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NON-NILPOTENT GROUPS IN WHICH EVERY PRODUCT OF FOUR ELEMENTS CAN BE REORDERED

M. MAJ AND S. E. STONEHEWER

1. Introduction. Let G be a group and $n(\geqq 2)$ an integer. We say that G belongs to the class of groups P_n if every product of n elements can be reordered, i.e. for all n-tuples $(x_1, x_2, \ldots, x_n)$, $x_i \in G$, there exists a non-trivial element σ in the symmetric group Σ_n such that

$$x_{\sigma(1)}x_{\sigma(2)}\ldots x_{\sigma(n)} = x_1x_2\ldots x_n.$$

Let P denote the union of the classes P_n, $n \geqq 2$. Clearly every finite group belongs to P and each class P_n is closed with respect to forming subgroups and factor groups.

Trivially P_2 is the class of abelian groups and in [**2**] P_3 was shown to be precisely those groups G for which the derived subgroup G' has order $\leqq 2$. Also the class P is known to coincide with the class of groups G possessing a subgroup N with $|G : N|$ and N' both finite [**3**]. The situation with regard to P_4 seems to be more complicated. Graham Higman [**4**] considered the problem and obtained two striking results. First, a group G with $G' \cong V_4$ (the 4-group) always belongs to P_4; and secondly a finite group G of odd order belongs to P_4 if and only if (i) G is abelian or (ii) $|G'| = 3$ or (iii) $|G'| = 5$ and G modulo its centre has order 25. Next it was shown in [**1**] that if a finite group G belongs to P_4, then G' is nilpotent. This was improved in [**5**] where *all* P_4-groups were shown to be metabelian.

The purpose of this work is to take the classification of P_4-groups a stage further and we shall give a complete description of the non-nilpotent groups in P_4. A contribution to the nilpotent case by P. Longobardi and the second author will appear elsewhere and a third and final contribution by all three authors giving the complete classification of P_4-groups will combine all the previous results. The non-nilpotent case, however, provides a convenient self-contained exercise, using ideas and methods peculiar to that case. The main result is:

THEOREM. *A group G belongs to P_4 if and only if one of the following holds:*

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(i) *G has an abelian subgroup of index 2;*
(ii) *G is nilpotent of class $\leqq 4$ and $G \in P_4$;*
(iii) *$G' \cong V_4$;*
(iv) *$G = B\langle a, x\rangle$, where $B \leqq Z(G)$, $|a| = 5$ and $a^x = a^2$.*

Notation is as follows.

C_n	a cyclic group of order n,
V_4	the 4-group,
Σ_n	the symmetric group of degree n,
G'	derived subgroup of G,
$Z(G)$	centre of G,
$Z_i(G)$	ith term of the upper central series of G,
C_G	centraliser in G,
$\|g\|$	order of element g,
g^x	$x^{-1}gx$,
$[x, y]$	$x^{-1}y^{-1}xy$.

After some technical lemmas, abelian-by-cyclic groups are studied in detail and then the finite non-nilpotent P_4-groups are classified. Local arguments allow us to pass to infinite groups.

2.1. Some technical preliminaries

2.1.1. Let $G \in P_4$ and A be an abelian subgroup of G containing G'. If $a, b \in A$ and $x, y \in G$, then at least one of the following holds:
(i) $[a, x] = 1$;
(ii) $[a, y] = 1$;
(iii) $[a, x^{-1}y] = 1$;
(iv) $[a, x] = [b, x]$;
(v) $[b, x] = 1$;
(vi) $[b, x] = [y, a]^x$;
(vii) $[b, xy] = [y, a]^x$;
(viii) $[y, a]^x = [b, y]$;
(ix) $[a, x] = [ab, y]$;
(x) $[a, x] = [b, y]$;
(xi) $[b, xy] = 1$;
(xii) $[b, xy] = [a, x]$;
(xiii) $[b, xy] = [a, x][y, a]$.

Proof. If $[x, y] = 1$, then the result follows without difficulty by considering all the possible rearrangements of the product $yaxb$. Thus if there are elements $c, d \in A$ such that $[xc, yd] = 1$, then, with xc, yd for x, y respectively, we obtain the same conclusion (observing that the commutators in (i)–(xiii) remain unchanged). Therefore we may assume that $[xc, yd] \neq 1$ for all $c, d \in A$, and again one easily checks that the only possible rearrangements of $yaxb$ lead to one of the listed relations. ∎

A special case of this result will be useful.

2.1.2. Let $G \in P_4$ and A be an abelian subgroup of G containing G'. Let $a, b \in A$, $x, y \in G$ and suppose that $[a, x]$, $[a, y]$ and $[a, x^{-1}y]$ are all different from 1 and $[b, y] = 1$.

(i) If $[b, x]$ has order 2 and commutes with x, then

$$[b,\ x] = [a,\ x],\ [a,\ y] \text{ or } [a,\ x][y,\ a].$$

(ii) If $[a, y]$ has order 2 and commutes with x, then

$$[b,\ x] = 1,\ [a,\ x],\ [a,\ y] \text{ or } [a,\ x][a,\ y].$$

Proof. (i) One checks easily from the hypotheses that the only possibilities in 2.1.1 are (iv), (vi), (vii), (xii) and (xiii), giving the result.

(ii) Again the only possibilities in 2.1.1 are (iv), (v), (vi), (vii), (xi), (xii) and (xiii), hence the result. ∎

We apply 2.1.2 immediately.

2.1.3. Let $G \in P_4$ be a finite 2-group and A be an abelian subgroup of G containing G'. If $G = A\langle x \rangle$, then one of the following holds:

(1) $[A, x^2] = 1$;

(2) $G' \cong V_4$;

(3) $G' \cong C_4$ and $G' \leqq Z(G)$.

Proof. Suppose that $x^2 \notin C_G(A)$ and choose an element b in $(Z_2(G) \cap A) \backslash Z(G)$ of minimal order. Then

$$1 = [b^2,\ x] = [b,\ x^2]$$

and $[b, x] \neq 1$. Now let a be an element of A such that $[a,\ x^2] \neq 1$. We claim that

(i) $[a, x^2] = [b, x]$ and

(ii) if $c \in A$ and $[c, x^2] = 1$ with $[c, x] \neq 1$, then $[c, x] = [a, x^2]$.

For, taking $y = x^2$ in 2.1.2(i), we have

$$[b,\ x] = [a,\ x],\ [a,\ x^2] \text{ or } [a,\ x][x^2,\ a](= [x,\ a]^x).$$

But if $[b,x] = [a,x]$, then $[a,x^2] = [b,x^2] = 1$, a contradiction. The third possibility coincides with the first and so (i) follows. Then take c and x^2 for b and y, respectively, in 2.1.2(ii). This is permissible, since $[a,x^2]$ has order 2 and commutes with x, by (i). If $[c,x] = [a,x]$ or $[a,x][a,x^2]$, then $1 = [c,x^2] = [a,x^2]$, a contradiction, and so (ii) holds.

From (i) we have

$$A = \langle a \rangle C_A(x^2).$$

We distinguish two possibilities:

Case (a). Suppose that $[a^2,x] \neq 1$. Then, by (i),

$$[a^2,\ x^2] = 1;$$

and, by (ii),

$$[a^2,\ x] = [a,\ x^2].$$

Therefore $[a,x] \in Z(G)$ and $|[a,x]| = 4$. Again by (ii),

$$[C_A(x^2),x] \leqq \langle [a^2,x] \rangle$$

and so $G' = [A,x] = \langle [a,x] \rangle$, i.e. (3) holds.

Case (b). Suppose that $[a^2,x] = 1$. Then

$$[b,\ x] = [a,\ x^2] = [a,\ x,\ x]$$

and

$$V_4 \cong \langle [a,\ x],\ [b,\ x] \rangle \lhd G.$$

By (ii), $[C_A(x^2),x] \leqq \langle [a,x^2] \rangle = \langle [b,x] \rangle$ and it follows that $G' = \langle [a,x],\ [b,x] \rangle$, i.e. (2) holds. ∎

2.2. Finite P_4-groups: Part 1. In this paragraph we obtain preliminary results for the later description (in 2.3) of finite P_4-groups.

Throughout, G will be a finite P_4-group and A will denote a maximal abelian subgroup of G containing G'.

We shall use the following observation (see [**1**]) repeatedly. Let a,b be elements of a P_4-group and $c = [a,b]$ with $c^2 \neq 1$. Since $a^{-1}b^{-1}ab$ can be rearranged, it is easy to check that, by conjugation,

(1) a, b or ab inverts or centralises c.

2.2.1. Suppose that $G = A\langle x \rangle$. Then

$$A \leqq C_G(x^4) \cup C_G(x^3) \cup Z_3(G).$$

Proof. Let $a \in A$ and let

$$x_1 = x^{-1}, \quad x_2 = a, \quad x_3 = x, \quad x_4 = xa.$$

By considering the rearrangements of the product

$$x_1 x_2 x_3 x_4,$$

it is easy to see that either

$$[a,\ x^2] = 1 \text{ or } [a,\ x^3] = 1$$

or one of the following holds:
(i) $ax^2a = xa^2x$, or
(ii) $ax^2a = xax^2ax^{-1}$, or
(iii) $ax^2a = x^3a^2x^{-1}$.
From (i) we obtain $[a,\ x^2] = [a^2,\ x]$ and hence $[a,\ x,\ x] = 1$, i.e. $a \in Z_2(G)$. If (ii) holds, we have

$$[ax^2a,\ x] = 1$$

and so $[a,x]^{x^2} = [x,a]$. Then $[a,x^4] = 1$. Finally suppose that (iii) holds. Thus

$$x^{-2}ax^2a^{-1} = xa^2x^{-1}a^{-2}$$

and hence

$$[a,\ x^2] = [a^2,\ x^{-1}] = [x,\ a^2]^{x^{-1}}.$$

By (1) it follows that either $|[a,x]| \leqq 2$ or x^2 inverts or centralises $[a,x]$. In the first case $[a^2,x] = 1$ and so $[a,x^2] = 1$. If $[a,x]^{x^2} = [x,a]$, then $[a,x^4] = 1$. If

$$[a,\ x,\ x^2] = 1,$$

then

$$[a,\ x^2]^x = [a,\ x^2] = [x,\ a^2]$$

and therefore $[a^2,x,x] = 1$. Thus

$$[a,\ x,\ x]^x = [a,\ x,\ x]^{-1} = [a,\ x,\ x] \text{ and } a \in Z_3(G). \qquad \blacksquare$$

2.2.2. Let $G = A\langle x\rangle$. Then $x^4 \in A$ or $x^3 \in A$ or G is nilpotent of class $\leqq 3$.

Proof. We have $Z(G) = C_A(x) = C_A(x^4) \cap C_A(x^3) \leqq A \cap Z_3(G)$. By 2.2.1,

$$A = C_A(x^4) \cup C_A(x^3) \cup (A \cap Z_3(G)).$$

If the result is false, then A is covered by 3 proper subgroups and so $A/Z(G) \cong V_4$ ([**6**]). Thus $[a^2, x] = 1$, all $a \in A$. Therefore if $a \in Z_3(G)$,

$$\begin{aligned}[a,\ x^4] &= [a,\ x^2][a,\ x^2]^{x^2}\\ &= [a,\ x,\ x][a,\ x,\ x]^{x^2}\\ &= [a,\ x,\ x]^2 = 1.\end{aligned}$$

Hence $A = C_A(x^4) \cup C_A(x^3)$, a contradiction. ∎

Now we make further applications of 2.1.1 to yield

2.2.3. Let $G = A\langle x\rangle$ and a be a p-element of A (p prime) with $[a,\ x^2] \neq 1$. Then x centralises the p-complement of A.

Proof. Let b be a p'-element of A and put $y = x^2$. Assume, for a contradiction, that $[b, x] \neq 1$. Then (xi) of 2.1.1 must hold, i.e. $[b, x^3] = 1$. Taking y in the notation of 2.1.1 to be x^{-2} here, we must have $[a, x^3] = 1$ and so

(2) $\qquad [ab,\ x^3] = 1.$

Now either $[a, x]^2 \neq 1$ or $[b, x]^2 \neq 1$ and hence

$$[ab, x]^2 \neq 1.$$

With abx and x replacing a, b respectively in (1), it follows that x^4 must centralise $[ab, x]$ and therefore, by (2), $[ab, x, x] = 1$. Thus

$$1 = [ab,\ x^3] = [ab,\ x]^3 = [a,\ x]^3[b,\ x]^3$$

and so $[a,\ x]^3 = [b,\ x]^3 = 1$, a contradiction. ∎

Further relations in the situation of 2.2.3 are contained in

2.2.4. Suppose that $G = A\langle x\rangle$ and a is an element of A such that $[a,\ x^2] \neq 1$. Then one of the following holds: (i) $[a, x, x, x] = 1$; (ii) $|[a, x]| = 2$; (iii) $|[a, x^2]| = 2$; (iv) $[a, x]^x = [a, x]^2$; (v) $[a, x]^x = [a, x]^{-2}$.

Proof. Suppose that neither (i) nor (ii) holds. By 2.2.1 we must have

$$[a,\ x^4] = 1 \text{ or } [a, x^3] = 1.$$

With ax and x for a, b in (1), we have that x^2 centralises or inverts $[a,x]$ and therefore if $[a,x^3]=1$, it follows that $[a,x,x]=1$, a contradiction. Thus

$$[a,x^4]=1.$$

Hence

$$[a,\ x^3]=[a,x^{-1}]=[x,a]^{x^{-1}}.$$

In the notation of 2.2.1, take $b=a^x$ and $y=x^2$. Then we have

$$|[a,\ x^2]|=2 \text{ or } [a,x]=[x^2,\ a] \text{ or } [x^2,\ a]=[x,\ a]^{x^{-1}}.$$

The second possibility gives (v). Therefore suppose that

$$[x^2,a]=[x,\ a]^{x^{-1}}.$$

Then $[x,a]^{x^2}[x,a]^x=[x,a]$. Recalling that

$$[a,\ x]^{x^2}=[a,\ x]^{\pm 1},$$

we obtain (iv). ∎

When G is not nilpotent we can describe G' precisely. Thus

2.2.5. Let $G=A\langle x\rangle$ and a be an element of A such that $[a,x^2]\neq 1$. If G is not nilpotent, then $G'=\langle [a,x]\rangle^G$.

Proof. Let $b\in A$. By 2.2.2, either $x^3\in A$ or $x^4\in A$. If $x^3\in A$, a consideration of the rearrangements of the product

$$x^{-1}a(bx)x$$

shows that $[b,x]\in\langle [a,x]\rangle^G$. If $x^4\in A$, then a similar consideration of

$$x^{-1}ax(bx)$$

gives the same conclusion. ∎

Now we turn our attention to the case when G/A is not necessarily cyclic.

2.2.6. Let $x,y\in G$ and $a\in A$.

(a) If $[a,x,y]=1$, then $[a,x,x]=1$ or $[a,y]=1$ or $[a,x,x,x]=[a,y,y]=1$. Now suppose that $x^2,y^2\in A$. Then

(b) $[a,x]\in C_G(x)\ \cup\ C_G(y)\ \cup\ C_G(xy)$;

(c) $a^2 \in C_G(x) \cup C_G(y) \cup C_G(xy)$; and
(d) if $[a^2, x] = [a^2, y] = 1$, then $[a, x, y] = [a, x, x] = [a, y, y] = 1$.

Proof. (a) Clearly $[a, x, y] = 1$ implies $[a, y, x] = 1$. Let

$$b = [a^{-1}, x].$$

Then $[b, y] = 1$ and $[b, xy] = [b, x]^y = [b, x]$. From 2.1.1 there are 13 possibilities:

(i) $[a, x] = 1$; or
(ii) $[a, y] = 1$; or
(iii) $[a, y] = [a^{-1}, x^{-1}]^y = [a, x]^{x^{-1}y} = [a, x]^{x^{-1}} = [a, x]$ and so $[a, x, x] = 1$; or
(iv) $[a, x] = [a^{-1}, x, x]$, i.e. $[a, x] = 1$; or
(v) $[a^{-1}, x, x] = 1$, i.e. $[a, x, x] = 1$; or
(vi) $[a^{-1}, x, x] = [y, a]$, i.e. $[a, x, x] = [a, y]$ and so $[a, x, x, x] = [a, y, y] = 1$; or
(vii) $[b, x] = [y, a]$ as in (vi); or
(viii) $[a, y] = 1$; or
(ix) $[a, x] = [a, y]$ as in (iii); or
(x) $[a, x] = 1$; or
(xi) $[a^{-1}, x, x] = 1$ as in (v); or
(xii) $[b, x] = [a, x]$ as in (iv); or finally
(xiii) $[a^{-1}, x, x] = [a, x][y, a]$, i.e. $[a^{-1}, x, x, x] = [a, x, x]$ and so $[a, x, x] = 1$.
Thus in all cases we obtain the required conclusion.

(b) Observe now that, by conjugation, x inverts $[a, x]$, y inverts $[a, y]$ and xy inverts $[a, xy]$. Taking b in 2.1.1 to be a^{-1} here, the only possibilities which do not immediately give our requirements are

(iv) $[a, x] = [a^{-1}, x]$ and so $1 = [a, x][a, x]^x = [a, x, x]$;
(vi) $[a^{-1}, x] = [y, a]^x$, i.e. $[a, x] = [y, a]$ which is inverted by x and y and therefore centralised by xy;
(vii) $[a^{-1}, xy] = [y, a]^x$, i.e. conjugating by xy,

$$[a, xy] = [a, y]$$

and so $[a, x] = 1$;
(viii) $[y, a]^x = [a^{-1}, y]$, i.e. $[a, y]^x = [a, y]$ and therefore $[a, x]^y = [a, x]$;
(x) $[a, x] = [a^{-1}, y]$ as in (vi);
(xi) $[a^{-1}, xy] = 1$, i.e. $[a, y][a, x]^y = 1$ and so $[a, x] = [a, y]$ which is centralised by xy;
(xii) $[a^{-1}, xy] = [a, x]$ which is centralised by x^2y and therefore by y;
(xiii) $[a^{-1}, xy] = [a, x][y, a]$, i.e. $[a^{-1}, x]^y = [a, x]$ and so xy centralises $[a, x]$.

(c) By (b), $[a,x] \in C_G(x) \cup C_G(y) \cup C_G(xy)$. If $[a,x,x] = 1$, then $1 = [a,x^2] = [a^2,x]$. If $[a,x,y] = 1$, then by (a) either $[a,x,x] = 1$ (whence again $[a^2,x] = 1$) or $[a,y,y] = 1$ and so similarly $[a^2,y] = 1$. Finally if $[a,x,xy] = 1$, then again by (a) either $[a,x,x] = 1$ or $[a,xy,xy] = 1$, i.e. $[a^2,xy] = 1$.

(d) From $1 = [a,x^2] = [a,x]^2[a,x,x] = [a^2,x][a,x,x]$, we get $[a,x,x] = 1$. Similarly $[a,y,y] = 1$. Taking b in 2.1.1 to be $[a,y]$ here, it follows without difficulty that $[a,x,y] = 1$. ∎

2.2.7. Suppose that G/A is a non-cyclic elementary abelian 2-group. Then the 2-complement of A is contained in $Z(G)$.

Proof. Let B be the 2-complement and C the 2-component of A. So $A = B \times C$. Choose $x \in G \setminus A$. It suffices to show that $[B,x] = 1$.

By hypothesis there exists $y \in G \setminus A$ such that

$$\langle xA,\ yA\rangle = \langle xA\rangle \times \langle yA\rangle.$$

From 2.2.6(c) we have

$$B = B^2 \leqq C_G(x) \cup C_G(y) \cup C_G(xy)$$

and hence (see [**6**]) B lies in the centraliser of x, y or xy. Suppose, for a contradiction, that $[B,x] \neq 1$. Then without loss of generality $[B,y] = 1$. Since $y \notin A$, there is a 2-element $c \in A$ such that $[c,y] \neq 1$. Let $b \in B$ with $[b,x] \neq 1$. Since $[b,x]$ and $[c,x]$ have coprime orders,

$$[b^{-1}c,\ x] \neq 1.$$

Similarly $[b^{-1}c, x^{-1}y] \neq 1$; and $[b^{-1}c, y] = [c,y] \neq 1$. Taking a in the notation of 2.1.1 to be $b^{-1}c$ here, it follows easily that either $[b,x]$ or $[b^2,x]$ has even order. Thus $[b^2,x] = 1$ and so $[b,x] = 1$, a contradiction. ∎

2.3. Finite P_4-groups: Part 2. In this paragraph we classify the finite, non-nilpotent P_4-groups. It will transpire that they are abelian-by-cyclic (see 2.3.2). Thus we begin with

2.3.1. Let $G = A\langle x\rangle$ be a finite P_4-group, where A is a maximal abelian subgroup of G containing G'. Then one of the following holds:

(i) $x^2 \in A$; or

(ii) G is nilpotent; or

(iii) $G' \cong V_4$; or

(iv) $G = B\langle a, x\rangle$, where $B \leqq Z(G)$, $a \in A$, $|a| = 5$ and $a^x = a^2$.

Proof. Suppose that G is not nilpotent and $x^2 \notin A$. Then there exists $a \in A$ such that a is a p-element, for some prime p, and $[a, x^2] \neq 1$. By 2.2.3

$$A = A_1 \times A_2$$

where A_1 is the p-complement of A and lies in $Z(G)$ and A_2 is the p-component of A. Since G is not nilpotent, A_2 is not contained in the hypercentre of G. Also $C_{A_2}(x^2) < A_2$. Using the fact that a group cannot be the set-theoretic union of 2 proper subgroups, we may assume that a does not lie in the hypercentre of G. In particular

$$(3) \qquad [a,\ x,\ x,\ x] \neq 1.$$

By 2.2.2, either $x^3 \in A$ or $x^4 \in A$.

Assume first that $x^3 \in A$. If $p = 3$, then G/A_1 is a 3-group and hence G is nilpotent, a contradiction. Therefore $p \neq 3$. If $|[a, x]| \neq 2$, then with ax and x for a, b in (1), 2.2, it follows that $[a, x, x^4] = 1$. Since $[a, x, x^3] = 1$, we have $[a, x, x] = 1$, contradicting (3). Thus $|[a, x]| = 2$. Then from $[a, x^3] = 1$ we obtain

$$[a,\ x]^{x^2} = [a,\ x^2] = [a, x][a,\ x]^x$$

and hence

$$\langle [a,\ x] \rangle^G = \langle [a, x], [a, x]^x \rangle \cong V_4.$$

Therefore, by 2.2.5, $G' \cong V_4$.

Now suppose that $x^4 \in A$. Then $p \neq 2$ since G is not nilpotent. Therefore by 2.2.4 and (3)

$$[a,\ x]^x = [a,\ x]^{\pm 2}.$$

Since $[a, x^4] = 1$, it follows that $[a, x^2]^{x^2} = [x^2, a]$ and so

$$[a,\ x]^{x^2} \neq [a,\ x].$$

Therefore by (1) in 2.2

$$[a,\ x]^{x^2} = [x,\ a]$$

and hence $[x, a] = [a, x]^4$ and $|[a, x]| = 5$. Let $c = [a, x]$. By 2.2.5, $G' = \langle c \rangle = \langle [c, x] \rangle$. If $b \in A$, then

$$[b,\ x] = [c^\alpha,\ x],$$

for some α, and so $bc^{-\alpha} \in Z(G)$. Therefore

$$A = (Z(G) \cap A)\langle c \rangle$$

and thus, with $B = Z(G) \cap A, G = B\langle c, x \rangle$. Since $c^x = c^2$ or $c^{x^{-1}} = c^2$, we have established (iv) (replacing a by c and x by x^{-1} if necessary). ∎

Now we can establish the nilpotency of the finite P_4-groups which are not abelian-by-cyclic.

2.3.2. Let G be a finite P_4-group and A be a maximal abelian subgroup of G containing G'. Suppose that G/A is not cyclic. Then G is nilpotent.

Proof. Suppose, for a contradiction, that G is not nilpotent. Then there exists $x \in G$ such that

$$A\langle x \rangle \text{ is not nilpotent.}$$

If G/A has exponent 2, then $G/Z(G)$ is a 2-group (by 2.2.7), contradicting the fact that G is not nilpotent. Therefore there is an element $y \in G$ such that $y^2 \notin A$. Thus we may assume that $x^2 \notin A$, since $A\langle y \rangle$ and $A\langle xy \rangle$ cannot both be nilpotent.

Let $H = A\langle x \rangle$. By 2.3.1, either $H' \cong V_4$ or $H' \cong C_5$. Thus $G/C_G(H')$ is cyclic. Now let $g \in C_G(H')$. For any $a \in A$, either $[a, x, x, x] = 1$ or $[a, g] = 1$, by 2.2.6(a). Therefore $A \subseteq Z_3(H) \cup C_G(g)$. Since H is not nilpotent, $A \nsubseteq Z_3(H)$ and hence $g \in C_G(A) = A$. Thus $C_G(H') \leqq A$ and so G/A is cyclic, a contradiction. ∎

Now we turn our attention to nilpotentfinite P_4-groups and show (in 2.3.4) that either they have class $\leqq 4$ or they have an abelian subgroup of index 2.

First we have

2.3.3. Let $G = A \times B$ be a finite P_4-group with A of odd order and B a 2-group. Then either A or B is abelian.

Proof. Since G is metabelian, $G' = A' \times B'$ is abelian; and, by [**4**], $A' \leqq Z(G)$. Suppose that A is not abelian and choose $b, x \in A$ such that $[b, x] \neq 1$. We claim that

(4) $\qquad C_B(B') = Z(B).$

Then since $B' \leqq C_B(B')$, it follows that B is abelian as required.

Suppose, for a contradiction, that (4) is false and choose $c \in C_B(B'), y \in B$ such that $|[c, y]| = 2$. Then

$$[b, y] = [b, c] = [c, x] = 1.$$

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Let $C = \langle G', b, c\rangle$ and $a = b^{-1}c$. Clearly C is abelian,

$$[a,\ x] = [b,\ x]^{-1},\ [a,\ y] = [c,\ y]$$

and

$$[a,\ x^{-1}y] = [c,\ y][b^{-1},\ x^{-1}]^{y} = [c,\ y][b,\ x].$$

Then an easy application of 2.1.2(ii) gives the required contradiction. ∎

This leads to information about finite nilpotent P_4-groups.

2.3.4. Let G be a finite nilpotent P_4-group. Then either G has class $\leqq 4$ or G has an abelian subgroup of index 2.

Proof. By Higman's characterisation of the finite P_4-groups of odd order [**4**] and 2.3.3, we may assume that G is a 2-group. Let A be a maximal abelian subgroup of G containing G'.

Consider first the case in which G has an element x with $x^2 \notin A$ and let $H = A\langle x\rangle$. Then $H \lhd G$ and $|H'| = 4$, by 2.1.3. Thus $H' \leqq Z_2(G)$ and so $[A, x] \leqq Z_2(G)$. Now let $y \in G$. If $y^2 \notin A$, then similarly $[A, y] \leqq Z_2(G)$. If $y^2 \in A$, then $(xy)^2 \notin A$ and so $[A, xy] \leqq Z_2(G)$, i.e. $[A, y] \leqq Z_2(G)$. Hence

$$G' \leqq A \leqq Z_3(G)$$

and G has class $\leqq 4$.

Now it remains to consider the case in which G/A is elementary abelian, but not cyclic. We claim that

(5) $$[a^4,\ x] = 1 \quad \text{for all} \quad a \in A,\ x \in G.$$

For, suppose $x \notin A$ and choose $y \in G$ such that

$$\langle x,\ y\rangle A/A = \langle xA\rangle \times \langle yA\rangle$$

has order 4. From 2.2.6(a) and (b) it follows easily that, for any $a \in A$,

$$[a,\ x,\ x,\ x] = 1 \quad \text{or} \quad [a,\ y] = 1 \quad \text{or} \quad [a,\ xy] = 1.$$

Therefore $A \subseteq Z_3(A\langle x\rangle) \cup C_A(y) \cup C_A(xy)$. By the maximality of A, $C_A(y)$ and $C_A(xy)$ are proper subgroups of A. Thus, by [**6**],

$$\text{either} \quad A \leqq Z_3(A\langle x\rangle) \quad \text{or} \quad a^2 \in C_A(y) \cap C_A(xy), \quad \text{all} \quad a \in A.$$

In the first case, $[A, x, x, x] = 1$ and then $[a^4, x] = 1$, for all $a \in A$, since $x^2 \in A$. In the second case, $[a^2, x] = 1$ for all $a \in A$. Therefore (5) is true.

It follows from 2.2.6(d) that $[a^2, g] \in Z(G)$, for all $a \in A$, $g \in G$. Hence, again by the same result, $[a, g] \in Z_2(G)$. Therefore $G' \leqq A \leqq Z_3(G)$ and so G has class $\leqq 4$. ∎

So far we have considered only consequences of $G \in P_4$. We end this paragraph with a complete characterisation of the finite P_4-groups which are not nilpotent.

2.3.5. Let G be a finite group. Then $G \in P_4$ if and only if one of the following holds:

(i) G has an abelian subgroup of index 2;

(ii) G is nilpotent of class $\leqq 4$ and $G \in P_4$;

(iii) $G' \cong V_4$;

(iv) $G = B\langle a, x\rangle$, where $B \leqq Z(G)$, $|a| = 5$ and $a^x = a^2$.

Proof. Suppose that $G \in P_4$ and let A be a maximal abelian subgroup of G containing G'. Suppose also that $|G/A| \neq 2$. If G/A is cyclic, then one of (ii), (iii), (iv) holds, by 2.3.1 and 2.3.4. If G/A is not cyclic, then G is nilpotent, by 2.3.2, and of class $\leqq 4$, by 2.3.4, i.e. (ii) holds.

Conversely, suppose that (i) holds. Then an easy exercise shows that $G \in P_4$. If (iii) holds, then Higman ([**4**]) has shown that $G \in P_4$. Finally suppose that (iv) holds. If $x^4 = 1$, then

$$\langle a,\ x\rangle = \langle a\rangle \rtimes \langle x\rangle$$

with x acting faithfully on $\langle a\rangle$. Embedding $\langle a,\ x\rangle$ in Σ_5 with $a = (1\,2\,3\,4\,5)$ and $x = (2\,3\,5\,4)$, Derek Holt (to whom we are most grateful) has shown, using CAYLEY on the Mathematics Institute computer at Warwick University, that $\langle a, x\rangle \in P_4$. Alternatively this can be established by a long and tedious hand calculation which we omit. Thus $G \in P_4$.

Now suppose that $x^4 \neq 1$ and let $H = \langle x^4\rangle$. Then $H \leqq Z(G)$ and (by the previous case) for any $x_1, x_2, x_3, x_4 \in \langle a, x\rangle$, there exists $\sigma \in \Sigma_4, \sigma \neq 1$, such that

$$x_1x_2x_3x_4 \equiv x_{\sigma(1)}x_{\sigma(2)}x_{\sigma(3)}x_{\sigma(4)} \bmod H.$$

We have $x_i = a^{\alpha_i}x^{\beta_i}$ for integers α_i, β_i, $1 \leqq i \leqq 4$. Thus there are integers γ, δ such that

$$x_1x_2x_3x_4 = a^{\gamma}x^{\beta} \text{ and}$$

$$x_{\sigma(1)}x_{\sigma(2)}x_{\sigma(3)}x_{\sigma(4)} = a^{\delta}x^{\beta},$$

where $\beta = \beta_1 + \beta_2 + \beta_3 + \beta_4$. Therefore $a^\gamma \equiv a^\delta \bmod H$ and so $a^\gamma = a^\delta$. Thus

$$x_1x_2x_3x_4 = x_{\sigma(1)}x_{\sigma(2)}x_{\sigma(3)}x_{\sigma(4)}$$

and $\langle a, x\rangle \in P_4$. Hence $G \in P_4$. ∎

2.4. Arbitrary P_4-groups. A finitely generated P_4-group is polycyclic ([**3**]) and hence residually finite. Then it is not difficult to extend 2.3.5 to infinite groups and to obtain our Theorem, stated in the introduction.

Proof of the Theorem. Let $G \in P_4$ and suppose, for a contradiction, that none of (i)–(iv) holds. Using local arguments it is not difficult to see that we may assume that G is finitely generated and therefore residually finite. Similarly it then follows easily that G has a finite quotient which does not satisfy any of (i)–(iv), contradicting 2.3.5.

For the converse, the argument of 2.3.5 applies. ∎

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Dipartimento Di Matematica
Ed Applicazioni,
Via Mezzocannone 8,
80134 Napoli,
Italy

Mathematics Institute
University of Warwick,
Coventry CV4 7AL,
England.