorphic to $C_{p^\infty} \times C_{p^\infty}$. Thus H sn$(\mathfrak{M}_\pi)G$, where $\pi = \{p\}$ and, of course, G is nilpotent.

As above, suppose $x = abz \in C = \mathrm{Core}_G H$. If $b \neq 1$ then $\mathbb{Q}_p \cong [b, A] \leqslant H$, a contradiction. If $a \neq 1$ then $\mathbb{Q} \cong [a, B] \leqslant H$, another contradiction. Thus $C = H \cap Z = Z^*$ and so $H/C \cong \mathbb{Z} \times \mathbb{Z}_p \notin \mathfrak{M}_\pi$.

It is immediate from part (2) of the theorem that if G is periodic nilpotent and H sn$(\mathfrak{X})G$, where $\mathfrak{X}$ is the class of groups having finite p-rank for all primes p, then $H/\mathrm{Core}_G H \in \mathfrak{X}$. It is also clear that "f.r." may be replaced by "min" or "minimax" in (2). (For periodic nilpotent groups, "min" and "minimax" are of course equivalent.) Example 2 again shows that the situation is quite different for G nonperiodic (even torsionfree).

Proof of the theorem. We begin by proving (3). Much of the reduction required in the other cases is similar and may then be omitted. The relevant facts about isolators are to be found in [**2**]. Suppose that the hypotheses of (3) are satisfied and note firstly that if X is an isolated subgroup of G then $Y = \mathrm{Core}_G X$ is also isolated, since the isolator of Y is normal in G and contained in X. We write $I_G(X)$ for the isolator of X in G.

Suppose the theorem false and let G be a counterexample of minimal nilpotency class with H a subgroup whose "normaliser series" in G has minimal length subject to G/C being of infinite rank. We may assume H is core-free in G and thus that G is torsionfree. Since every series from H to G has its factors of finite rank and since normalisers of isolated subgroups are isolated, the inductive hypothesis gives us an isolated normal subgroup K of G, containing both H and the centre Z of G, such that H/D has finite rank, where $D = \mathrm{Core}_K H$. We may thus assume $H \lhd K \lhd G$.

Now let $I = I_G(HZ)$ and let $N/Z = \mathrm{Core}_{G/Z}(I/Z)$. By the inductive hypothesis, G/N has finite rank (and is torsionfree). Also $N \leqslant I \leqslant K$ and thus N normalises H. Now $N/H \cap N$ is f.r. and $H \cap N$ is isolated in G. Writing "$X \sim Y$" to denote that $I_G(X) = I_G(Y)$ we have $HZ \sim I$ and hence $H' = (HZ)' \sim I'$. Since H is isolated it follows that $N' \leqslant I' \leqslant H$ and thus $N' = 1$ and N is abelian. Replacing H by $H \cap N$, we may assume $H \leqslant N$. Also, since $H \cap Z = 1$ we have Z of finite rank. Next, for each non-negative integer n and for each subgroup X of N, let $X_n = [X, {}_nG]$. Suppose t is maximal such that N_t has infinite rank and write $M = I_G(N_t)$. Then $M \leqslant N$ and $M \lhd G$, while $[M, G]$ has finite rank (since $[M, G] \sim N_{t+1}$ and G is torsionfree).

There exists a f.g. subgroup $G_0=\langle g_1,\ldots,g_k\rangle$ of G such that $NG_0\sim G$. For each $i=1,\ldots,k$, let J_i be the kernel of the homomorphism from M to M defined by $a\to[a,g_i]$ for all $a\in M$ and let $J=\bigcap_{i=1}^{k}J_i$. Then M/J has finite rank and so J has infinite rank. But $[J,NG_0]=1$ and, since centralisers are isolated in G, we have $J=Z$, contradicting the fact that Z has finite rank. Thus (3) is proved.

In order to prove (4) we need to consdier π-isolators instead of "absolute" isolators. The argument is much the same (except for obvious amendments) up to the point where we define t to be maximal such that $N_t=[N,{}_tG]\notin\mathfrak{M}_\pi$. Now H is π-isolated and core-free, and so G is π-free. Since $[M,G]\overset{\pi}{\sim}N_{t+1}$, we thus have $[M,G]\in\mathfrak{M}_\pi$. The proof proceeds as for (3) (using, at the appropriate stage, the fact that centralisers are π-isolated in G).

For the proof of (1) and (2), let N be the core of HZ in G and again assume that H is core-free. Since $N=(H\cap N)Z$ we quickly reduce to the case where $N=HZ$, which is abelian. The definition of $M=N_t$ is the obvious one. For (1), write $G=NG_0$, where $G_0=\langle g_1,\ldots,g_k\rangle$ and obtain the contradiction that Z is infinitely generated.

In case (2), we may write $G=\bigcup_{i=1}^{\infty}NF_i$ (an ascending union), where each F_i is finite and r-generated, for some fixed r. Suppose $[M,G]$ has rank s and, for each $i=1,2,\ldots,$ let L_i be the centraliser of F_i in M. Arguing as before, we deduce that M/L_i has rank at most s^r. Let $L=\bigcap_{i=1}^{\infty}L_i$ and note that $L_{i+1}\leqslant L_i$ for all i. Thus every finite subgroup of M/L has rank at most s^r and so M/L has rank at most s^r. But $L=M\cap Z$ and we have a contradiction. This completes the proof of the theorem.

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