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EXTENSIONS BY INVERSE SEMIGROUPS AND λ-SEMIDIRECT PRODUCTS

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Abstract. The 'canonical embedding approach' was introduced by the second author and, subsequently, it has been applied several times to prove the embeddability of certain regular extensions by groups into semidirect products by groups. In the present paper this technique is generalized so that it is suitable to handle regular extensions by inverse semigroups. As an application, B. Billhardt's embedding theorem on regular extensions of semilattices by inverse semigroups is reproved.

0. Introduction. As a possible way to generalize McAlister's P-theorem for orthodox semigroups, the second author initiated the study of embeddings of E-unitary regular semigroups into semidirect product of bands by groups. In [**12**] a 'canonical embedding approach' was developed for this purpose and was applied to prove the embeddability of E-unitary regular semigroups with regular bands of idempotents. Furthermore, this approach was used in [**13**] and [**3**], and the same ideas also appeared in [**8**] in connection with regular extensions of Clifford semigroups by groups. The canonical embedding approach was generalized to the case of regular extensions of orthodox semigroups by groups in [**14**] and was applied to regular extensions of regular orthogroups by groups in [**15**]. Although the roots of the investigations of regular extensions by inverse semigroups go back to L. O'Carroll [**10**] and C. H. Houghton [**6**], the newer results on regular extensions by groups directed attention to the problem of whether similar results could be achieved for extensions by inverse semigroups. The first results of this kind are due to B. Billhardt [**1**] and [**2**]. In particular, he raised the question of which regular extensions of regular orthogroups by inverse semigroups are embeddable into a λ-semidirect product of a regular orthogroup by an inverse semigroup. In order to facilitate these investigations, the goal of the present paper is to develop a canonical embedding approach for regular extensions by inverse semigroups (Section 2). As an application, we reprove the main result in [**1**] (Section 3). Note that an alternative proof can be found in [**5**]. The proof presented here is an adaptation to the more general case of a proof of McAlister's P-theorem that was used several times by the second author in lectures to illustrate the canonical embedding approach (see Remark 3.6).

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1. Preliminaries. In this section we recall the notions and results needed in the paper. For any undefined notions and notation the reader is referred to [7].

If S is a regular semigroup then an *inverse unary operation* is defined to be a mapping ${}^{\dagger}\colon S \to S$ with the property that $s^{\dagger} \in V(s)$ for every $s \in S$. In particular, if S is an inverse semigroup then the unique inverse unary operation is denoted in the usual way by ${}^{-1}$.

A semigroup S is termed *locally inverse* if, for each idempotent e in S, the submonoid eSe is inverse. A semigroup is called *E-solid* if, for any idempotents $e, f, g \in S$ with $e \mathcal{R} f \mathcal{L} g$, there exists an idempotent $h \in S$ such that $e \mathcal{L} h \mathcal{R} g$. Throughout the paper, we will consider only regular locally inverse and regular E-solid semigroups, so, for short, we will omit the attribute 'regular' from these expressions.

Now we recall several notions and results on e-varieties.

A class of regular semigroups is termed an *e-variety* if it is closed under taking direct products, regular subsemigroups and homomorphic images. For example, the classes of all groups, completely regular semigroups, inverse semigroups, orthodox semigroups, locally inverse semigroups and E-solid semigroups form e-varieties. Note that a class of completely regular semigroups or of inverse semigroups constitutes an e-variety if and only if it is a variety of unary semigroups in the usual sense where a completely regular semigroup is considered as a unary semigroup in which the unary operation maps each element to its inverse within the maximal subgroup containing it, and an inverse semigroup is considered as a unary semigroup in which the unary operation maps each element to its unique inverse.

Given a non-empty set A, we will 'double' it as follows. Consider a set A' disjoint from A and a bijection ${}'\colon A \to A'$, $a \mapsto a'$. The union $A \cup A'$ will be denoted by $\overline{A}$. If S is a regular semigroup, then a mapping $\vartheta\colon \overline{A} \to S$ is called *matched* if $a'\vartheta \in V(a\vartheta)$ for every $a \in A$. Let $\mathbf{C}$ be a class of regular semigroups, T a member of $\mathbf{C}$ and $\iota\colon \overline{A} \to T$ a matched mapping. We say that (T, ι) is *a bifree object in* $\mathbf{C}$ *on* A if, for any S in $\mathbf{C}$ and any matched mapping $\vartheta\colon \overline{A} \to S$, there is a unique homomorphism $\phi\colon T \to S$ such that $\iota\phi = \vartheta$. In cases when ι is obvious we omit it, and we term T the bifree object in $\mathbf{C}$ on A. Note that, in any class of regular semigroups, there exists, up to isomorphism, at most one bifree object on any non-empty set. Moreover, if a class $\mathbf{C}$ admits a bifree object on any non-empty set then each member of $\mathbf{C}$ is a homomorphic image of a bifree object in $\mathbf{C}$.

A remarkable result by Y. T. Yeh [17] says that an e-variety $\mathbf{V}$ has a bifree object on a set of at least two elements, or, equivalently, on any non-empty set if and only if $\mathbf{V}$ consists either of locally inverse semigroups or of E-solid semigroups.

Let $\mathbf{V}$ be an e-variety of locally inverse or of E-solid semigroups. One can see that if $\mathbf{V}$ is non-trivial and (T, ι) is a bifree object in $\mathbf{V}$ on A then ι restricted to A or A' is necessarily injective. The bifree object in $\mathbf{V}$ on A will be denoted by $BF\mathbf{V}(A)$, and, without loss of generality, we will assume that $A, A' \subseteq BF\mathbf{V}(A)$ provided $\mathbf{V}$ is non-trivial.

It is clear from the definition of a bifree object that, in every variety of inverse semigroups, the bifree objects coincide with the free objects. However, this is not the case in varieties of completely regular semigroups. In a variety of completely regular semigroups, one can see that the free object on a given set is, up to isomorphism, a proper subsemigroup in the bifree object on the same set. If $\mathbf{V}$ is a variety of inverse or of completely regular semigroups then $F\mathbf{V}(A)$ will stand for the free object in $\mathbf{V}$ on A, and $A \subseteq F\mathbf{V}(A)$ will be assumed if $\mathbf{V}$ is non-trivial.

By the *kernel* of an inverse semigroup congruence θ on S we mean the subsemigroup $\{s \in S\colon s\theta \in E_{S/\theta}\}$ which we denote by $\operatorname{Ker}\theta$. Obviously, we have $E_S \subseteq \operatorname{Ker}\theta$. If S is a regular semigroup then $\operatorname{Ker}\theta$ is a regular subsemigroup of S, and, by Lallement's lemma, we have $\operatorname{Ker}\theta = \{s \in S\colon s\,\theta\, e \text{ for some } e \in E_S\}$. As usual, the least inverse semigroup congruence on S is denoted by γ_S or, simply, by γ.

If $\phi\colon S \to T$ is a homomorphism then the congruence on S induced by ϕ — called also the kernel of the homomorphism ϕ — will be denoted by $\ker\phi$. If S' is a subsemigroup in S then by $\phi|_{S'}$ we mean the restriction $\phi \cap (S' \times S'\phi)$ of ϕ.

Let K be a semigroup and T an inverse semigroup. If S is a semigroup and θ is a congruence on S such that S/θ is isomorphic to T and the kernel of θ is isomorphic to K then we call (S, θ) an *extension of* K *by* T. If, moreover, S is regular then (S, θ) is termed a *regular extension of* K *by* T. In this case, K is necessarily regular. If (S, θ) and (S', θ') are extensions by inverse semigroups then a homomorphism $\phi\colon S \to S'$ is defined to be a *homomorphism of* (S, θ) *into* (S', θ') and is denoted by $\phi\colon (S, \theta) \to (S', \theta')$, if $\theta \subseteq \ker(\phi\theta'^{\natural})$. If $\theta = \ker(\phi\theta'^{\natural})$ then we term ϕ a *semi-injective homomorphism*. If $\phi\colon (S, \theta) \to (S', \theta')$ is a semi-injective homomorphism which is injective as a mapping $S \to S'$ then we call it an *embedding of* (S, θ) *into* (S', θ').

Now we recall the definition of a λ-semidirect product of a semigroup by an inverse semigroup, and formulate the basic properties of this construction we need later.

Let K be a semigroup and T an inverse semigroup. Denote by $\operatorname{End} K$ the endomorphism monoid of K (where, according to the convention in [**7**], an endomorphism, just as any mapping, is written on the right, and so in a product of mappings the left factor is applied first). We say that T acts on K by endomorphisms on the left if an antihomomorphism $\varepsilon\colon T \to \operatorname{End} K$, $t \mapsto \varepsilon_t$ is given, that is, $\varepsilon_u\varepsilon_t = \varepsilon_{tu}$ $(t, u \in T)$ holds for the mapping ε. For brevity, we will say only that T acts on K, and we will denote $a\varepsilon_t$ by ${}^t a$ $(a \in K, t \in T)$. The *λ-semidirect product* $K *_\lambda T$ is defined on the underlying set

$$\{(a, t) \in K \times T\colon {}^{tt^{-1}}a = a\}$$

by the multiplication

$$(a, t)(b, u) = ({}^{(tu)(tu)^{-1}}a \cdot {}^t b, tu) \qquad (a, b \in K,\ t, u \in T).$$

A straightforward calculation shows that $K *_\lambda T$ is a semigroup. The following properties of a λ-semidirect product will be important for us. Note that statement (i) below was proved in the special cases where K is inverse, and where K is an E-solid semigroup by B. Billhardt [**1**] and M. Kuřil [**9**], respectively.

Proposition 1.1. *Let K be a semigroup and T an inverse semigroup acting on K.*

(i) *If K is a regular [E-solid, locally inverse, orthodox, inverse] semigroup then the λ-semidirect product $K *_\lambda T$ is also a regular [E-solid, locally inverse, orthodox, inverse] semigroup. Furthermore, we have*

$$E_{K*_\lambda T} = \{(e, i)\colon\ e \in E_K,\ i \in E_T \text{ and } {}^i e = e\}$$

and

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$$V_{K*_\lambda T}((a,t)) = \{(b,t^{-1})\colon\ b \in V_K({}^{t^{-1}}a) \text{ and } {}^{t^{-1}t}b = b\}.$$

(ii) *The second projection* $\pi_2\colon K *_\lambda T \to T,\ (a,t) \mapsto t$ *is a homomorphism of* $K *_\lambda T$ *onto* T *with* $\mathrm{Ker}\,(\ker \pi_2) = \{(a,i)\colon\ a \in K,\ i \in E_T \text{ and } {}^{i}a = a\}$. *Moreover, if* K *is regular then* $\mathrm{Ker}\,(\ker \pi_2)$ *is isomorphic to the strong semilattice of the regular subsemigroups* $K_i = \{a \in K\colon\ {}^{i}a = a\}$ $(i \in E_T)$ *with the surjective structure homomorphisms* $\varepsilon_j|_{K_i}\colon K_i \to K_j$ $(i,j \in E_T,\ i \geq j)$.

Proof. The proof is straightforward and it is left to the reader. □

Observe that a λ-semidirect product by an inverse semigroup can be considered as an extension in the following sense. If K is a regular semigroup and T is an inverse semigroup acting on K then the second projection $\pi_2 = \pi_2^{K*_\lambda T}$ of $K *_\lambda T$ induces the congruence $\ker \pi_2$ on $K *_\lambda T$, and $(K *_\lambda T, \ker \pi_2)$ is a regular extension of a strong semilattice of regular subsemigroups of K by T. We will refer to this extension as a *λ-semidirect product extension of* K *by* T.

Now we recall the basic notions concerning graphs and semigroupoids needed later.

A *graph* $\mathcal{X}$ consists of a set of *objects* denoted by $\mathrm{Obj}(\mathcal{X})$ and, for every pair $i, j \in \mathrm{Obj}(\mathcal{X})$, a set of *arrows* from i to j which is denoted by $\mathcal{X}(i,j)$ and is called a *hom-set*. The arrows a, b are called *coterminal* if $a, b \in \mathcal{X}(i,j)$ for some $i, j \in \mathrm{Obj}(\mathcal{X})$ and are termed *consecutive* provided $a \in \mathcal{X}(i,j)$ and $b \in \mathcal{X}(j,k)$ for some $i, j, k \in \mathrm{Obj}(\mathcal{X})$. The different hom-sets are supposed to be disjoint. The set of all arrows will be denoted by $\mathrm{Arr}(\mathcal{X})$. If $B \subseteq \mathrm{Arr}(\mathcal{X})$ then the graph $\mathcal{Y}$ defined by

$$\mathrm{Obj}(\mathcal{Y}) = \{i, j \in \mathrm{Obj}(\mathcal{X})\colon\ B \cap \mathcal{X}(i,j) \text{ is not empty}\}$$

and

$$\mathcal{Y}(i,j) = B \cap \mathcal{X}(i,j) \quad (i, j \in \mathrm{Obj}(\mathcal{Y}))$$

is termed the *subgraph of* $\mathcal{X}$ *determined by* B.

Just as we doubled a set above, we may double a graph as follows. Given a graph $\mathcal{X}$, we consider a graph $\mathcal{X}'$ such that $\mathrm{Obj}(\mathcal{X}') = \mathrm{Obj}(\mathcal{X})$, $\mathrm{Arr}(\mathcal{X}')$ is disjoint from $\mathrm{Arr}(\mathcal{X})$ and a bijection $'\colon \mathcal{X}(i,j) \to \mathcal{X}'(j,i),\ a \mapsto a'$ is fixed for every $i, j \in \mathrm{Obj}(\mathcal{X})$. Define the graph $\overline{\mathcal{X}}$ by $\mathrm{Obj}(\overline{\mathcal{X}}) = \mathrm{Obj}(\mathcal{X})$ and $\overline{\mathcal{X}}(i,j) = \mathcal{X}(i,j) \cup \mathcal{X}'(i,j)$ $(i, j \in \mathrm{Obj}(\mathcal{X}))$. Notice that the bijections $'$ of the hom-sets of $\mathcal{X}$ onto those of $\mathcal{X}'$ determine a bijection of $\mathrm{Arr}(\mathcal{X})$ onto $\mathrm{Arr}(\mathcal{X}')$. Therefore, if we have a graph $\mathcal{X}$ and we double both $\mathcal{X}$ as a graph and $A = \mathrm{Arr}(\mathcal{X})$ as a set then we will assume that $A' = \mathrm{Arr}(\mathcal{X}')$ and the bijection $'\colon A \to A'$ is the one induced by the bijections between the hom-sets of $\mathcal{X}$ and $\mathcal{X}'$. So $\overline{A} = \mathrm{Arr}(\overline{\mathcal{X}})$ also follows.

A *semigroupoid* is a graph $\mathcal{C}$ equipped with a composition which assigns to every pair of consecutive arrows $a \in \mathcal{C}(i,j)$, $b \in \mathcal{C}(j,k)$ an arrow $ab \in \mathcal{C}(i,k)$ such that the composition is associative, that is, for any arrows $a \in \mathcal{C}(i,j)$, $b \in \mathcal{C}(j,k)$ and $c \in \mathcal{C}(k,l)$, we have $(ab)c = a(bc)$.

By a *regular semigroupoid* we mean a semigroupoid in which, for every arrow a in $\mathcal{C}$, there exists an arrow c such that $aca = a$. Just as with semigroups, one can see that, in this case, for every arrow a in $\mathcal{C}$, there also exists an *inverse* b of a in the sense that $aba = a$ and $bab = b$ hold. A mapping $^{\dagger}\colon \mathrm{Arr}(\mathcal{C}) \to \mathrm{Arr}(\mathcal{C})$ is called an *inverse unary operation* on $\mathcal{C}$ if it assigns an inverse of a to each arrow a. If every arrow in $\mathcal{C}$

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admits a unique inverse then $\mathcal{C}$ is termed an *inverse semigroupoid*, and the unique inverse of an arrow a is denoted by a^{-1}.

2. The canonical embedding. Let $\mathbf{V}$ be an e-variety of locally inverse or of E-solid semigroups and (S, θ) a regular extension by an inverse semigroup. In this section we construct a member $K_{\mathbf{V}}^{(S,\theta)} \in \mathbf{V}$, a λ-semidirect product $K_{\mathbf{V}}^{(S,\theta)} *_{\lambda} (S/\theta)$ and a semi-injective homomorphism $\kappa_{\mathbf{V}}^{(S,\theta)}: (S, \theta) \to (K_{\mathbf{V}}^{(S,\theta)} *_{\lambda} (S/\theta), \ker(\pi_2))$ such that (S, θ) is embeddable into a λ-semidirect product of a member of $\mathbf{V}$ by an inverse semigroup if and only if $\kappa_{\mathbf{V}}^{(S,\theta)}$ is injective.

First we define a semigroupoid $\mathcal{C} = \mathcal{C}(S, \theta)$ corresponding to an extension (S, θ) by an inverse semigroup which we call the *derived semigroupoid* of (S, θ) since it is closely related to the derived semigroupoid of the homomorphism $\theta^{\natural}$, cf. [**16**].

Let (S, θ) be an extension by an inverse semigroup. For brevity, denote S/θ by T. Let $\mathcal{C}$ be the graph with

$$\mathrm{Obj}(\mathcal{C}) = T,$$

and

$$\mathcal{C}(a, b) = \{(a, s, b) \in T \times S \times T\colon\ a \cdot s\theta = b \text{ and } b \cdot (s\theta)^{-1} = a\} \quad (a, b \in T).$$

One can equip $\mathcal{C}$ with the following multiplication: if $(a, s, b) \in \mathcal{C}(a, b)$ and $(b, t, c) \in \mathcal{C}(b, c)$ then

$$(a, s, b) \circ (b, t, c) = (a, st, c).$$

Obviously, $(a, st, c) \in \mathcal{C}(a, c)$, and this multiplication is associative. Thus $\mathcal{C} = (\mathcal{C}; \circ)$ forms a semigroupoid. It is straightforward to see that $V((a, s, b)) = \{(b, s', a)\colon s' \in V(s)\}$.

Now suppose that (S, θ) is a regular extension. Then we immediately obtain that $\mathcal{C}$ is a regular semigroupoid. Let us choose and fix an inverse unary operation † on S. This determines an inverse unary operation, also denoted by †, on $\mathcal{C}$ by letting $(a, s, b)^{\dagger} = (b, s^{\dagger}, a)$ for every $(a, s, b) \in \mathrm{Arr}(\mathcal{C})$.

Now we use this semigroupoid $\mathcal{C}$ to construct the semigroup $K_{\mathbf{V}}^{(S,\theta)}$. Roughly speaking, $K_{\mathbf{V}}^{(S,\theta)}$ is the bifree object in $\mathbf{V}$ 'on the semigroupoid $\mathcal{C}$'. Let $\mathbf{V}$ be an e-variety of locally inverse or of E-solid semigroups. Consider the bifree object $BF\mathbf{V}(A)$ in $\mathbf{V}$ on $A = \mathrm{Arr}(\mathcal{C})$ and the congruence $\tau_{\mathbf{V}}^{(S,\theta)}$ on it generated by

$$\begin{aligned} \nu = &\{(\mathrm{a}', \mathrm{a}^{\dagger})\colon\ \mathrm{a} \in A\} \\ &\cup \{(\mathrm{ab}, \mathrm{c})\colon\ \mathrm{a}, \mathrm{b}, \mathrm{c} \in A \text{ such that } \mathrm{c} = \mathrm{a} \circ \mathrm{b} \text{ in } \mathcal{C}\}. \end{aligned}$$

Put $K_{\mathbf{V}}^{(S,\theta)} = BF\mathbf{V}(A)/\tau_{\mathbf{V}}^{(S,\theta)}$. Clearly, we have $K_{\mathbf{V}}^{(S,\theta)} \in \mathbf{V}$.

Now we define an action of T on $K_{\mathbf{V}}^{(S,\theta)}$ as follows. The semigroup T naturally acts on the semigroupoid $\mathcal{C}$ by defining

$${}^{t}(a, s, b) = (ta, s, tb) \quad (t \in T,\ (a, s, b) \in A),$$

because one can immediately check that we have ${}^{t}(\mathrm{a} \circ \mathrm{b}) = {}^{t}\mathrm{a} \circ {}^{t}\mathrm{b}$ for every $t \in T$ and consecutive arrows $\mathrm{a}, \mathrm{b} \in A$, and we have ${}^{t}({}^{u}\mathrm{a}) = {}^{tu}\mathrm{a}$ for every $t, u \in T$ and $\mathrm{a} \in A$.

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Moreover, the definition of the inverse unary operation † ensures that ${}^t(a^\dagger) = ({}^ta)^\dagger$ for every $t \in T$ and $a \in A$. On the other hand, this action induces an action of T on the doubled graph $\overline{\mathcal{C}}$ by setting

$$ {}^t(a') = ({}^ta)' \quad (t \in T,\ a \in A). $$

For every $t \in T$, define $\xi_t\colon BF\mathbf{V}(A) \to BF\mathbf{V}(A)$ to be the unique extension of the matched mapping $\overline{A} \to BF\mathbf{V}(A)$, $a \mapsto {}^ta$ $(a \in \overline{A})$ to an endomorphism of $BF\mathbf{V}(A)$. By unicity of the ξ_t's, we clearly have

$$ \xi_u\xi_t = \xi_{tu} \quad (t, u \in T). \tag{1} $$

Thus an action ξ of T on $BF\mathbf{V}(A)$ is defined by $t \mapsto \xi_t$. The properties of the actions of T on $\mathcal{C}$ and $\overline{\mathcal{C}}$ mentioned above ensure that, for every $t \in T$, the images of ν-related elements under ξ_t are ν-related, whence $\tau_{\mathbf{V}}^{(S,\theta)} \subseteq \ker(\xi_t(\tau_{\mathbf{V}}^{(S,\theta)})^\natural)$ follows. This implies that there exists a unique endomorphism $\varepsilon_t\colon K_{\mathbf{V}}^{(S,\theta)} \to K_{\mathbf{V}}^{(S,\theta)}$ with $(\tau_{\mathbf{V}}^{(S,\theta)})^\natural\varepsilon_t = \xi_t(\tau_{\mathbf{V}}^{(S,\theta)})^\natural$. Unicity of the ε_t's ensures by (1) that $\varepsilon_u\varepsilon_t = \varepsilon_{tu}$ for every $t, u \in T$, and so $t \mapsto \varepsilon_t$ defines an action ε of T on $K_{\mathbf{V}}^{(S,\theta)}$.

The semigroups $K_{\mathbf{V}}^{(S,\theta)}$ and T together with the action ε of T on $K_{\mathbf{V}}^{(S,\theta)}$ define a λ-semidirect product $K_{\mathbf{V}}^{(S,\theta)} *_\lambda T$. Now we define a semi-injective homomorphism $\kappa_{\mathbf{V}}^{(S,\theta)}\colon (S, \theta) \to (K_{\mathbf{V}}^{(S,\theta)} *_\lambda T, \ker(\pi_2))$. Put

$$ s\kappa_{\mathbf{V}}^{(S,\theta)} = \big((s\theta(s\theta)^{-1}, s, s\theta)\tau_{\mathbf{V}}^{(S,\theta)}, s\theta\big) \quad (s \in S). $$

It is routine to check that $\kappa_{\mathbf{V}}^{(S,\theta)}\colon S \to K_{\mathbf{V}}^{(S,\theta)} *_\lambda T$ is a homomorphism. For, if $s, t \in S$ then

$$ \begin{aligned} s\kappa_{\mathbf{V}}^{(S,\theta)} \cdot t\kappa_{\mathbf{V}}^{(S,\theta)} &= \big((s\theta(s\theta)^{-1}, s, s\theta)\tau_{\mathbf{V}}^{(S,\theta)}, s\theta\big)\big((t\theta(t\theta)^{-1}, t, t\theta)\tau_{\mathbf{V}}^{(S,\theta)}, t\theta\big) \\ &= \big({}^{s\theta t\theta(s\theta t\theta)^{-1}}((s\theta(s\theta)^{-1}, s, s\theta)\tau_{\mathbf{V}}^{(S,\theta)}) \cdot {}^{s\theta}((t\theta(t\theta)^{-1}, t, t\theta)\tau_{\mathbf{V}}^{(S,\theta)}), s\theta t\theta\big) \\ &= \big((s\theta t\theta(s\theta t\theta)^{-1}, s, s\theta t\theta(t\theta)^{-1})\tau_{\mathbf{V}}^{(S,\theta)} \cdot (s\theta t\theta(t\theta)^{-1}, t, s\theta t\theta)\tau_{\mathbf{V}}^{(S,\theta)}, s\theta t\theta\big) \\ &= \big((s\theta t\theta(s\theta t\theta)^{-1}, st, s\theta t\theta)\tau_{\mathbf{V}}^{(S,\theta)}, s\theta t\theta\big) \\ &= \big(((st)\theta((st)\theta)^{-1}, st, (st)\theta)\tau_{\mathbf{V}}^{(S,\theta)}, (st)\theta\big) \\ &= (st)\kappa_{\mathbf{V}}^{(S,\theta)}. \end{aligned} $$

Moreover, we obviously have $\ker(\kappa_{\mathbf{V}}^{(S,\theta)}\pi_2) = \theta$.

REMARK 2.1. (i) Notice that the definition of $K_{\mathbf{V}}^{(S,\theta)}$ and that of the action of $T = S/\theta$ on $K_{\mathbf{V}}^{(S,\theta)}$ involves an arbitrarily chosen inverse unary operation †. We will show in Corollary 2.3 that they are, up to isomorphism, independent of the choice of †.

(ii) When S is an inverse semigroup and $\mathbf{V}$ is a variety of inverse semigroups then $^{-1}$ is the unique inverse unary operation on S and $BF\mathbf{V}(A) = F\mathbf{V}(A)$. Therefore the above definition of $K_{\mathbf{V}}^{(S,\theta)}$ and the action of T on it can be interpreted also in the following way: consider the (unary) congruence $\tau_{\mathbf{V}}^{(S,\theta)}$ on $F\mathbf{V}(A)$ generated by $\nu = \{(ab, c)\colon a, b, c \in A \text{ such that } c = a \circ b \text{ in } \mathcal{C}\}$, put $K_{\mathbf{V}}^{(S,\theta)} = F\mathbf{V}(A)/\tau_{\mathbf{V}}^{(S,\theta)}$ and extend the natural action of T on the semigroupoid $\mathcal{C}$ to an action (by unary endomorphisms) of T on $F\mathbf{V}(A)$; it induces an action of T on $K_{\mathbf{V}}^{(S,\theta)}$.

We are ready to formulate the main result of this section. It proves a universal property of $\kappa_{\mathbf{V}}^{(S,\theta)}$ among certain homomorphisms of the extension (S,θ) into λ-semidirect product extensions of members in **V** by inverse semigroups. The following homomorphisms play a crucial role here.

Let $L, \overline{L}$ be any semigroups and let $U, \overline{U}$ be inverse semigroups acting on L and $\overline{L}$, respectively. If $\phi_1: L \to \overline{L}$ and $\phi_2: U \to \overline{U}$ are homomorphisms such that, for every $l \in L$ and $u \in U$, we have $({}^u l)\phi_1 = {}^{u\phi_2}(l\phi_1)$, then the mapping $\phi: (L *_\lambda U, \ker(\pi_2^{L*_\lambda U})) \to (\overline{L} *_\lambda \overline{U}, \ker(\pi_2^{\overline{L}*_\lambda \overline{U}}))$ defined by $(l,u)\phi = (l\phi_1, u\phi_2)$ is easily seen to be a homomorphism. If a homomorphism $\phi: (L *_\lambda U, \ker(\pi_2^{L*_\lambda U})) \to (\overline{L} *_\lambda \overline{U}, \ker(\pi_2^{\overline{L}*_\lambda \overline{U}}))$ is of this form for some ϕ_1 and ϕ_2 then we term it a *splitting homomorphism* of the λ-semidirect products, and denote it also by (ϕ_1, ϕ_2).

Theorem 2.2. *Let S be a regular semigroup and let θ be an inverse semigroup congruence on S. Let **V** be an e-variety of locally inverse or of E-solid semigroups. Suppose that $L \in \mathbf{V}$, U is an inverse semigroup acting on L and $\psi: (S,\theta) \to (L *_\lambda U, \ker(\pi_2^{L*_\lambda U}))$ is a homomorphism. Then there exists a unique splitting homomorphism*

$$\phi: (K_{\mathbf{V}}^{(S,\theta)} *_\lambda (S/\theta), \ker(\pi_2^{K_{\mathbf{V}}^{(S,\theta)} *_\lambda (S/\theta)})) \to (L *_\lambda U, \ker(\pi_2^{L*_\lambda U}))$$

such that $\psi = \kappa_{\mathbf{V}}^{(S,\theta)}\phi$.

Proof. For brevity, we will write τ, K and κ for $\tau_{\mathbf{V}}^{(S,\theta)}$, $K_{\mathbf{V}}^{(S,\theta)}$ and $\kappa_{\mathbf{V}}^{(S,\theta)}$, respectively. Furthermore, we denote the i-th $(i = 1,2)$ projection $\pi_i^{K*_\lambda T}$ and $\pi_i^{L*_\lambda U}$ by π_i and $\overline{\pi}_i$, respectively. We begin by proving the uniqueness of ϕ. Assume that $\phi = (\phi_1, \phi_2): (K *_\lambda T, \ker \pi_2) \to (L *_\lambda U, \ker \overline{\pi}_2)$ is a splitting homomorphism satisfying $\psi = \kappa\phi$. Obviously, this equality is equivalent to the equalities $\psi\overline{\pi}_i = \kappa\pi_i\phi_i$ $(i = 1,2)$. It follows from the definition of κ that $\kappa\pi_2 = \theta^\natural$. Hence we immediately see that ϕ_2 is the unique homomorphism of T into U with

$$\theta^\natural\phi_2 = \psi\overline{\pi}_2. \tag{2}$$

On the other hand, the equality $\psi\overline{\pi}_1 = \kappa\pi_1\phi_1$ is equivalent to the fact that

$$((s\theta(s\theta)^{-1}, s, s\theta)\tau)\phi_1 = s\psi\overline{\pi}_1 \tag{3}$$

for every $s \in S$. Hence it follows that

$$((a,s,b)\tau)\phi_1 = {}^{a\phi_2}(s\psi\overline{\pi}_1) \tag{4}$$

for any $(a,s,b) \in A$. For, if $(a,s,b) \in A$ then $a(s\theta) = b$ and $b(s\theta)^{-1} = a$ whence we obtain that $(a,s,b) = {}^a(s\theta(s\theta)^{-1}, s, s\theta)$. Applying (3) and the fact that (ϕ_1, ϕ_2) is a splitting homomorphism, we see that

$$\begin{aligned}((a,s,b)\tau)\phi_1 &= \big(({}^a(s\theta(s\theta)^{-1}, s, s\theta))\tau\big)\phi_1 \\ &= \big({}^a((s\theta(s\theta)^{-1}, s, s\theta)\tau)\big)\phi_1 \\ &= {}^{a\phi_2}((s\theta(s\theta)^{-1}, s, s\theta)\tau)\phi_1 \\ &= {}^{a\phi_2}(s\psi\overline{\pi}_1).\end{aligned}$$

Put $\Phi = \tau^\natural \phi_1$. Since $\tau^\natural$ is surjective, in order to show that ϕ_1 is uniquely determined, it suffices to verify that Φ is uniquely determined. However, this is not difficult to check because, by definition, $a'\tau = a^\dagger\tau$ $(a \in A)$, and so, by (4), we see that Φ is the unique extension of the matched mapping $\vartheta\colon \overline{A} \to L$, $(a, s, b)\vartheta = {}^{a\phi_2}(s\psi\overline{\pi}_1)$, $(a, s, b)'\vartheta = {}^{b\phi_2}(s^\dagger\psi\overline{\pi}_1)$ $((a, s, b) \in A)$. The fact that ϑ is, indeed, matched can be checked by a straightforward calculation. Let $s \in S$. Since $s^\dagger \in V(s)$, we have $s^\dagger\psi\overline{\pi}_2 = (s\psi\overline{\pi}_2)^{-1}$. Moreover, $(s\psi\overline{\pi}_1, s\psi\overline{\pi}_2) \in L *_\lambda U$ whence ${}^{s\psi\overline{\pi}_2(s\psi\overline{\pi}_2)^{-1}}(s\psi\overline{\pi}_1) = s\psi\overline{\pi}_1$ follows. Since $\psi\colon S \to L *_\lambda U$ is a homomorphism, we infer that

$$\begin{aligned} s\psi\overline{\pi}_1 &= (s\psi \cdot s^\dagger\psi \cdot s\psi)\overline{\pi}_1 \\ &= {}^{s\psi\overline{\pi}_2(s\psi\overline{\pi}_2)^{-1}}\big({}^{s\psi\overline{\pi}_2(s\psi\overline{\pi}_2)^{-1}}(s\psi\overline{\pi}_1) \cdot {}^{s\psi\overline{\pi}_2}(s^\dagger\psi\overline{\pi}_1)\big) \cdot {}^{s\psi\overline{\pi}_2(s\psi\overline{\pi}_2)^{-1}}(s\psi\overline{\pi}_1) \\ &= s\psi\overline{\pi}_1 \cdot {}^{s\psi\overline{\pi}_2}(s^\dagger\psi\overline{\pi}_1) \cdot s\psi\overline{\pi}_1. \end{aligned}$$

Similarly, we also obtain that $s^\dagger\psi\overline{\pi}_1 = s^\dagger\psi\overline{\pi}_1 \cdot {}^{(s\psi\overline{\pi}_2)^{-1}}(s\psi\overline{\pi}_1) \cdot s^\dagger\psi\overline{\pi}_1$ which implies ${}^{s\psi\overline{\pi}_2}(s^\dagger\psi\overline{\pi}_1) = {}^{s\psi\overline{\pi}_2}(s^\dagger\psi\overline{\pi}_1) \cdot s\psi\overline{\pi}_1 \cdot {}^{s\psi\overline{\pi}_2}(s^\dagger\psi\overline{\pi}_1)$. Therefore $s\psi\overline{\pi}_1 \in V\big({}^{s\psi\overline{\pi}_2}(s^\dagger\psi\overline{\pi}_1)\big)$. If $(a, s, b) \in A$ then $b = a(s\theta)$, and so $b\phi_2 = a\phi_2 \cdot s\psi\overline{\pi}_2$ yields by (2). This implies ${}^{a\phi_2}(s\psi\overline{\pi}_1) \in V\big({}^{b\phi_2}(s^\dagger\psi\overline{\pi}_1)\big)$, completing the proof that ϑ is matched. Thus the uniqueness of ϕ is proved.

The existence of ϕ will follow if we show that the only pair (ϕ_1, ϕ_2) possible by the above argument exists and forms a splitting homomorphism. Let $\phi_2\colon T \to U$ be the unique homomorphism such that (2) is valid. Consider the matched mapping ϑ defined above. Let Φ be the unique extension of ϑ to a homomorphism $BF\mathbf{V}(A) \to L$. We intend to show that $\tau \subseteq \ker\Phi$. For this purpose, it is enough to verify that $\nu \subseteq \ker\Phi$. By definition, $(a, s, b)'\Phi = (b, s^\dagger, a)\Phi = (a, s, b)^\dagger\Phi$. So the first set in the definition of ν is clearly contained in $\ker\Phi$. The same property of the second set follows in the following manner. Let $(a, s, b), (b, t, c) \in A$. Then we have $a(s\theta) = b$ and $a((st)\theta)((st)\theta)^{-1} = a$. This implies by (2) that $a\phi_2 \cdot s\psi\overline{\pi}_2 = b\phi_2$ and $a\phi_2 \cdot ((st)\psi\overline{\pi}_2)((st)\psi\overline{\pi}_2)^{-1} = a\phi_2$. On the other hand, we have $(a, s, b) \circ (b, t, c) = (a, st, c)$ in $\mathcal{C}$. Since $(st)\psi\overline{\pi}_1 = {}^{((st)\psi\overline{\pi}_2)((st)\psi\overline{\pi}_2)^{-1}}(s\psi\overline{\pi}_1) \cdot {}^{s\psi\overline{\pi}_2}(t\psi\overline{\pi}_1)$, we obtain that $(a, st, c)\Phi = {}^{a\phi_2}((st)\psi\overline{\pi}_1) = {}^{a\phi_2}(s\psi\overline{\pi}_1) \cdot {}^{b\phi_2}(t\psi\overline{\pi}_1) = (a, s, b)\Phi \cdot (b, t, c)\Phi$. Thus we have proved that $\tau \subseteq \ker\Phi$. This ensures the existence of a homomorphism ϕ_1 such that $\tau^\natural\phi_1 = \Phi$.

To complete the proof, it remains to show that

$$({}^t k)\phi_1 = {}^{t\phi_2}(k\phi_1) \quad (t \in T,\ k \in K).$$

This is equivalent to $\xi_t\Phi = \Phi\eta_{t\phi_2}$ $(t \in T)$ where ξ is the action of T on $BF\mathbf{V}(A)$ introduced before the theorem and η is used to denote the action of U on L. For, $k = w\tau$ for some $w \in BF\mathbf{V}(A)$, and so $({}^t k)\phi_1 = ({}^t(w\tau))\phi_1 = (({}^t w)\tau)\phi_1 = w\xi_t\Phi$ and ${}^{t\phi_2}(k\phi_1) = {}^{t\phi_2}((w\tau)\phi_1) = w\Phi\eta_{t\phi_2}$. Since $\xi_t\Phi$ and $\Phi\eta_{t\phi_2}$ are homomorphisms of $BF\mathbf{V}(A)$, their equality follows if their restrictions to $\overline{A}$ coincide. The latter property can be easily checked. For every $(a, s, b) \in A$, we have $(a, s, b)\xi_t\Phi = (ta, s, tb)\Phi = {}^{(ta)\phi_2}(s\psi\overline{\pi}_1) = {}^{t\phi_2 \cdot a\phi_2}(s\psi\overline{\pi}_1) = {}^{t\phi_2}({}^{a\phi_2}(s\psi\overline{\pi}_1)) = (a, s, b)\Phi\eta_{t\phi_2}$, and a similar calculation applies for $(a, s, b)' \in A'$. The proof of the theorem is complete. □

Now we turn to proving the property of the construction $K_{\mathbf{V}}^{(S,\theta)} *_\lambda (S/\theta)$ formulated in Remark 2.1(i). Let $\tau_{\mathbf{V}}^{(S,\theta)}$, $K_{\mathbf{V}}^{(S,\theta)}$, ε, $\kappa_{\mathbf{V}}^{(S,\theta)}$ and $\hat{\tau}_{\mathbf{V}}^{(S,\theta)}$, $\hat{K}_{\mathbf{V}}^{(S,\theta)}$, $\hat{\varepsilon}$, $\hat{\kappa}_{\mathbf{V}}^{(S,\theta)}$ be two quadruples constructed as described before Theorem 2.2 by means of two different

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inverse unary operations. Then Theorem 2.2 implies the existence of splitting homomorphisms

$$\phi = (\phi_1, \phi_2) \colon (K_{\mathbf{V}}^{(S,\theta)} *_\lambda (S/\theta), \ker(\pi_2)) \to (\hat{K}_{\mathbf{V}}^{(S,\theta)} *_\lambda (S/\theta), \ker(\hat{\pi}_2))$$

with $\hat{\kappa}_{\mathbf{V}}^{(S,\theta)} = \kappa_{\mathbf{V}}^{(S,\theta)}\phi$ and

$$\hat{\phi} = (\hat{\phi}_1, \hat{\phi}_2) \colon (\hat{K}_{\mathbf{V}}^{(S,\theta)} *_\lambda (S/\theta), \ker(\hat{\pi}_2)) \to (K_{\mathbf{V}}^{(S,\theta)} *_\lambda (S/\theta), \ker(\pi_2))$$

with $\kappa_{\mathbf{V}}^{(S,\theta)} = \hat{\kappa}_{\mathbf{V}}^{(S,\theta)}\hat{\phi}$ where π_2 and $\hat{\pi}_2$ stand for the second projections of $K_{\mathbf{V}}^{(S,\theta)} *_\lambda (S/\theta)$ and $\hat{K}_{\mathbf{V}}^{(S,\theta)} *_\lambda (S/\theta)$, respectively. Clearly, ϕ_2 and $\hat{\phi}_2$ are identical on S/θ. Moreover, $\phi\hat{\phi}$ and $\hat{\phi}\phi$ are splitting homomorphisms of $(K_{\mathbf{V}}^{(S,\theta)} *_\lambda (S/\theta), \ker(\pi_2))$ and $(\hat{K}_{\mathbf{V}}^{(S,\theta)} *_\lambda (S/\theta), \ker(\hat{\pi}_2))$, respectively, into themselves with $\kappa_{\mathbf{V}}^{(S,\theta)}\phi\hat{\phi} = \kappa_{\mathbf{V}}^{(S,\theta)}$ and $\hat{\kappa}_{\mathbf{V}}^{(S,\theta)}\hat{\phi}\phi = \hat{\kappa}_{\mathbf{V}}^{(S,\theta)}$. Thus, by Theorem 2.2, both $\phi\hat{\phi}$ and $\hat{\phi}\phi$ are identical which implies that ϕ_1 and $\hat{\phi}_1$ are inverses of each other. Hence ϕ_1 is an isomorphism of $K_{\mathbf{V}}^{(S,\theta)}$ onto $\hat{K}_{\mathbf{V}}^{(S,\theta)}$. Since ϕ is splitting, we have $\varepsilon_t\phi_1 = \phi_1\hat{\varepsilon}_t$ for every $t \in T$.

Let L, $\overline{L}$ and U be semigroups such that U acts both on L and $\overline{L}$. We say that a homomorphism $\psi \colon L \to \overline{L}$ respects the actions of U if $({}^u l)\psi = {}^u(l\psi)$ for every $u \in U$ and $l \in L$. It is easy to see that ψ possesses this property if and only if $(\psi, \iota) \colon (L *_\lambda U, \ker(\pi_2^{L *_\lambda U})) \to (\overline{L} *_\lambda U, \ker(\pi_2^{\overline{L} *_\lambda U}))$, where ι is the identity automorphism of U, is a splitting homomorphism.

With this terminology, the statement we have verified is the following.

Corollary 2.3. *Let $\tau_V^{(S,\theta)}$, $K_V^{(S,\theta)}$, ε, $\kappa_V^{(S,\theta)}$ and $\hat{\tau}_V^{(S,\theta)}$, $\hat{K}_V^{(S,\theta)}$, $\hat{\varepsilon}$, $\hat{\kappa}_V^{(S,\theta)}$ be two quadruples constructed as described before Theorem 2.2 by means of different inverse unary operations. Then these constructions are equivalent in the sense that there exists an isomorphism $\phi_1 \colon K_V^{(S,\theta)} \to \hat{K}_V^{(S,\theta)}$ which respects the actions ε and $\hat{\varepsilon}$.* □

This corollary makes it possible to call $\kappa_{\mathbf{V}}^{(S,\theta)}$ the *canonical homomorphism of (S, θ) into a λ-semidirect product extension of a member in* **V** *by an inverse semigroup*. If $\kappa_{\mathbf{V}}^{(S,\theta)}$ is injective then we say that (S, θ) is *canonically embeddable into a λ-semidirect product extension of a member in* **V** *by an inverse semigroup*.

One sees immediately that if ψ in Theorem 2.2 is injective then $\kappa_{\mathbf{V}}^{(S,\theta)}$ is also injective. This establishes the following crucial property of $\kappa_{\mathbf{V}}^{(S,\theta)}$.

Corollary 2.4. *Let S be a regular semigroup, and let θ be an inverse semigroup congruence on S. Let* **V** *be an e-variety of locally inverse or of E-solid semigroups. Then (S, θ) is embeddable into a λ-semidirect product extension of a member in* **V** *by an inverse semigroup if and only if (S, θ) is canonically embeddable, or, equivalently, if and only if the relations $s\,\theta\,t$ in S and $(s\theta(s\theta)^{-1}, s, s\theta)\,\tau_V^{(S,\theta)}\,(t\theta(t\theta)^{-1}, t, t\theta)$ in $BF\mathbf{V}(A)$ imply $s = t$ for every $s, t \in S$.* □

3. An application.

In this section we apply the canonical embedding approach introduced in the former section to reprove the following result by B. Billhardt [**1**]:

Theorem 3.1. *Let S be an inverse semigroup and θ an idempotent pure congruence on S. Then the extension (S, θ) can be embedded into a λ-semidirect product extension of a semilattice by S/θ.*

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Proof. Let S be an inverse semigroup and θ an idempotent pure congruence on S. Denote the factor semigroup S/θ by T and the variety of semilattices (as a variety of inverse semigroups) by $\mathbf{S}$. Construct the derived semigroupoid $\mathcal{C} = \mathcal{C}(S, \theta)$ of (S, θ) and the congruence $\tau = \tau_{\mathbf{S}}^{(S,\theta)}$ on the free semilattice $F\mathbf{S}(A)$; see Remark 2.1. By Corollary 2.4, we have to prove that, for every $s, t \in S$, the relations $s \,\theta\, t$ in S and $(s\theta(s\theta)^{-1}, s, s\theta) \,\tau\, (t\theta(t\theta)^{-1}, t, t\theta)$ in $F\mathbf{S}(A)$ imply $s = t$.

Assume that $s, t \in S$ satisfy the premisses of the implication to be proved. Put $\alpha = s\theta = t\theta$ and $\epsilon = s\theta(s\theta)^{-1} = t\theta(t\theta)^{-1}$. Then we have $(\epsilon, s, \alpha) \,\tau\, (\epsilon, t, \alpha)$ in $F\mathbf{S}(A)$ where $A = \mathrm{Arr}(\mathcal{C})$. Observe that $\mathcal{C}$ is an inverse semigroupoid, and we have $(\beta, u, \gamma)^{-1} = (\gamma, u^{-1}, \beta)$ for every arrow (β, u, γ) in $\mathcal{C}$.

Recall from [**11**] that $F\mathbf{S}(A)$ can be interpreted in the usual way as the factor semigroup of $A^{\oplus} = (\overline{A}^{+}; {}')$, the free semigroup with involution on the set A, by the fully invariant congruence $\rho = \rho(\mathbf{S}, A)$. Here $\overline{A}^{+}$ is the free semigroup on $\overline{A} = \mathrm{Arr}(\overline{\mathcal{C}})$, and the involution $'$ is the unique extension of the bijection $' \colon A \to A'$ to an involution of $\overline{A}^{+}$. The fully invariant congruence ρ is the following relation. Given a word w in $A^{\oplus}$, its content $A(w)$ is defined to be the set of all elements of A which appear in w with or without $'$. For every pair of words $u, v \in A^{\oplus}$, we have $u \,\rho\, v$ if and only if $A(u) = A(v)$.

From the definition of the relation τ, it is routine to see that, for every pair of words $x, y \in A^{\oplus}$, we have $(x\rho) \,\tau\, (y\rho)$ if and only if there exists a finite sequence $x = w_0, w_1, \ldots, w_n = y$ of words in $A^{\oplus}$ such that, for every i $(0 \leq i < n)$, the word w_{i+1} is obtained from w_i by one of the following steps:

(S0) $w_i = ua'v$, $w_{i+1} = ua^{-1}v$ for some $u, v \in (A^{\oplus})^1$ and $a \in A$,
(S0′) $w_i = ua^{-1}v$, $w_{i+1} = ua'v$ for some $u, v \in (A^{\oplus})^1$ and $a \in A$,
(S1) $w_i = uabv$, $w_{i+1} = ucv$ for some $u, v \in (A^{\oplus})^1$ and $a, b, c \in A$ with $a \circ b = c$ in $\mathcal{C}$,
(S1′) $w_i = ucv$, $w_{i+1} = uabv$ for some $u, v \in (A^{\oplus})^1$ and $a, b, c \in A$ with $a \circ b = c$ in $\mathcal{C}$,
(S2) $w_{i+1} \,\rho\, w_i$.

Therefore, in order to prove the implication formulated above, we have to show that if $(\epsilon, s, \alpha) = w_0, w_1, \ldots, w_n = (\epsilon, t, \alpha)$ is a sequence of words in $A^{\oplus}$ such that, for every i $(0 \leq i < n)$, the word w_{i+1} is obtained from w_i by one of the steps (S0)–(S2), then $s = t$. From now on, we fix such a sequence $(\epsilon, s, \alpha) = w_0, w_1, \ldots, w_n = (\epsilon, t, \alpha)$, and intend to prove the equality $s = t$.

Let R be an $\mathcal{R}$-class of S such that $\epsilon \cap R$ is not empty. Now we define two sub-semigroupoids in $\mathcal{C}$. Let $\mathcal{U}_R$ and $\mathcal{V}_R$ be the subgraphs in $\mathcal{C}$ determined by the sets of arrows

$$U = \{(\beta, u, \gamma) \in \mathrm{Arr}(\mathcal{C})\colon \text{ there exists } b \in \beta \cap R \text{ such that } b^{-1}b = uu^{-1}\}$$

and

$$V = \{(\beta, v, \gamma) \in \mathrm{Arr}(\mathcal{C})\colon \text{ there exists } u \in S \text{ such that } u \leq v \text{ and } (\beta, u, \gamma) \in U\},$$

respectively. It is clear by definition that $\mathcal{U}_R$ is a subgraph in $\mathcal{V}_R$. Notice that the assumption that $\epsilon \cap R$ is non-empty ensures that $\mathcal{U}_R$ is non-empty. For, if $e \in \epsilon \cap R$ then $e\theta = \epsilon$ whence e is idempotent since θ is idempotent pure. Thus it is the unique idempotent in the $\mathcal{R}$-class R. Therefore (ϵ, e, ϵ) is easily seen to be an arrow in $\mathcal{U}_R$.

We will find it convenient to have several properties equivalent to the defining properties of $\mathcal{U}_R$ and $\mathcal{V}_R$, respectively.

LEMMA 3.2. (i) *For any* $(\beta, u, \gamma) \in T \times S \times T$, *the following properties are equivalent:*

(a) $(\beta, u, \gamma) \in \mathrm{Arr}(\mathcal{U}_R)$,
(b) $\beta \cdot u\theta = \gamma$ *and there exists* $b \in \beta \cap R$ *such that* $b^{-1}b = uu^{-1}$,
(b′) $\beta \cdot u\theta = \gamma$, $\beta \cap R = \{b\}$ *and* $b^{-1}b = uu^{-1}$,
(c) $\gamma \cdot (u\theta)^{-1} = \beta$ *and there exists* $c \in \gamma \cap R$ *such that* $c^{-1}c = u^{-1}u$,
(c′) $\gamma \cdot (u\theta)^{-1} = \beta$, $\gamma \cap R = \{c\}$ *and* $c^{-1}c = u^{-1}u$.

Moreover, if (b) *or* (b′) *holds then* $c = bu$ *satisfies properties* (c) *and* (c′), *and conversely, if* (c) *or* (c′) *holds then* $b = cu^{-1}$ *satisfies properties* (b) *and* (b′).
(ii) *For any* $(\beta, v, \gamma) \in T \times S \times T$, *the following properties are equivalent:*
(d) $(\beta, v, \gamma) \in \mathrm{Arr}(\mathcal{V}_R)$,
(e) *there exists* $u \in S$ *such that* $u \leq v$ *and* $(\beta, u, \gamma) \in \mathrm{Arr}(\mathcal{U}_R)$.

Proof. (i) Recall (see [**11**] Proposition III.4.2) that, in an inverse semigroup, the intersection of an idempotent pure congruence and $\mathcal{R}$ is the equality relation. This immediately implies that properties (b) and (b′), and, similarly, properties (c) and (c′) are equivalent.

The implication (a) $\Rightarrow$ (b) is clear. We show (b’) $\Rightarrow$ (c) by verifying the first statement in the last sentence of (i). First notice that $\gamma \cdot (u\theta)^{-1} = \beta(u\theta)(u\theta)^{-1} = \beta \cdot (uu^{-1})\theta = \beta \cdot (b^{-1}b)\theta = \beta\beta^{-1}\beta = \beta$. Furthermore, if $c = bu$ then $(bu)\theta = \beta \cdot u\theta = \gamma$ whence $c \in \gamma$ follows, and since $cc^{-1} = (bu)(bu)^{-1} = buu^{-1}b^{-1} = bb^{-1}bb^{-1} = bb^{-1}$, we also see that $b\,\mathcal{R}\,c$, and so $c \in R$. Finally, we have $c^{-1}c = (bu)^{-1}(bu) = u^{-1}b^{-1}bu = u^{-1}uu^{-1}u = u^{-1}u$. Dually, one can check that the second statement in the last sentence of (i) is valid, and so the implication (c’) $\Rightarrow$ (b) also holds. Since (b),(c) imply $(\beta, u, \gamma) \in \mathrm{Arr}(\mathcal{C})$, we see that (b),(c) $\Rightarrow$ (a) follows.

(ii) The implication (d) $\Rightarrow$ (e) is obvious. In order to prove the reverse implication, we have to verify that if $(\beta, u, \gamma) \in \mathrm{Arr}(\mathcal{U}_R)$ and $v \in S$ with $u \leq v$ then $(\beta, v, \gamma) \in \mathrm{Arr}(\mathcal{C})$. Since $u \leq v$, we have $u = uu^{-1}v$ and $u = vu^{-1}u$. Therefore, since $(\beta, u, \gamma) \in \mathrm{Arr}(\mathcal{C})$, we see that $\beta \cdot v\theta = \beta(u\theta)(u\theta)^{-1} \cdot v\theta = \beta \cdot (uu^{-1}v)\theta = \beta \cdot u\theta = \gamma$ and $\gamma \cdot (v\theta)^{-1} = \gamma(u\theta)^{-1}(u\theta) \cdot (v\theta)^{-1} = \gamma \cdot (u^{-1}uv^{-1})\theta = \gamma \cdot ((vu^{-1}u)\theta)^{-1} = \gamma \cdot (u\theta)^{-1} = \beta$. □

The following properties of $\mathcal{U}_R$ and $\mathcal{V}_R$ will be important for us.

LEMMA 3.3. *The graphs* $\mathcal{U}_R$ *and* $\mathcal{V}_R$ *form inverse subsemigroupoids in* $\mathcal{C}$.

Proof. Taking into account the assertions (i) and (ii) in the previous lemma, one easily sees that $\mathcal{U}_R$ and $\mathcal{V}_R$, respectively, are closed under taking inverses.

Now we show that $\mathcal{U}_R$ is closed under multiplication. Let $(\beta, u, \gamma), (\gamma, v, \delta) \in \mathrm{Arr}(\mathcal{U}_R)$. Since $(\beta, u, \gamma) \circ (\gamma, v, \delta) = (\beta, uv, \delta) \in \mathrm{Arr}(\mathcal{C})$, we see that $\beta \cdot (uv)\theta = \delta$. Moreover, by applying the assumptions and Lemma 3.2, we obtain that $\{b\} = \beta \cap R$, $\{c\} = \gamma \cap R$, $b^{-1}b = uu^{-1}$ and $vv^{-1} = c^{-1}c = u^{-1}u$. Therefore $(uv)(uv)^{-1} = uvv^{-1}u^{-1} = uu^{-1}uu^{-1} = uu^{-1} = b^{-1}b$. Thus property (b′) ensures that $(\beta, uv, \delta) \in \mathrm{Arr}(\mathcal{U}_R)$. Thus $\mathcal{U}_R$ is, indeed, closed under multiplication. This immediately implies the same property of $\mathcal{V}_R$. □

LEMMA 3.4. *For any elements $\beta, \gamma \in T$, there exists at most one arrow in $\mathcal{U}_R(\beta, \gamma)$.*

Proof. Let $(\beta, u, \gamma) \in \mathrm{Arr}(\mathcal{U}_R)$. Then, by making use of property (b′), we obtain that $b^{-1}b = uu^{-1}$ and $u\theta = (uu^{-1})\theta \cdot u\theta = (b^{-1}b)\theta \cdot u\theta = \beta^{-1} \cdot \beta \cdot u\theta = \beta^{-1}\gamma$ where $\{b\} = \beta \cap R$. If (β, v, γ) is also in $\mathrm{Arr}(\mathcal{U}_R)$ then, similarly, we have $b^{-1}b = vv^{-1}$ and $v\theta = \beta^{-1}\gamma$. Thus $u\theta \cap \mathcal{R}\, v$, and so, since θ is idempotent pure, we see that $u = v$. □

LEMMA 3.5. *The semigroupoid $\mathcal{V}_R$ is closed under taking divisors in $\mathcal{C}$, that is, if $\mathrm{a}, \mathrm{b}, \mathrm{c} \in \mathrm{Arr}(\mathcal{C})$ such that $\mathrm{c} = \mathrm{a} \circ \mathrm{b}$ and $\mathrm{c} \in \mathrm{Arr}(\mathcal{V}_R)$ then $\mathrm{a}, \mathrm{b} \in \mathrm{Arr}(\mathcal{V}_R)$ follows.*

Proof. Suppose that $(\beta, u, \gamma), (\gamma, v, \delta) \in \mathrm{Arr}(\mathcal{C})$ such that $(\beta, u, \gamma) \circ (\gamma, v, \delta) = (\beta, uv, \delta) \in \mathrm{Arr}(\mathcal{V}_R)$. Then there exists $x \leq uv$ in S such that $(\beta, x, \delta) \in \mathrm{Arr}(\mathcal{U}_R)$. In order to show that $(\beta, u, \gamma) \in \mathrm{Arr}(\mathcal{V}_R)$, it suffices to verify that $xv^{-1} \leq u$ and $(\beta, xv^{-1}, \gamma) \in \mathrm{Arr}(\mathcal{U}_R)$. The inequality $x \leq uv$ implies $xv^{-1} \leq uvv^{-1} \leq u$ and $x^{-1}x \leq v^{-1}v$. On the other hand, since $(\beta, x, \delta), (\delta, v^{-1}, \gamma) \in \mathrm{Arr}(\mathcal{C})$, we have $(\beta, xv^{-1}, \gamma) \in \mathrm{Arr}(\mathcal{C})$, and so $\beta \cdot (xv^{-1})\theta = \gamma$. Furthermore, $(\beta, x, \delta) \in \mathrm{Arr}(\mathcal{U}_R)$ whence $\{b\} = \beta \cap R$ and $b^{-1}b = xx^{-1}$ follow. Hence we obtain that $(xv^{-1})(xv^{-1})^{-1} = xv^{-1}vx^{-1} = xx^{-1} = b^{-1}b$. Thus $(\beta, xv^{-1}, \gamma) \in \mathrm{Arr}(\mathcal{U}_R)$, and so $(\beta, u, \gamma) \in \mathrm{Arr}(\mathcal{V}_R)$ is proved. Dually, one can also verify that $(\gamma, v, \delta) \in \mathrm{Arr}(\mathcal{V}_R)$. □

Now we are prepared to complete the proof of Theorem 3.1. For every i $(0 \leq i \leq n)$, define $\mathcal{X}_i$ to be the subgraph in $\mathcal{C}$ determined by the set of arrows

$$\bigcup_{j=0}^{i} \{\mathrm{a}, \mathrm{a}^{-1} \colon \ \mathrm{a} \in A(w_j)\}.$$

Let R be the $\mathcal{R}$-class of S containing the element ss^{-1}. Then $ss^{-1} \in \epsilon \cap R$, so that the semigroupoid $\mathcal{U}_R$ contains the arrow (ϵ, s, α).

The most important property of the graphs $\mathcal{X}_i$ $(0 \leq i \leq n)$ is that they are contained in $\mathcal{V}_R$. This is verified in the following way. First notice that if $\mathrm{a} \in \mathrm{Arr}(\mathcal{V}_R)$ then $\mathrm{a}^{-1} \in \mathrm{Arr}(\mathcal{V}_R)$ immediately follows since $\mathcal{V}_R$ is an inverse subsemigroupoid in $\mathcal{C}$ by Lemma 3.3. To prove the assertion, we proceed by induction on i. Clearly, we have $\mathrm{Arr}(\mathcal{X}_0) = \{(\epsilon, s, \alpha), (\epsilon, s, \alpha)^{-1}\} \subseteq \mathrm{Arr}(\mathcal{U}_R) \subseteq \mathrm{Arr}(\mathcal{V}_R)$. Assume that $\mathrm{Arr}(\mathcal{X}_i) \subseteq \mathrm{Arr}(\mathcal{V}_R)$ for some i with $0 \leq i < n$. According to whether w_{i+1} is obtained from w_i by one of the steps (S0),(S0′) and (S2), by (S1), or by (S1′), the following three possibilities occur by definition:

(i) $\quad \mathrm{Arr}(\mathcal{X}_{i+1}) = \mathrm{Arr}(\mathcal{X}_i)$,

(ii) $\quad \mathrm{Arr}(\mathcal{X}_{i+1}) = \mathrm{Arr}(\mathcal{X}_i) \cup \{c, c^{-1}\} \quad (\mathrm{a}, \mathrm{b} \in \mathrm{Arr}(\mathcal{X}_i)$ with $c = \mathrm{a} \circ \mathrm{b})$,

(iii) $\quad \mathrm{Arr}(\mathcal{X}_{i+1}) = \mathrm{Arr}(\mathcal{X}_i) \cup \{\mathrm{a}, \mathrm{a}^{-1}, \mathrm{b}, \mathrm{b}^{-1}\} \quad (c \in \mathrm{Arr}(\mathcal{X}_i)$ with $c = \mathrm{a} \circ \mathrm{b})$.

By applying the induction hypothesis and, in cases (ii) and (iii), the facts that $\mathcal{V}_R$ is closed under multiplication, and under taking divisors, respectively, we infer that $\mathrm{Arr}(\mathcal{X}_{i+1}) \subseteq \mathrm{Arr}(\mathcal{V}_R)$.

In particular, this implies that $\mathrm{Arr}(\mathcal{X}_n) \subseteq \mathrm{Arr}(\mathcal{V}_R)$, and so $(\epsilon, t, \alpha) \in \mathrm{Arr}(\mathcal{V}_R)$. Hence, by definition, there exists $x \leq t$ in S such that $(\epsilon, x, \alpha) \in \mathrm{Arr}(\mathcal{U}_R)$. However, we have seen above that $(\epsilon, s, \alpha) \in \mathrm{Arr}(\mathcal{U}_R)$ whence it follows by Lemma 3.4 that

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$x = s$. Thus we have shown that $s \leq t$ holds in S. By symmetry, the reverse inequality is also valid, completing the proof of the equality $s = t$. □

Remark 3.6. In the special case where θ is a group congruence, or, equivalently, when S is an E-unitary inverse semigroup and θ is the least group congruence on S, then the proof can be considerably simplified and made more transparent as follows. In this special case, $\mathcal{C}$ is necessarily a semigroupoid whose 'local subsemigroups' $\mathcal{C}(\alpha, \alpha)$ ($\alpha \in T$) are semilattices. Thus, by Simon's Lemma ([**4**], cf. also [**16**]), the following holds: for any two coterminal paths $p = \mathrm{a}_1\mathrm{a}_2 \ldots \mathrm{a}_k$ and $q = \mathrm{b}_1\mathrm{b}_2 \ldots \mathrm{b}_m$ on the graph $\mathcal{C}$ which 'span' the same subgraph in $\mathcal{C}$, that is, which satisfy $A(p) = A(q)$, we have the same product $\mathrm{a}_1 \circ \mathrm{a}_2 \circ \ldots \circ \mathrm{a}_k = \mathrm{b}_1 \circ \mathrm{b}_2 \circ \ldots \circ \mathrm{b}_m$. Given a finite, 'symmetric', connected subgraph $\mathcal{X}$ in $\mathcal{C}$ and two objects β, γ in $\mathrm{Obj}(\mathcal{X})$, this allows us to define an arrow $e(\mathcal{X}, \beta, \gamma)$ as the product of any (β, γ)-path spanning $\mathcal{X}$. (A subgraph is termed 'symmetric' if it is closed under taking inverses.) Therefore, instead of introducing subgraphs $\mathcal{U}_R$ and $\mathcal{V}_R$ and proving Lemmas 3.2–3.5, all we have to do is to assign $e(\mathcal{X}_i, 1, \alpha)$ to $\mathcal{X}_i$ for every $i\,(0 \leq i \leq n)$ and to observe that $e(\mathcal{X}_{i+1}, 1, \alpha) = e(\mathcal{X}_i, 1, \alpha)$ for each $i\ (0 \leq i < n)$. Note that $e(\mathcal{X}_i, 1, \alpha)$ $(0 \leq i \leq n)$ is just the unique element of $\mathcal{U}_R(1, \alpha)$.

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